INVERSE SATAKE ISOMORPHISM AND CHANGE OF WEIGHT

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Abstract. Let G be any connected reductive p-adic group. Let $K \subset G$ be any special parahoric subgroup and V, V' be any two irreducible smooth $\overline{\mathbb{F}}_p[K]$-modules. The main goal of this article is to compute the image of the Hecke bi-module $\mathrm{End}_{\overline{\mathbb{F}}_p[K]}(\text{c-Ind}_K^G V, \text{c-Ind}_K^G V')$ by the generalized Satake transform and to give an explicit formula for its inverse, using the pro-p Iwahori Hecke algebra of G. This immediately implies the "change of weight theorem" in the proof of the classification of mod p irreducible admissible representations of G in terms of supersingular ones. A simpler proof of the change of weight theorem, not using the pro-p Iwahori Hecke algebra or the Lusztig-Kato formula, is given when G is split (and in the appendix when G is quasi-split, for almost all K).

Contents

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1. Introduction

1.1. Throughout this paper, F is a local nonarchimedean field with finite residue field k of characteristic p, $\mathbf{G}$ is a connected reductive F-group, and C is an algebraically closed field of characteristic p. In our previous paper [AHHV17], we gave a classification of irreducible admissible smooth C-representations of $G = \mathbf{G}(F)$ in terms of supercuspidal representations of Levi subgroups of G. The most subtle ingredient in our proofs is the so-called "change of weight theorem", which we deduced from the existence of certain elements in the image of the mod p Satake transform. The main goal of this paper is to determine its image entirely and give an explicit formula for the inverse of the mod p Satake transform, we call it the *inverse Satake theorem*, from which the change of weight is an immediate consequence.

To be a bit more precise, the mod p Satake transform can be defined for the Hecke algebra of a single irreducible representation V of a special parahoric subgroup, as well as more generally for the Hecke bimodule of a pair (V, V') of such irreducible representations. The image of the mod p Satake transform was known in case of a single irreducible representation V of a special parahoric subgroup, cf. [HV15], [Her11b]. However, for the change of weight theorem it is essential to allow pairs (V, V') with $V \ncong V'$.

In earlier work [Her11a, Prop. 5.1], we established the inverse Satake theorem when $\mathbf{G}$ is split with simply-connected derived subgroup and $V = V'$ by deducing it from the Lusztig-Kato formula, which is an inverse formula for the usual Satake transform in characteristic zero. (See also the related work of Ollivier [Oll15].) In this paper we establish the inverse Satake theorem in characteristic p for arbitrary $\mathbf{G}$ and pairs (V, V') by using the pro-p Iwahori Hecke algebra.

1.2. We now explain our results in more detail. Let $\mathbf{S}$ be a maximal split torus of $\mathbf{G}$, $\mathbf{Z}$ its centralizer, $\mathbf{B} = \mathbf{ZU}$ a minimal parabolic subgroup and Δ the set of simple roots defined by $(\mathbf{G}, \mathbf{B}, \mathbf{S})$. Put $Z = \mathbf{Z}(F)$ and $U = \mathbf{U}(F)$. Let $X_*(\mathbf{S})$ be the group of cocharacters of $\mathbf{S}$ and $v_Z \colon Z \to X_*(\mathbf{S}) \otimes \mathbb{R}$ be the usual homomorphism (see Section 2.1). Put $Z^+ = \{z \in Z \mid \langle \alpha, v_Z(z)\rangle \geq 0 \text{ for any } \alpha \in \Delta\}$, so that Z^+ contracts U under conjugation.

Let K be a special parahoric subgroup of G corresponding to a special point of the apartment of S and put $Z^0 = Z \cap K$ (the unique parahoric subgroup of Z), $U^0 = U \cap K$. Let V be an irreducible smooth C-representation of K. It is parameterized by a pair $(\psi_V, \Delta(V))$, where $\psi_V : Z^0 \to C^\times$ describes the action of Z^0 on the line V_{U^0} and $\Delta(V) \subset \Delta$ is a certain subset (see §2.2). Let $\text{c-Ind}_K^G V$ denote the compact induction of V. If V' denotes another irreducible smooth C-representation of K, we define the Hecke bimodule $\mathcal{H}_G(V, V') := \mathrm{Hom}_{CG}(\text{c-Ind}_K^G V, \text{c-Ind}_K^G V')$. This is non-zero if and only if ψ_V is Z-conjugate to $\psi_{V'}$. Once we fix a linear isomorphism $\iota \colon V_{U^0} \simeq V'_{U^0}$, $\mathcal{H}_G(V, V')$ has a canonical C-basis $\{T_z = T_z^{V',V}\}$, where z runs through a system of representatives of $Z_G^+(V, V')/Z^0$ and $Z_G^+(V, V')$ is a certain union of cosets of Z^0 in $Z^+ \cap Z_{\psi_V, \psi_{V'}}$, where $Z_{\psi_V,\psi_{V'}} = \{z \in Z \mid z \cdot \psi_V = \psi_{V'}\}$ (see (2.9)). The element $T_z^{V',V}$ is determined up to scalar by the condition $\operatorname{supp} T_z^{V',V} = KzK$ and normalized by ι (see §2.6).

Similarly, we have the Hecke bimodule $\mathcal{H}_Z(V_{U^0}, V'_{U^0})$ with C-basis $\{\tau_z = \tau_z^{V'_{U^0}, V_{U^0}}\}$, where z runs through a system of representatives of $Z_{\psi_V,\psi_{V'}}/Z^0$. Then we have the mod p Satake transform $S^G \colon \mathcal{H}_G(V, V') \hookrightarrow \mathcal{H}_Z(V_{U^0}, V'_{U^0})$ which is C-linear and injective [HV15]:

$$S^G(f)(z)(\overline{v}) = \sum_{u \in U^0 \backslash U} \overline{f(uz)(v)}, \quad \text{for } f \in \mathcal{H}_G(V, V'), z \in Z \text{ and } v \in V,$$

where $v \mapsto \overline{v} : V \to V_{U^0}$ (resp. $V' \to V'_{U^0}$) is the quotient map from V (resp. V') onto its U^0-coinvariants, and we realize $\mathcal{H}_G(V, V')$ as a set of compactly supported functions on G with a certain K-bi-equivariance.

1.3. For $\alpha \in \Delta$, let M'_α be the subgroup of G generated by the root subgroups $U_{\pm\alpha}$ for the roots $\pm\alpha$. (Note that this need not be the F-points of a closed subgroup of $\mathbf{G}$.) Then $(Z \cap M'_\alpha)/(Z^0 \cap M'_\alpha) \simeq \mathbb{Z}$ and we let $a_\alpha \in Z \cap M'_\alpha$ be a lift of a generator such that $\langle \alpha, v_Z(a_\alpha)\rangle < 0$ [AHHV17, III.16 Notation]. Let $\Delta'(V)$ be the set of $\alpha \in \Delta(V)$ such that ψ_V is trivial on $Z^0 \cap M'_\alpha$. The element $\tau_{a_\alpha}^{V_{U^0}, V_{U^0}}$ is independent of the choice of a_α if $\alpha \in \Delta'(V)$. For $z \in Z_G^+(V, V')$, note that

$$Z_z^+(V, V') := Z^+ \cap z \prod_{\alpha \in \Delta'(V) \cap \Delta'(V')} a_\alpha^{\mathbb{N}}$$

is a finite subset of $Z_G^+(V, V')$ by Lemma 2.13.

Theorem 1.1 (Inverse Satake theorem, Theorem 2.12)**.** *A C-basis of the image of S^G is given by the elements*

$$\tau_z^{V'_{U^0}, V_{U^0}} \prod_{\alpha \in \Delta'(V') \setminus \Delta'(V)} (1 - \tau_{a_\alpha}^{V_{U^0}, V_{U^0}}) \tag{1.1}$$

for z running through a system of representatives of $Z_G^+(V,V')/Z^0$ in $Z_G^+(V,V')$.

A C-basis of $\mathcal{H}_G(V,V')$ is given by the elements

$$\varphi_z = \sum_{x\in Z_z^+(V,V')} T_x^{V',V}$$

for z running through a system of representatives of $Z_G^+(V,V')/Z^0$.

For $z \in Z_G^+(V,V')$ we have:

$$S^G(\varphi_z) = \tau_z^{V'_{U^0},V_{U^0}} \prod_{\alpha\in\Delta'(V')\setminus\Delta'(V)} (1-\tau_{a_\alpha}^{V_{U^0},V_{U^0}}).$$

When $\Delta'(V') \subset \Delta'(V)$, the convention is that $\prod_{\alpha\in\Delta'(V')\setminus\Delta'(V)}(1-\tau_{a_\alpha}^{V_{U^0},V_{U^0}}) = 1$.

There is a Satake transform $S_M^G : \mathcal{H}_G(V,V') \to \mathcal{H}_M(V_{N\cap K}, V'_{N\cap K})$ for any parabolic subgroup $\mathbf{P} = \mathbf{MN}$ containing $\mathbf{B}$ with Levi subgroup $\mathbf{M}$ containing $\mathbf{Z}$ [HV12, Prop. 2.2, 2.3] with $M = \mathbf{M}(F)$ and $N = \mathbf{N}(F)$. We compute also $S_M^G(\varphi_z)$ (Theorem 2.19).

1.4. From the above theorem, we can easily deduce the following result which implies the change of weight theorem (cf. Section 2.5). Suppose that V, V' satisfies that $\psi_V = \psi_{V'}$ and $\Delta(V) = \Delta(V') \sqcup \{\alpha\}$ for some $\alpha \in \Delta$. Let $Z^+_{\psi_V}$ the subset of Z^+ consisting of the elements which normalize ψ_V. Define c_α by

$$c_\alpha = \begin{cases} 1 & \text{if } \alpha \in \Delta'(V), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.2 (Theorem 2.3)**.** *Let $z \in Z^+_{\psi_V}$ such that $\langle \alpha, v_Z(z)\rangle > 0$. Then there exist G-equivariant homomorphisms $\varphi : \operatorname{c-Ind}_K^G V \to \operatorname{c-Ind}_K^G V'$ and $\varphi' : \operatorname{c-Ind}_K^G V' \to \operatorname{c-Ind}_K^G V$ satisfying*

$$S^G(\varphi\circ\varphi') = \tau_{z^2}^{V'_{U^0},V'_{U^0}} - c_\alpha\,\tau_{z^2 a_\alpha}^{V'_{U^0},V'_{U^0}}, \quad S^G(\varphi'\circ\varphi) = \tau_{z^2}^{V_{U^0},V_{U^0}} - c_\alpha\,\tau_{z^2 a_\alpha}^{V_{U^0},V_{U^0}}.$$

In Section 6 we give a simple proof of Theorem 1.2 (and hence of the change of weight theorem) when $\mathbf{G}$ is split. It is more elementary than the other proofs we know in this case. In particular, we do not use the pro-p Iwahori Hecke algebra or the Lusztig-Kato formula. In the proof we first reduce to the case where $\mathbf{G}$ has simply-connected derived subgroup and connected center, and $v_Z(z)$ is minuscule. We construct many parabolically induced representations which contain V but not V'. From this we deduce that if $\varphi = T_z^{V',V}$ and $\varphi' = T_z^{V,V'}$, then $S^G(\varphi'\circ\varphi)$ is so constrained that it is forced to be equal to $\tau_{z^2}^{V_{U^0},V_{U^0}} - \tau_{z^2 a_\alpha}^{V_{U^0},V_{U^0}}$.

In the appendix, two of us (N.A. and F.H.) show that the simple proof of the change of weight theorem can be made to work, with some effort, for all quasi-split groups $\mathbf{G}$, at least for most choices of special parahoric subgroup K. We do not know a simple proof for general $\mathbf{G}$ (or for the remaining choices of K when $\mathbf{G}$ is quasi-split), partly because the method seems less powerful in the case where $c_\alpha = 0$.

1.5. We briefly explain the strategy of the proof of Theorem 1.1. In [Her11a] when $\mathbf{G}$ is split and the derived subgroup is simply-connected, we assumed $V = V'$ and first made a reduction to the case where $\dim V = 1$. Since $\mathbf{G}$ is split, the character V of K can be extended to a character of G which allows us to reduce to the case where V is trivial and use the characteristic zero formula of Lusztig-Kato. This argument cannot work for general $\mathbf{G}$

since a character of K need not extend to G. For example, this can happen when $G = D^\times$ where D is a (non-commutative) division algebra over F.

In our proof, we treat arbitrary pairs (V, V'). First we make a reduction to the case where $\Delta(V') \subset \Delta(V)$ using properties of Satake transform and the convolution of Hecke operators (Lemmas 3.1, 3.2). When $\Delta(V') \subset \Delta(V)$, using a calculation in [AHHV17, §IV], we can express the inverse of the Satake transform using an alcove-walk basis of the pro-p Iwahori Hecke algebra (Proposition 5.1). Combining this with an explicit calculation of the alcove-walk basis (Proposition 4.30), we get Theorem 1.1. More details are given below.

1.6. Let $\mathcal{H}_G$ be the Hecke $\mathbb{Z}$-algebra of the pro-p Iwahori group $I = K(1)U^0_{\mathrm{op}}$, where $K(1)$ is the pro-p radical of K and $U^0_{\mathrm{op}} = K \cap U_{\mathrm{op}}$, where $\mathbf{U}_{\mathrm{op}}$ is the opposite to $\mathbf{U}$ (with respect to $\mathbf{Z}$). We also let $Z(1) = Z \cap K(1)$. Until the end of this introduction we assume $\Delta(V') \subset \Delta(V)$ and $z \in Z^+_G(V, V')$. We now explain how the theory of $\mathcal{H}_G$ allows us to prove

$$\tau_z^{V'_{U^0}, V_{U^0}} = S^G(\varphi_z)$$

in Theorem 1.1, hence the inverse Satake theorem.

Once we choose a non-zero element $v \in V_{U^0}$ and let $v' \in V'_{U_0}$ correspond to v under our fixed isomorphism $\iota : V_{U^0} \simeq V'_{U^0}$, we define embeddings

$$\text{c-Ind}_K^G V \xrightarrow{I_v} \mathfrak{X}_G, \quad \text{c-Ind}_K^G V' \xrightarrow{I_{v'}} \mathfrak{X}_G, \quad \text{c-Ind}_{Z^0}^Z V_{U^0} \xrightarrow{j_v} \mathfrak{X}_Z, \quad \text{c-Ind}_{Z^0}^Z V'_{U^0} \xrightarrow{j_{v'}} \mathfrak{X}_Z,$$

of $\text{c-Ind}_K^G V$ and $\text{c-Ind}_K^G V'$ in the parabolically induced representation $\mathfrak{X}_G = \mathrm{Ind}_B^G(\text{c-Ind}_{Z(1)}^Z C)$ and of $\text{c-Ind}_{Z^0}^Z V_{U^0}$ and $\text{c-Ind}_{Z^0}^Z V'_{U^0}$ in $\mathfrak{X}_Z = \text{c-Ind}_{Z(1)}^Z C$. We have

$$I_v = (\mathrm{Ind}_B^G j_v) \circ I_V, \quad I_{v'} = (\mathrm{Ind}_B^G j_{v'}) \circ I_{V'}$$

for the canonical $C[G]$-embedding $\text{c-Ind}_K^G V \xrightarrow{I_V} \mathrm{Ind}_B^G(\text{c-Ind}_{Z^0}^Z V_{U^0})$ [HV12], and similarly for $I_{V'}$. The representation $\text{c-Ind}_K^G V$ is generated by the I-invariant element f_v, which is supported on K and is such that $f_v(1)$ lies in $V^{U^0_{\mathrm{op}}}$ and maps to $v \in V_{U^0}$. Similarly for $f_{v'} \in \text{c-Ind}_K^G V'$.

Then, $I_v(f_v), I_{v'}(f_{v'})$ lie in the $(\mathcal{H}_Z, \mathcal{H}_G)$-bimodule $\mathfrak{X}_G^I = (\mathrm{Ind}_B^G(\text{c-Ind}_{Z(1)}^Z C))^I$. Let $\tau(z) \in \mathcal{H}_Z$ be the characteristic function of $zZ(1)$.

The first key ingredient is Proposition 5.3 (which generalizes [AHHV17, IV.19 Thm.]):

We give an explicit element $h_z \in \mathcal{H}_G$ such that $\tau(z)I_v(f_v) = I_{v'}(f_{v'})h_z$.

We deduce (Proposition 5.1): there exists an intertwiner $\phi_z : \text{c-Ind}_K^G V \to \text{c-Ind}_K^G V'$ defined by

$$\phi_z(f_v) = f_{v'} h_z.$$

Moreover, $\tau_z^{V'_{U^0}, V_{U^0}} = S^G(\phi_z)$. The second key ingredient is the computation of $f_{v'} h_z \in (\mathrm{Ind}_K^G V')^I$ on Z^+:

The function $f_{v'} h_z$ vanishes on $Z^+ \setminus Z^0 Z^+_z(V, V')$ and is equal to v' on $Z^+_z(V, V')$.

We prove that it implies $\varphi_z = \phi_z$ (proof of Proposition 5.10).

1.7. We develop in Section 4 the theory of the pro-p Iwahori Hecke algebra $\mathcal{H}_G$ behind the computation of $f_{v'}h_z|_{Z^+}$.

Let $\mathcal{N}$ be the G-normalizer of Z, $W(1) = \mathcal{N}/Z(1)$ the pro-p Iwahori Weyl group, $\lambda_x \in W(1)$ the image of $x \in Z$ and Z_k the image on Z^0 in $W(1)$. It is well known that the natural map $W(1) \to I\backslash G/I$ is bijective. The element $h_z \in \mathcal{H}_G$ is given as a product (Propositions 5.1, 5.3):

$$h_z = E'_{\lambda_z w_{V,V'}^{-1}} T^*_{w_{V,V'}},$$

where $(E'_w)_{w\in W(1)}$ is a certain alcove walk basis of $\mathcal{H}_G$ (which depends on V'), $(T^*_w)_{w\in W(1)}$ a non alcove walk basis of $\mathcal{H}_G$, and $w_{V,V'} \in W(1)$ is a lift of the product in $\mathcal{N}/Z$ of the longest elements of the finite Weyl groups associated to $\Delta(V)$ and $\Delta(V')$.

The two bases are related by triangular matrices to the classical Iwahori-Matsumoto basis $(T_w)_{w\in W(1)}$ of $\mathcal{H}_G$, where T_w is the characteristic function of InI for $n \in \mathcal{N}$ lifting w. We have

$$T^*_w = \sum_{u\in W(1), u\le w} c^*(w,u) T_u$$

with coefficients $c^*(w,u) \in C$ and $c^*(w,w) = 1$, where $\le$ is the Bruhat (pre)order on $W(1)$ associated to B (see (4.5)). Let M be the Levi subgroup of G containing Z associated to $\Delta(V')$; an index M indicates an object relative to M instead of G. It was a surprise to discover (partially following an idea of Ollivier [Oll14]) that the coefficients of the expansion of the alcove walk element $E'_{\lambda_z w_{V,V'}^{-1}}$ in the classical basis of $\mathcal{H}_G$ are given by the coefficients $c^{M,*}(\lambda_z, u)$ of the expansion of the non alcove walk basis element $T^{M,*}_{\lambda_z} \in \mathcal{H}_M$ in the classical basis $(T^M_w)_{w\in W_{M(1)}}$ of $\mathcal{H}_M$. Recall that $\mathcal{H}_M$ is not a subalgebra of $\mathcal{H}_G$, and that the restriction to $W_M(1)$ of the Bruhat order $\le$ on $W(1)$ is not equal to the Bruhat order $\le^M$ associated to $B_M = M \cap B$. We show (Proposition 4.30):

$$E'_{\lambda_z w_{V,V'}^{-1}} = \sum_{u\in W_M(1),\ u\le^M \lambda_z} c^{M,*}(\lambda_z, u) T_{u w_{V,V'}^{-1}}.$$

We carry out a detailed study of the sum $\sum_{t\in Z_k} c^*(w, tu)T_t$ modulo $q = \#k$ for $w, u \in W(1), u \le w$. In particular, we show (Theorems 4.23, 4.39), for a character $\psi : Z_k \to C^\times$:

$$\textit{For } x \in Z^+ \textit{ and } \lambda_x \le \lambda_z, \textit{ we have } \sum_{t\in Z_k} c^*(\lambda_z, t\lambda_x)\psi(t) = \begin{cases} 1 & \text{if } \ x \in Z^0 z \prod_{\alpha\in\Delta'_\psi} a_\alpha^{\mathbb{N}}, \\ 0 & \textit{otherwise.}\end{cases}$$

Here $\Delta'_\psi = \{\alpha \in \Delta \mid \psi \text{ is trivial on } \ Z^0 \cap M'_\alpha\}$. With a "little more" we deduce that on Z^+,

$$f_{v'} E'_{\lambda_z w_{V,V'}^{-1}} T^*_{w_{V,V'}} = f_{v'} \sum_{x\in Z_z^+(V,V')} \sum_{t\in Z_k} c^{M,*}(\lambda_z, t\lambda_x)\psi_{V'}^{-1}(t) T_{\lambda_x} = f_{v'} \sum_{x\in Z_z^+(V,V')} T_{\lambda_x}.$$

By the "little more", we mean: if $u \in W_M(1)$ and $f_{v'} T_{u w_{V,V'}^{-1}} T^*_{w_{V,V'}}$ does not vanish on Z^+ then $u \in Z^+/Z(1)$ (see (5.4)). The two conditions $u \in Z^+/Z(1)$ and $u \le^M \lambda_z$ are equivalent to $u = \lambda_x$ for $x \in Z^0 Z_z^+(V,V')$ (Proposition 4.3). For $x \in Z^0 Z_z^+(V,V')$, we have $f_{v'} T_{\lambda_x w_{V,V'}^{-1}} T^*_{w_{V,V'}} = f_{v'} T_{\lambda_x w_{V,V'}^{-1}} T_{w_{V,V'}}$ on Z^+ (see (5.5)). Then we use the braid relation $T_{\lambda_x w_{V,V'}^{-1}} T_{w_{V,V'}} = T_{\lambda_x}$, that $f_{v'} T_{t\lambda_x} = \psi_{V'}^{-1}(t) f_{v'} T_{\lambda_x}$ for $t \in Z_k$, and that $\Delta^M_{\psi_{V'}^{-1}} = \Delta'(V') = \Delta'(V) \cap \Delta'(V')$.

From $f_{v'}h_z = f_{v'}\sum_{x\in Z_z^+(V,V')} T_{\lambda_x}$ on Z^+ – and checking easily that $f_{v'}T_{\lambda_x}$ is supported on KxI with value v' at x, and $Z^+\cap KxI = Z^0x$, for all $x\in Z_z^+(V,V')$ – we obtain the desired value of $f_{v'}h_z$ on Z^+ (§1.6).

2. Change of weight and Inverse Satake isomorphism

2.1. **Notation.** Throughout this paper we follow the notation given in [AHHV17]. As in loc. cit., let F be a nonarchimedean field with ring of integers $\mathcal{O}$ and residue field k of characteristic p and cardinality q. Let $\mathrm{ord}_F : F^\times \to \mathbb{Z}$ denote the normalized valuation of F. A linear algebraic F-group is denoted with a boldface letter like $\mathbf{H}$ and the group of its F-points with the corresponding ordinary letter $H = \mathbf{H}(F)$; we use the similar convention for groups over k. Let $\mathbf{G}$ be a connected reductive F-group.

We fix a triple $(\mathbf{S},\mathbf{B},x_0)$ where $\mathbf{S}$ is a maximal torus in $\mathbf{G}$, $\mathbf{B}$ a minimal F-parabolic subgroup of $\mathbf{G}$ containing $\mathbf{S}$ with unipotent radical $\mathbf{U}$ and Levi subgroup the centralizer $\mathbf{Z}$ of $\mathbf{S}$ in $\mathbf{G}$, and x_0 a special point in the apartment corresponding to S in the adjoint Bruhat-Tits building of G.

We write $\boldsymbol{\mathcal{N}}$ for the normalizer of $\mathbf{S}$ in $\mathbf{G}$. If $X^*(\mathbf{S})$ is the group of characters of $\mathbf{S}$ and $X_*(\mathbf{S})$ is the group of cocharacters, we write $\langle\ ,\ \rangle : X^*(\mathbf{S})\times X_*(\mathbf{S})\to\mathbb{Z}$ for the natural pairing. We let $\Phi\subset X^*(\mathbf{S})$ be the set of roots of $\mathbf{S}$ in $\mathbf{G}$ and we write Δ for the set of simple roots in the set Φ^+ of positive roots with respect to $\mathbf{B}$. For $\alpha\in\Phi$, the corresponding coroot in $X_*(\mathbf{S})$ is denoted by $\alpha^\vee$. For $\alpha,\beta\in\Phi$, we say that α is orthogonal to β if and only if $\langle\alpha,\beta^\vee\rangle = 0$. The Weyl group $W_0 := \mathcal{N}/Z \simeq \boldsymbol{\mathcal{N}}/\mathbf{Z}$ is isomorphic to the Weyl group of Φ.

We say that P is a parabolic subgroup of G to mean that $P = \mathbf{P}(F)$ where $\mathbf{P}$ is an F-parabolic subgroup of $\mathbf{G}$. If P contains B, we write $P = MN$ to mean that N is the unipotent radical of P and M the (unique) Levi component containing Z; we write $P_{\mathrm{op}} = MN_{\mathrm{op}}$ for the parabolic subgroup opposite to P with respect to M. The parabolic subgroups containing B are in one-to-one correspondence with the subsets of Δ; we denote by $P_J = M_JN_J$ the group corresponding to $J\subset\Delta$ (when $J=\{\alpha\}$ we write simply $P_\alpha = M_\alpha N_\alpha$).

The apartment corresponding to S in the adjoint Bruhat-Tits building of G is an affine space $x_0 + V_{\mathrm{ad}}$ where $V_{\mathrm{ad}} := X_*(\mathbf{S}_{\mathrm{ad}})\otimes\mathbb{R}$ and $\mathbf{S}_{\mathrm{ad}}$ is the torus image of $\mathbf{S}$ in the adjoint group $\mathbf{G}_{\mathrm{ad}}$ of $\mathbf{G}$. The group $\mathcal{N}$ acts by affine automorphisms on the apartment, its subgroup Z acting by translation by $\nu = -v$ where $v : Z\to V_{\mathrm{ad}}$ is the composite of the map $v_Z : Z\to X_*(\mathbf{S})\otimes\mathbb{R}$ defined in [HV15, 3.2] and of the natural quotient map $X_*(\mathbf{S})\otimes\mathbb{R}\twoheadrightarrow X_*(\mathbf{S}_{\mathrm{ad}})\otimes\mathbb{R}$. (We recall that v_Z is determined by the requirement that $\langle\chi, v_Z\rangle = \mathrm{ord}_F\circ\chi$ for all F-rational characters χ of $\mathbf{Z}$.) The root system of $\mathbf{S}_{\mathrm{ad}}$ in $\mathbf{G}_{\mathrm{ad}}$ identifies with Φ. The coroot of $\alpha\in\Phi$ in V_{ad} is the image of the coroot $\alpha^\vee\in X_*(\mathbf{S})\otimes\mathbb{R}$ by the quotient map, and is still denoted by $\alpha^\vee$.

As in [AHHV17, I.5] we write K for the special parahoric subgroup of G fixing x_0 and $K(1)$ for the pro-p radical of K. For a subgroup H of G, we put $H^0 := H\cap K$ and $\overline{H} := (H\cap K)/(H\cap K(1))$. The group S^0 is the maximal compact subgroup of S, Z^0 is the unique parahoric subgroup of Z and $Z(1) := Z\cap K(1)$ is the unique pro-p Sylow subgroup of Z^0. The group $G_k := \overline{G} = \overline{K}$ is naturally the group of k-points of a connected reductive k-group $\mathbf{G}_k$, of minimal parabolic subgroup $B_k := \overline{B}$ with Levi decomposition $B_k = Z_kU_k$ where $Z_k := \overline{Z}$ and $U_k := \overline{U}$. The set of simple roots of the maximal split torus $S_k = \overline{S}$ of G_k with respect to B_k is in natural bijection with Δ and will be identified with Δ. For $J\subset\Delta$, the corresponding parabolic subgroup $P_{J,k}$ of G_k containing B_k is $\overline{P}_J$; its Levi decomposition is $P_{J,k} = M_{J,k}N_{J,k}$ where $M_{k,J} = \overline{M}_J$ and $N_{J,k} = \overline{N}_J$. We write $P_{J,k,\mathrm{op}} = M_{J,k}N_{J,k,\mathrm{op}}$ for the parabolic group opposite to $P_{J,k}$ with respect to $M_{J,k}$.

We fix an algebraically closed field C of characteristic p. In this paper, a representation means a smooth representation on a C-vector space.

2.2. **The Satake transform S_M^G.** Let V be an irreducible representation of the special parahoric subgroup K of G; the normal pro-p subgroup $K(1)$ of K acts trivially on V and the action of K on V factors through the finite reductive group G_k. Seeing V as an irreducible representation of G_k, we attach to V a character ψ_V of Z_k and a subset $\Delta(V) \subset \Delta$ as in [AHHV17, III.9]; the space of U_k-coinvariants V_{U_k} of V is a line on which Z_k acts by ψ_V and the G_k-stabilizer of the kernel of the natural map $V \to V_{U_k}$ is $P_{\Delta(V),k}$. The pair $(\psi_V, \Delta(V))$, called the parameter of V, determines V. The character ψ_V can be seen as the character of Z^0 acting on the space U^0-coinvariants V_{U^0} of V.

Let $P = MN$ be the parabolic subgroup of G containing B corresponding to $J \subset \Delta$. Then M^0 is a special parahoric subgroup of M and V_{N^0} is an irreducible representation of M^0 with parameter $(\psi_V, J \cap \Delta(V))$ [AHHV17, III.10].

The compact induction $\text{c-Ind}_K^G V$ of V to G is the representation of G by right translation on the space of functions $f : G \to V$ with compact support satisfying $f(kg) = kf(g)$ for all $k \in K, g \in G$. We view the intertwining algebra $\mathrm{End}_{CG}(\text{c-Ind}_K^G V)$ as the convolution algebra $\mathcal{H}_G(V)$ of compactly supported functions $\varphi : G \to \mathrm{End}_C(V)$ satisfying $\varphi(k_1 g k_2) = k_1 \varphi(g) k_2$ for all $k_1, k_2 \in K, g \in G$. The action of $\varphi \in \mathcal{H}_G(V)$ on $f \in \text{c-Ind}_K^G(V)$ is given by convolution

$$(\varphi * f)(g) = \sum_{x \in G/K} \varphi(x)(f(x^{-1}g)). \tag{2.1}$$

We have also the algebra $\mathrm{End}_{CM}(\text{c-Ind}_{M^0}^M(V_{N^0})) \simeq \mathcal{H}_M(V_{N^0})$. The Satake transform is a natural injective algebra homomorphism [AHHV17, III.3]

$$S_M^G : \mathcal{H}_G(V) \hookrightarrow \mathcal{H}_M(V_{N^0});$$

it induces an homomorphism between the centers $\mathcal{Z}_G(V) \to \mathcal{Z}_M(V_{N^0})$; both homomorphisms are localizations at a central element [AHHV17, I.5].

For a representation σ of M, the parabolic induction $\mathrm{Ind}_P^G \sigma$ of σ to G is the representation of G by right translation on the space of functions $f : G \to \sigma$ satisfying $f(mngk) = mf(g)$ for all $m \in M, n \in N, g \in G, k$ in some open compact subgroup of G depending on f. The canonical isomorphism

$$\mathrm{Hom}_{CG}(\text{c-Ind}_K^G V, \mathrm{Ind}_P^G \sigma) \xrightarrow{\sim} \mathrm{Hom}_{CM}(\text{c-Ind}_{M^0}^M V_{N^0}, \sigma)$$

is $\mathcal{H}_G(V)$-equivariant via S_M^G [HV12, §2].

2.3. **The Satake transform $S^G = S_Z^G$.** As in [AHHV17, III.4], the algebra $\mathcal{H}_Z(V_{U^0})$ is easily described. The unique parahoric subgroup Z^0 of Z being normal, for $z \in Z$ we have the character $z \cdot \psi_V$ of Z^0 defined by $(z \cdot \psi_V)(x) = \psi_V(z^{-1}xz), x \in Z^0$. Let

$$Z_{\psi_V} = \{z \in Z \mid z \cdot \psi_V = \psi_V\}$$

be the Z-normalizer of ψ_V. For $z \in Z_{\psi_V}$, there is a unique function $\tau_z \in \mathcal{H}_Z(V_{U^0})$ of support zZ^0 with $\tau_z(z) = \mathrm{id}_{V_{U^0}}$. A basis of $\mathcal{H}_Z(V_{U^0})$ is given by the functions τ_z where z runs through a system of representatives of Z_{ψ_V}/Z^0 in Z_{ψ_V}. The multiplication satisfies $\tau_{z_1} * \tau_{z_2} = \tau_{z_1 z_2}$. The function τ_z belongs to the center $\mathcal{Z}_Z(V_{U^0})$ if and only if $\psi_V(z^{-1}xzx^{-1}) = 1$ for all $x \in Z_{\psi_V}$. We write also $\tau_z = \tau_z^{V_{U^0}}$.

Let

$$Z^+ = \{z \in Z \mid \langle \alpha, v_Z(z) \rangle \geq 0 \text{ for all } \alpha \in \Delta\}.$$

be the dominant submonoid of Z. For a subset H of Z we write $H^+ = H \cap Z^+$.

When $M = Z$ we put $S^G = S^G_Z$. The image of S^G is

$$S^G(\mathcal{H}_G(V)) = \bigoplus_z C\tau_z \tag{2.2}$$

for z in a system of representatives of $Z^+_{\psi_V}/Z^0$ in $Z^+_{\psi_V}$ (see [Her11b] when $\mathbf{G}$ is unramified and [HV15] in general). For another irreducible representation V' of K with $\psi_V = \psi_{V'}$, we have a canonical Z^0-equivariant isomorphism $\mathrm{End}_C(V_{U^0}) \simeq \mathrm{End}_C(V'_{U^0})$ and hence a canonical isomorphism $i_Z : \mathcal{H}_Z(V_{U^0}) \xrightarrow{\simeq} \mathcal{H}_Z(V'_{U^0})$ (sending the function $\tau_z \in \mathcal{H}_Z(V_{U^0})$ to the function $\tau_z \in \mathcal{H}_Z(V'_{U^0})$ for all $z \in Z_{\psi_V}$). It induces a canonical isomorphism

$$i_G : \mathcal{H}_G(V) \xrightarrow{\simeq} \mathcal{H}_G(V') \tag{2.3}$$

satisfying $S^G \circ i_G = i_Z \circ S^G$.

2.4. **The elements a_α.** Let G' be the group generated by U and U_{op} (this is not the group of F-points of a linear algebraic group in general). The action of $\mathcal{N}$ on the apartment $x_0 + V_{\mathrm{ad}}$ induces an isomorphism from $(\mathcal{N} \cap G')/(Z^0 \cap G')$ onto the affine Weyl group W^{aff} of a reduced root system

$$\Phi_a = \{\alpha_a := e_\alpha \alpha \mid \alpha \in \Phi\} \tag{2.4}$$

on V_{ad}, where e_α for $\alpha \in \Phi$ are positive integers [Vig16, Lemma 3.9], [Bou02, VI.2.1]. The map $\alpha \to \alpha_a$ gives a bijection from Δ to a set Δ_a of simple roots of Φ_a; the coroot in $X_*(\mathbf{S}_{\mathrm{ad}}) \otimes \mathbb{R}$ associated to α_a is $\alpha_a^\vee = e_\alpha^{-1}\alpha^\vee$; the homomorphism $\nu = -v : Z \to V_{\mathrm{ad}}$ induces a quotient map $Z \cap G' \twoheadrightarrow \oplus_{\alpha \in \Delta} \mathbb{Z}\alpha_a^\vee$ with kernel $Z^0 \cap G'$. An element $z \in Z$ belongs to Z^+ if and only if $\nu(z)$ lies in the closed antidominant Weyl chamber

$$\mathfrak{D}^- = \{x \in V_{\mathrm{ad}} \mid \langle \alpha_a, x \rangle \le 0 \text{ for } \alpha \in \Delta\}. \tag{2.5}$$

For $\alpha \in \Delta$ we also have M'_α and the quotient map $Z \cap M'_\alpha \twoheadrightarrow \mathbb{Z}\alpha_a^\vee$ with kernel $Z^0 \cap M'_\alpha$ induced by ν [AHHV17, III.16].

Definition 2.1. For a character $\psi : Z^0 \to C^\times$ and $\alpha \in \Delta$, let

$$\Delta'_\psi = \{\alpha \in \Delta \mid \psi \text{ is trivial on } Z^0 \cap M'_\alpha\},$$
$$a_\alpha \in Z \cap M'_\alpha \text{ such that } \nu(a_\alpha) = \alpha_a^\vee.$$

If $\alpha \in \Delta'_\psi$, then $Z \cap M'_\alpha$ is contained in the Z-normalizer Z_ψ of ψ,

$$\tau_\alpha := \tau_{a_\alpha} \in \mathcal{H}_Z(\psi)$$

does not depend on the choice of a_α, and belongs to the center $\mathcal{Z}_Z(\psi)$ [AHHV17, III.16]. The set Δ'_ψ is included in the subset $\Delta(\psi)$ of Δ defined by (4.18) (cf. Remark 4.33).

2.5. **Change of weight.** Let V' and V be two irreducible representations of K with parameters $\psi_V = \psi_{V'}, \Delta(V) = \Delta(V') \sqcup \{\alpha\}$ where $\alpha \in \Delta - \Delta(V')$, let $\chi : \mathcal{Z}_G(V) \to C$ be a character of the center of $\mathcal{H}_G(V)$, let $P = MN$ denote the smallest parabolic subgroup of G containing B such that χ factors through S^G_M, and let $\Delta(\chi)$ be the subset of Δ corresponding to P (denoted by $\Delta_0(\chi)$ in [AHHV17, III.4 Notation]). We have the homomorphism χ' : $\mathcal{Z}_G(V') \to C$ corresponding to χ via the isomorphism (2.3).

Theorem 2.2 (Change of weight)**.** *Assume $\alpha \notin \Delta(\chi)$. The representations $\chi \otimes_{\mathcal{Z}_G(V)} \operatorname{c-Ind}_K^G V$ and $\chi' \otimes_{\mathcal{Z}_G(V')} \operatorname{c-Ind}_K^G V'$ of G are isomorphic unless*

$$\alpha \text{ is orthogonal to } \Delta(\chi),\ \psi_V \text{ is trivial on } Z^0 \cap M'_\alpha,\ \chi(\tau_\alpha) = 1.$$

The change of weight theorem was proved in [AHHV17, IV.2 Corollary] (generalizing [Her11a] for GL_n and [Abe13] for split groups) and was one of the key tools in establishing a classification result for irreducible representations of G over C. The change of weight theorem is a simple consequence of the next theorem. Define

$$c_\alpha = \begin{cases} 1 & \text{if } \psi_V \text{ is trivial on } Z^0 \cap M'_\alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Theorem 2.3. *Let $z \in Z^+_{\psi_V}$ such that $\langle \alpha, v(z) \rangle > 0$. Then there exist G-equivariant homomorphisms $\varphi : \operatorname{c-Ind}_K^G V \to \operatorname{c-Ind}_K^G V'$ and $\varphi' : \operatorname{c-Ind}_K^G V' \to \operatorname{c-Ind}_K^G V$ satisfying*

$$S^G(\varphi \circ \varphi') = \tau_{z^2}^{V'_{U^0}} - c_\alpha\, \tau_{z^2 a_\alpha}^{V'_{U^0}}, \quad S^G(\varphi' \circ \varphi) = \tau_{z^2}^{V_{U^0}} - c_\alpha\, \tau_{z^2 a_\alpha}^{V_{U^0}}.$$

We will prove in Proposition 2.17 that Theorem 2.3 follows from the inverse Satake theorem (Theorem 2.12) for the pair (V, V') and for the pair (V', V). We now recall why Theorem 2.3 implies Theorem 2.2 (compare with the proof of [AHHV17, IV.2 Corollary]).

Proof of Theorem 2.2. As in §2.3 we can canonically identify $\mathcal{H}_G(V)$ with $\mathcal{H}_G(V')$ and similarly $\mathcal{Z}_G(V)$ with $\mathcal{Z}_G(V')$, denoting them $\mathcal{H}_G$ and $\mathcal{Z}_G$ for short. We also identify χ and χ'. Pick any $z \in Z^+_{\psi_V}$ such that $\langle \alpha, v(z) \rangle > 0$, $\langle \beta, v(z) \rangle = 0$ for all $\beta \in \Delta - \{\alpha\}$, and such that $\tau_{z^2} \in \mathcal{Z}_Z(\psi_V)$ (cf. [AHHV17, III.4]). As S^G is injective and compatible with compositions, the homomorphisms φ, φ' of Theorem 2.3 for our chosen z are $\mathcal{Z}_G$-equivariant and induce G-equivariant homomorphisms between $\chi \otimes_{\mathcal{Z}_G} \operatorname{c-Ind}_K^G V$ and $\chi \otimes_{\mathcal{Z}_G} \operatorname{c-Ind}_K^G V'$ with composition in either direction equal to $\chi(\tau_{z^2} - c_\alpha\, \tau_{z^2 a_\alpha}) \in C$. It suffices to show that $\chi(\tau_{z^2} - c_\alpha\, \tau_{z^2 a_\alpha}) \neq 0$. First, $\chi(\tau_{z^2}) \neq 0$ by [AHHV17, III.4 Lemma] and as $\alpha \notin \Delta(\chi)$, so we are done if $c_\alpha = 0$. For the same reason, if $c_\alpha = 1$ and α is not orthogonal to $\Delta(\chi)$, then $\chi(\tau_{z^2 a_\alpha}) = 0$ and we are done. Finally, if $c_\alpha = 1$, α is orthogonal to $\Delta(\chi)$, and $\chi(\tau_\alpha) \neq 1$, then $\chi(\tau_{z^2} - c_\alpha\, \tau_{z^2 a_\alpha}) = \chi(\tau_{z^2})(1 - \chi(\tau_\alpha)) \neq 0$. ☐

2.6. **Intertwiners from $\operatorname{c-Ind}_K^G V$ to $\operatorname{c-Ind}_K^G V'$.** Let V and V' be two irreducible representations of K. We extend to the space of intertwiners $\operatorname{Hom}_{CG}(\operatorname{c-Ind}_K^G V, \operatorname{c-Ind}_K^G V')$ our previous discussion on $\operatorname{End}_{CG}(\operatorname{c-Ind}_K^G V)$ in §2.2. We view $\operatorname{Hom}_{CG}(\operatorname{c-Ind}_K^G V, \operatorname{c-Ind}_K^G V')$ as the space $\mathcal{H}_G(V, V')$ of compactly supported functions $\varphi : G \to \operatorname{Hom}_C(V, V')$ satisfying $\varphi(k_1 g k_2) = k_1 \varphi(g) k_2$ for all $k_1, k_2 \in K, g \in G$. For $z \in Z$, we write

$$\Delta_z = \{\alpha \in \Delta \mid \langle \alpha, v(z) \rangle = 0\}. \tag{2.7}$$

Remark 2.4. When $z, z' \in Z^+$, we have $\Delta_{z'z} = \Delta_{z'} \cap \Delta_z$.

The quotient map $p : V \twoheadrightarrow V_{U^0}$ induces a Z^0-equivariant isomorphism between the lines $V^{U^0_{\mathrm{op}}} \xrightarrow{\sim} V_{U^0}$; similarly for V'. We fix compatible linear isomorphisms

$$\iota^{\mathrm{op}} : V^{U^0_{\mathrm{op}}} \xrightarrow{\sim} (V')^{U^0_{\mathrm{op}}} \text{ and } \iota : V_{U^0} \xrightarrow{\sim} V'_{U^0}. \tag{2.8}$$

When $V = V'$ we suppose that ι^{op} and ι are the identity maps. We now recall the description of $\mathcal{H}_G(V, V')$. By the Cartan decomposition [HV15, 6.4 Prop.], the map $Z \to K\backslash G/K, z \mapsto KzK$ induces a bijection $Z^+/Z^0 \xrightarrow{\sim} K\backslash G/K$. Recalling from §2.2 the parameters $(\psi_V, \Delta(V))$

of V and $(\psi_{V'}, \Delta(V'))$ of V', a double coset KzK with $z \in Z^+$ supports a non zero function of $\mathcal{H}_G(V,V')$ if and only if z lies in

$$Z_G^+(V,V') = \{z \in Z^+ \mid z \cdot \psi_V = \psi_{V'} \text{ and } \Delta_z \cap (\Delta(V) \triangle \Delta(V')) = \varnothing\} \tag{2.9}$$

$$= \{z \in Z^+ \mid z \cdot \psi_V = \psi_{V'} \text{ and } \langle \alpha, v(z)\rangle > 0 \text{ for all } \alpha \in \Delta(V) \triangle \Delta(V')\}. \tag{2.10}$$

where $\Delta(V)\triangle\Delta(V') = (\Delta(V) \setminus \Delta(V')) \cup (\Delta(V') \setminus \Delta(V))$ is the symmetric difference.

The space of such functions has dimension 1 and contains a unique function T_z such that the restriction of $T_z(z)$ to $V^{U^0_{\mathrm{op}}}$ is ι^{op}. The function T_z is also denoted by $T_z = T_z^{V',V}$ or $T_z^{V',V,\iota}$.

Proposition 2.5 ([HV15, 7.7])**.** *A basis of $\mathcal{H}_G(V,V')$ consists of the T_z for z running through a system of representatives of $Z_G^+(V,V')/Z^0$ in $Z_G^+(V,V')$.*

We will write that $(T_z)_{z \in Z_G^+(V,V')/Z^0}$ is a basis of $\mathcal{H}_G(V,V')$.

These considerations apply also to the group Z and to the representations V_{U^0}, V'_{U^0} of Z^0. We write $Z_{\psi_V,\psi_{V'}} = \{z \in Z \mid z \cdot \psi_V = \psi_{V'}\}$. Then the function $\tau_z \in \mathcal{H}_Z(V_{U^0}, V'_{U^0})$ of support $Z^0 z$ and value ι at z for $z \in Z_{\psi_V,\psi_{V'}}$ is denoted also by $\tau_z^{V'_{U^0},V_{U^0}}$ or $\tau_z^{V'_{U^0},V_{U^0},\iota}$. A basis of $\mathcal{H}_Z(V_{U^0}, V'_{U^0})$ is $(\tau_z)_{z\in Z_{\psi_V,\psi_{V'}}/Z^0}$.

Example 2.6. If $V = V'$, then $Z_G^+(V,V) = Z^+_{\psi_V}$. If $\psi_V = \psi_{V'}$, then $Z_G^+(V,V') = Z_G^+(V',V) \subset Z^+_{\psi_V}$. If $\Delta(V) = \Delta(V')$, then $Z_G^+(V,V') = Z^+_{\psi_V,\psi_{V'}}$.

Remark 2.7. (i) We have $\mathcal{H}_G(V,V') \neq 0$ if and only if $Z_{\psi_V,\psi_{V'}}$ is not empty [HV15, 7.8 Prop.]. In this case $\Delta'_{\psi_V} = \Delta'_{\psi_{V'}}$ (Definition 2.1) because $Z^0 \cap M'_\alpha$ is a normal subgroup of Z.

(ii) Let $z \in Z_{\psi_V,\psi_{V'}}$, $\alpha \in \Delta'_{\psi_V} = \Delta'_{\psi_{V'}}$ and $a_\alpha \in Z \cap M'_\alpha$ (Definition 2.1). Then $a_\alpha z a_\alpha^{-1} z^{-1} \in Z^0 \cap M'_\alpha$ ($Z \cap M'_\alpha$ is also a normal subgroup of Z) hence $z a_\alpha = t a_\alpha z \in Z_{\psi_V,\psi_{V'}}$, some $t \in Z^0 \cap M'_\alpha$. The convolution satisfies

$$\tau_z^{V'_{U^0},V_{U^0},\iota}\tau_\alpha^{V_{U^0},V_{U^0}} = \tau_{za_\alpha}^{V'_{U^0},V_{U^0},\iota} = \tau_{ta_\alpha z}^{V'_{U^0},V_{U^0},\iota} = \tau_\alpha^{V'_{U^0},V'_{U^0}}\tau_z^{V'_{U^0},V_{U^0},\iota}.$$

Let V'' be a third irreducible representation of K. The composition of intertwiners corresponds to the convolution. We fix compatible linear ${\iota'}^{\mathrm{op}} : (V')^{U^0_{\mathrm{op}}} \xrightarrow{\sim} (V'')^{U^0_{\mathrm{op}}}$ and $\iota' : V'_{U^0} \xrightarrow{\sim} V''_{U^0}$ and we define as above $T_z^{V'',V'} = T_z^{V'',V',\iota'}$ when $z \in Z_G^+(V',V'')$ and $T_z^{V'',V} = T_z^{V'',V,\iota'\circ\iota}$ when $z \in Z_G^+(V,V'')$.

For $g \in G$ we note that $(T_{z'}^{V'',V'} * T_z^{V',V})(g)$ equals

$$\sum_{x\in Kz'K/K} T_{z'}^{V'',V'}(x) \circ T_z^{V',V}(x^{-1}g) = \sum_{x \in K/(K\cap z'Kz'^{-1})} T_{z'}^{V'',V'}(xz') \circ T_z^{V',V}(z'^{-1}x^{-1}g).$$

Remark 2.8. (i) When $\psi_{V'} = \psi_{V''}$ and $\Delta(V) \cap \Delta(V') \subset \Delta(V'') \subset \Delta(V) \cup \Delta(V')$, we have $Z_G^+(V,V') \subset Z_G^+(V,V'')$.

(ii) For $z \in Z_G^+(V,V')$, $z' \in Z_G^+(V',V'')$ we have $z'z \in Z_G^+(V,V'')$ because $z'z\cdot\psi_V = z'\cdot\psi_{V'} = \psi_{V''}$, $\Delta_{z'z} = \Delta_z \cap \Delta_{z'}$ (as $z, z' \in Z^+$), and $\Delta(V)\triangle\Delta(V'') \subset (\Delta(V)\triangle\Delta(V')) \cup (\Delta(V')\triangle\Delta(V''))$.

(iii) For $z \in Z_{\psi_V,\psi_{V'}}$, $z' \in Z_{\psi_{V'},\psi_{V''}}$ we have $\tau_{z'}^{V''_{U^0},V'_{U^0},\iota'}\tau_z^{V'_{U^0},V_{U^0},\iota} = \tau_{z'z}^{V''_{U^0},V_{U^0},\iota'\circ\iota}$.

We will later use the following lemma concerning the support of $S^G(T_z^{V',V})$.

Lemma 2.9. *If $z \in Z_G^+(V,V')$, $z' \in Z$ and $S^G(T_z^{V',V})(z') \neq 0$, then $v_Z(z') \in v_Z(z) + \mathbb{R}_{\leq 0}\Delta^\vee$.*

Proof. Letting w_G denote the Kottwitz homomorphism, we have $\ker w_G = Z^0 G'$ [Vig16, Rk. 3.37]. If $S^G(T_z)(z') \neq 0$, then $z' \in Z \cap UKzK$, hence $w_G(z') = w_G(z)$, so $z' \in z \ker(w_G|_Z) = zZ^0(Z \cap G')$. By [AHHV17, II.6 Prop.] with $I = \varnothing$ it follows that $Z \cap G'$ is generated by all $Z \cap M'_\alpha$ for $\alpha \in \Delta$. As $v_Z(Z \cap M'_\alpha) = \mathbb{Z} v_Z(a_\alpha) \subset \mathbb{R}\alpha^\vee$, we see that $v_Z(z') \in v_Z(z) + \mathbb{R}\Delta^\vee$. By [HV15, 6.10 Prop.] we deduce $v_Z(z') \in v_Z(z) + \mathbb{R}_{\leq 0}\Delta^\vee$. □

Remark 2.10. In fact, we know that $v_Z(a_\alpha) = -e_\alpha^{-1}\alpha^\vee$ [AHHV17, IV.11 Example 3]. So the the proof shows that $v_Z(z') \in v_Z(z) + \sum_{\alpha\in\Delta} \mathbb{Z}_{\leq 0} e_\alpha^{-1}\alpha^\vee$. This improves on [Her11b, Lemma 3.6] when **G** is unramified and [HV15, 6.10 Prop.] when **G** is general.

2.7. **The generalized Satake transform.** Let $P = MN$ be a parabolic subgroup of G containing B.

Definition 2.11 ([HV12, Prop. 2.2 and 2.3], [HV15, Prop. 7.9])**.** The generalized Satake transform is the injective linear homomorphism
$$S_M^G : \mathcal{H}_G(V, V') \hookrightarrow \mathcal{H}_M(V_{N^0}, V'_{N^0})$$
defined as follows. Let $\varphi \in \mathcal{H}_G(V, V'), m \in M$ and let $p : V \twoheadrightarrow V_{N^0}$, $p' \colon V' \twoheadrightarrow V'_{N^0}$ denote the natural quotient maps. Then S_M^G is determined by the relation
$$(S_M^G\varphi)(m) \circ p = p' \circ \sum_{x\in N^0\backslash N} \varphi(xm).$$

For $\varphi \in \mathcal{H}_G(V, V')$ and $\varphi' \in \mathcal{H}_G(V', V'')$ we have $S_M^G(\varphi' * \varphi) = S_M^G(\varphi') * S_M^G(\varphi)$ [HV12, Formula (6)].

When $M = Z$, we write $S^G = S_Z^G$.

2.8. **Inverse Satake theorem.** We now give our main result. Let V and V' be irreducible representations of K. Our main theorem determines the image of the Satake transform
$$S^G : \mathcal{H}_G(V, V') \hookrightarrow \mathcal{H}_Z(V_{U^0}, V'_{U^0})$$
and moreover gives an explicit formula for the inverse of S^G on a basis of the image. (Of course this theorem is only interesting when $\mathcal{H}_G(V, V') \neq 0$. See Remark 2.7 for when this happens.)

We fix compatible isomorphisms $\iota^{\mathrm{op}} : V^{U^0_{\mathrm{op}}} \to V'^{U^0_{\mathrm{op}}}$ and $\iota : V_{U^0} \to V'_{U^0}$ as in (2.8) and $a_\alpha \in Z \cap M'_\alpha$ for $\alpha \in \Delta$ (Definition 2.1). Recalling Δ'_ψ (Definition 2.1), we denote

$$\Delta'(V) = \Delta(V) \cap \Delta'_{\psi_V} = \{\alpha \in \Delta(V) \mid \psi_V \text{ is trivial on } Z^0 \cap M'_\alpha\}. \tag{2.11}$$

Theorem 2.12 (Inverse Satake theorem)**.** *A basis of the image of S^G is given by the elements*
$$\tau_z \prod_{\alpha\in\Delta'(V')\backslash\Delta'(V)} (1 - \tau_{a_\alpha}) \tag{2.12}$$
for z running through a system of representatives of $Z_G^+(V, V')/Z^0$ in $Z_G^+(V, V')$. The inverse of S^G sends (2.12) *to*
$$\varphi_z^{V',V} := \sum_{x\in Z_z^+(V,V')} T_x^{V',V}, \quad \textit{where} \quad Z_z^+(V, V') := Z^+ \cap z \prod_{\alpha\in\Delta'(V)\cap\Delta'(V')} a_\alpha^{\mathbb{N}}.$$

The function $\varphi_z^{V',V}$ is well defined for $z \in Z_G^+(V, V')$ because of the following lemma.

Lemma 2.13. *For $z \in Z_G^+(V, V')$, the set $Z_z^+(V, V')$ is finite and contained in $Z_G^+(V, V')$.*

Proof. For $z \in Z$, the set $Z^+ \cap z\prod_{\alpha\in\Delta} a_\alpha^{\mathbb{N}}$ is finite. Indeed, $z\prod_{\alpha\in\Delta} a_\alpha^{n(\alpha)}$, $n(\alpha) \in \mathbb{N} = \{0,1,\dots\}$ lies in Z^+ if and only if

$$\langle \beta_a, \nu(z)\rangle + \sum_{\alpha\in\Delta} n(\alpha)\langle \beta_a, \alpha_a^\vee\rangle \leq 0 \text{ for all } \beta \in \Delta.$$

These inequalities admit only finitely solutions $n(\alpha) \in \mathbb{N}$ for $\alpha \in \Delta$, because the matrix $(d_\beta\langle \beta_a, \alpha_a^\vee\rangle)_{\alpha,\beta\in\Delta}$ is positive definite for some $d_\beta > 0$.

For $z \in Z_G^+(V,V')$, an element $x = z\prod_{\alpha\in\Delta'(V)\cap\Delta'(V')} a_\alpha^{n(\alpha)}$ of $Z_z^+(V,V')$ lies in $Z_{\psi_V,\psi_{V'}}$ as $a_\alpha \in Z_{\psi_V}$ for $\alpha \in \Delta'_{\psi_V}$ (see §2.1). For

$$\alpha \in \Delta'(V) \cap \Delta'(V') = \Delta(V) \cap \Delta(V') \cap \Delta'_{\psi_V}$$

and $\beta \in \Delta(V) \triangle \Delta(V')$ we have $\langle \beta_a, \alpha_a^\vee\rangle \leq 0$. By (2.10), $z \in Z_G^+(V,V')$ satisfies $\langle \beta_a, \nu(z)\rangle < 0$, so the same is true for x. Hence $x \in Z_G^+(V,V')$. □

Remark 2.14. When $V = V'$, and $\mathbf{G}$ is split with simply-connected derived subgroup, the inverse Satake theorem was obtained by [Her11a, Prop. 5.1] using the Lusztig-Kato formula. The proof of the inverse Satake theorem for arbitrary G and an arbitrary pair (V,V') uses the pro-p Iwahori Hecke algebra. It is inspired by the work of Ollivier [Oll15].

Remark 2.15. When $V = V'$ the image of S^G was known, see (2.2). The description of the image of S^G for a pair (V,V') with $V \not\simeq V'$ was an open question in [HV15, §7.9]. Theorem 2.12 shows that the image of S^G for a pair (V,V') with $V \not\simeq V'$ is not always contained in the subspace of functions in $\mathcal{H}_Z(V_{U^0}, V'_{U^0})$ supported in Z^+. This was noticed for many split groups in [Her11a, Prop. 6.13].

Remark 2.16. We establish a similar theorem for S_M^G in the next section (Corollary 2.21), at least when $\Delta'(V') \subset \Delta'(V) \cup \Delta_M$.

We mentioned earlier that Theorem 2.3 (hence the change of weight theorem) follows from the inverse Satake theorem; it is now the time to justify this assertion.

Proposition 2.17. *The inverse Satake theorem* (*Theorem 2.12*) *implies Theorem 2.3* (*and hence the change of weight theorem*).

Our first proof only uses the "image of S^G" part of Theorem 2.12 (for $V \not\cong V'$), whereas our second proof uses the explicit formula in Theorem 2.12 (but only for $V = V'$).

First proof. As in Theorem 2.3, we suppose that the parameters of the irreducible representations V and V' of K satisfy $\psi_V = \psi_{V'}$ and $\Delta(V) = \Delta(V') \sqcup \{\alpha\}$. In the proof, we will use only that we know the image of the Satake homomorphisms for (V,V') and for (V',V).

As in Theorem 2.3, let $z \in Z_{\psi_V}^+$ satisfying $\langle \alpha, v(z)\rangle > 0$. This is equivalent to $z \in Z_G^+(V,V') = Z_G^+(V',V)$ (Example 2.6). By the definition of c_α (2.6) and of $\Delta'(V)$ (2.11),

$$\Delta'(V) \setminus \Delta'(V') = \begin{cases} \{\alpha\} & \text{if } c_\alpha = 1, \\ \varnothing & \text{if } c_\alpha = 0. \end{cases}$$

The inverse Satake theorem (Theorem 2.12) gives two functions $\varphi_z^{V',V} \in \mathcal{H}_G(V,V')$ and $\varphi_z^{V,V'} \in \mathcal{H}_G(V',V)$ satisfying

$$S^G(\varphi_z^{V',V}) = \tau_z^{V'_{U^0},V_{U^0}} \quad \text{and} \quad S^G(\varphi_z^{V,V'}) = \tau_z^{V_{U^0},V'_{U^0}} - c_\alpha\,\tau_{za_\alpha}^{V_{U^0},V'_{U^0}}.$$

By Remark 2.7, the two convolution products are

$$\begin{aligned} S^G(\varphi_z^{V',V} * \varphi_z^{V,V'}) &= S^G(\varphi_z^{V',V}) S^G(\varphi_z^{V,V'}) = \tau_z^{V'_{U^0},V_{U^0}} (\tau_z^{V_{U^0},V'_{U^0}} - c_\alpha \, \tau_{za_\alpha}^{V_{U^0},V'_{U^0}}) \\ &= \tau_{z^2}^{V'_{U^0},V'_{U^0}} - c_\alpha \, \tau_{z^2 a_\alpha}^{V'_{U^0},V'_{U^0}}, \\ S^G(\varphi_z^{V,V'} * \varphi_z^{V',V}) &= S^G(\varphi_z^{V,V'}) S^G(\varphi_z^{V',V}) = (\tau_z^{V_{U^0},V'_{U^0}} - c_\alpha \, \tau_{za_\alpha}^{V_{U^0},V'_{U^0}}) \tau_z^{V'_{U^0},V_{U^0}} \\ &= \tau_{z^2}^{V_{U^0},V_{U^0}} - c_\alpha \, \tau_{za_\alpha z}^{V_{U^0},V_{U^0}} = \tau_{z^2}^{V_{U^0},V_{U^0}} - c_\alpha \, \tau_{z^2 a_\alpha}^{V_{U^0},V_{U^0}}. \end{aligned}$$

In the second product we used that $\tau_\alpha \in \mathcal{Z}_Z(V_{U^0})$. □

Second proof. In this proof, we prove Theorem 2.3 for $z \in Z^+_{\psi_V}$ such that $\langle \alpha, v(z) \rangle > 0$ and $\langle \beta, v(z) \rangle = 0$ for any $\beta \in Z^+$. As we mentioned after Theorem 2.3, this implies Theorem 2.2.

In this proof, we use Theorem 2.12 only for $V = V'$. We also use Lemma 3.1 and 3.2 from the next section. The argument is almost the same as the proof in [Her11a, Abe13].

Set $\varphi = T_z^{V',V} \in \mathcal{H}_G(V, V')$ and $\varphi' = T_z^{V,V'} \in \mathcal{H}_G(V', V)$. By the assumption on z, we have $\Delta_z = \Delta \setminus \{\alpha\}$. On the other hand, we have $\alpha \notin \Delta(V')$. Hence $\Delta(V') \subset \Delta_z$. By Lemma 3.2, we have $\varphi' * \varphi = T_{z^2}^{V,V}$.

We calculate $S^G(T_{z^2}^{V,V})$ using Theorem 2.12. From Lemma 2.18 below and Theorem 2.12, we get the following:

- If $\alpha \in \Delta'(V)$, then

$$\begin{aligned} \tau_{z^2}^{V_{U^0},V_{U^0}} &= \sum_{z' \in Z^+_{z^2}(V,V)} S^G(T_{z'}^{V,V}) \\ &= S^G(T_{z^2}^{V,V}) + \sum_{z' \in Z^+_{z^2 a_\alpha}(V,V)} S^G(T_{z'}^{V,V}) \\ &= S^G(T_{z^2}^{V,V}) + \tau_{z^2 a_\alpha}^{V_{U^0},V_{U^0}}. \end{aligned}$$

  Hence $S^G(T_{z^2}^{V,V}) = \tau_{z^2}^{V_{U^0},V_{U^0}} - \tau_{z^2 a_\alpha}^{V_{U^0},V_{U^0}}$.
- If $\alpha \notin \Delta'(V)$, then $\tau_{z^2}^{V_{U^0},V_{U^0}} = S^G(T_{z^2}^{V,V})$.

Therefore we get $S^G(\varphi' * \varphi) = \tau_{z^2}^{V_{U^0},V_{U^0}} - c_\alpha \tau_{z^2 a_\alpha}^{V_{U^0},V_{U^0}}$.

Since $\Delta(V') \subset \Delta_z$, Lemma 3.1 implies $S^G(\varphi) = \tau_z^{V'_{U^0},V_{U^0}}$. Hence $S^G(\varphi') \tau_z^{V'_{U^0},V_{U^0}} = \tau_{z^2}^{V_{U^0},V_{U^0}} - c_\alpha \tau_{z^2 a_\alpha}^{V_{U^0},V_{U^0}}$. Canceling $\tau_z^{V'_{U^0},V_{U^0}}$ and keeping in mind that τ_α is central, we get $S^G(\varphi') = \tau_z^{V_{U^0},V'_{U^0}} - c_\alpha \tau_{za_\alpha}^{V_{U^0},V'_{U^0}}$. Hence we have

$$\begin{aligned} S^G(\varphi * \varphi') &= \tau_z^{V'_{U^0},V_{U^0}} (\tau_z^{V_{U^0},V'_{U^0}} - c_\alpha \tau_{za_\alpha}^{V_{U^0},V'_{U^0}}) \\ &= \tau_{z^2}^{V'_{U^0},V'_{U^0}} - c_\alpha \tau_{z^2 a_\alpha}^{V'_{U^0},V'_{U^0}}. \end{aligned}$$ □

Lemma 2.18. *Let $\alpha \in \Delta$, $z \in Z^+$ such that $\langle \alpha, v(z) \rangle > 0$ and $\langle \beta, v(z) \rangle = 0$ for $\beta \in \Delta \setminus \{\alpha\}$.*

(i) *We have $z^2 a_\alpha \in Z^+$.*

(ii) *We have* $z_1 \in Z^+ \cap z^2 \prod_{\beta\in\Delta} a_\beta^{\mathbb{N}}$ *if and only if* $z_1 = z^2$ *or* $z_1 \in Z^+ \cap z^2 a_\alpha \prod_{\beta\in\Delta} a_\beta^{\mathbb{N}}$*. In particular, for any irreducible representation* V *of* K*, we have*

$$Z^+_{z^2}(V,V) = \begin{cases} \{z^2\} \sqcup Z^+_{z^2 a_\alpha}(V,V) & (\alpha \in \Delta'(V)),\\ \{z^2\} & (\alpha \notin \Delta'(V)). \end{cases}$$

Proof. Let $\beta \in \Delta$. If $\beta \neq \alpha$, then $\langle \beta_a, v(a_\alpha)\rangle = \langle \beta_a, -\alpha_a^\vee\rangle \geq 0$. Hence $\langle \beta_a, v(z^2 a_\alpha)\rangle \geq \langle \beta_a, v(z^2)\rangle \geq 0$. For $\beta = \alpha$, we have $\langle \alpha_a, v(a_\alpha)\rangle = \langle \alpha_a, -\alpha_a^\vee\rangle = -2$. Hence $\langle \alpha_a, v(z^2 a_\alpha)\rangle = 2\langle \alpha_a, v(z)\rangle - 2 \geq 0$.

For (ii), the "if" part is trivial. We prove the "only if" part. Let $z_1 \in z^2 \prod_{\beta\in\Delta} a_\beta^{\mathbb{N}} \cap Z^+$ and take $n(\beta) \in \mathbb{N}$ such that $z_1 = z^2 \prod_{\beta\in\Delta} a_\beta^{n(\beta)}$. Assume that $z_1 \notin z^2 a_\alpha \prod_{\beta\in\Delta} a_\beta^{\mathbb{N}} \cap Z^+$, namely $n(\alpha) = 0$. Then for $\gamma \in \Delta \setminus \{\alpha\}$, we have

$$0 \leq \langle \gamma_a, v(z_1)\rangle = \langle \gamma_a, v(z^2)\rangle - \sum_{\beta\in\Delta\setminus\{\alpha\}} n(\beta)\langle \gamma_a, \beta_a^\vee\rangle$$

Hence

$$\sum_{\beta\in\Delta\setminus\{\alpha\}} n(\beta)\langle \gamma_a, \beta_a^\vee\rangle \leq 2\langle \gamma_a, v(z)\rangle = 0$$

from the assumption on z. Since the matrix $(d_\gamma \langle \gamma_a, \beta_a^\vee\rangle)_{\beta,\gamma\in\Delta\setminus\{\alpha\}}$ is positive definite for some $d_\gamma > 0$, we get $n(\beta) = 0$ for all $\beta \in \Delta \setminus \{\alpha\}$. Hence $z_1 = z^2$. $\square$

2.9. **Inverse Satake for Levi subgroups.** Let $P = MN$ be a parabolic subgroup containing B. By the inverse Satake theorem (Theorem 2.12) for $S^G = S^G_Z$, we can get the following formula for S^G_M. Let V, V' be irreducible K-representations. We denote the function $T_z^{V_{N^0}, V'_{N^0}} \in \mathcal{H}_M(V_{N^0}, V'_{N^0})$ for M by $T_z^{V'_{N^0}, V_{N^0}, M}$. Also, for $X \subset \Delta$ we write $a_X := \prod_{\gamma\in X} a_\gamma$.

Theorem 2.19. *For* $z \in Z^+_G(V,V')$*, we have*

$$\sum_{x\in Z_z^+(V,V')} S^G_M(T_x^{V',V}) = \sum_{X\subset\Delta'(V')\setminus(\Delta'(V)\cup\Delta_M)} (-1)^{\#X} \sum_{x\in Z^{+,M}_{za_X}(V_{N^0}, V'_{N^0})} T_x^{V'_{N^0}, V_{N^0}, M}.$$

Remark 2.20. In the theorem we have $za_X \in Z^+_M(V_{N^0}, V'_{N^0})$ since $z \in Z^+_G(V,V') \subset Z^+_M(V_{N^0}, V'_{N^0})$ and $\langle \beta_a, \gamma_a^\vee\rangle \leq 0$ for any $\beta \in \Delta_M$ and $\gamma \in X \subset \Delta \setminus \Delta_M$.

Proof of Theorem 2.19. Apply S^M to both sides of the formula given in the theorem. For the left-hand side, we have

$$\begin{aligned} S^M\left(\sum_{x\in Z_z^+(V,V')} S^G_M(T_x^{V',V})\right) &= \sum_{x\in Z_z^+(V,V')} S^G(T_x^{V',V})\\ &= \tau_z \prod_{\alpha\in\Delta'(V')\setminus\Delta'(V)} (1-\tau_\alpha) \end{aligned}$$

by Theorem 2.12. For the right-hand side, applying Theorem 2.12 to M and using an inclusion-exclusion formula, we have

$$\begin{aligned}
&S^M\left(\sum_{X\subset\Delta'(V')\setminus(\Delta'(V)\cup\Delta_M)}(-1)^{\#X}\sum_{x\in Z^{+,M}_{za_X}(V_{N^0},V'_{N^0})}T_x^{V'_{N^0},V_{N^0},M}\right)\\
&=\sum_{X\subset\Delta'(V')\setminus(\Delta'(V)\cup\Delta_M)}(-1)^{\#X}\sum_{x\in Z^{+,M}_{za_X}(V_{N^0},V'_{N^0})}S^M(T_x^{V'_{N^0},V_{N^0},M})\\
&=\sum_{X\subset\Delta'(V')\setminus(\Delta'(V)\cup\Delta_M)}(-1)^{\#X}\tau_{za_X}\prod_{\alpha\in\Delta'(V'_{N^0})\setminus\Delta'(V_{N^0})}(1-\tau_\alpha)\\
&=\tau_z\prod_{\alpha\in\Delta'(V')\setminus(\Delta'(V)\cup\Delta_M)}(1-\tau_\alpha)\prod_{\alpha\in\Delta'(V'_{N^0})\setminus\Delta'(V_{N^0})}(1-\tau_\alpha)\\
&=\tau_z\prod_{\alpha\in\Delta'(V')\setminus\Delta'(V)}(1-\tau_\alpha),
\end{aligned}$$

noting also that $\Delta'(V'_{N^0})\setminus\Delta'(V_{N^0})=(\Delta_M\cap\Delta'(V'))\setminus\Delta'(V)$ (since $\Delta(V_{N^0})=\Delta_M\cap\Delta(V)$ by [AHHV17, III.10 Lemma]). Since S^M is injective, we get the theorem. $\square$

In a special case the formula is simple. In particular this happens when $V\simeq V'$.

Corollary 2.21. *If $\Delta'(V')\subset\Delta'(V)\cup\Delta_M$, then we have for $z\in Z^+_G(V,V')$,*

$$\sum_{x\in Z^+_z(V,V')}S^G_M(T_x^{V',V})=\sum_{x\in Z^{+,M}_z(V_{N^0},V'_{N^0})}T_x^{V'_{N^0},V_{N^0},M},$$

and the image of S^G_M is spanned by $\{T_z^{V'_{N^0},V_{N^0},M}\mid z\in Z^+_G(V,V')\}$.

Proof. The first part is immediate. For the last part fix $z\in Z^+_G(V,V')$. We note that $Z^{+,M}_z(V_{N^0},V'_{N^0})\subset Z^+_z(V,V')\subset Z^+_G(V,V')$. Let $\preceq$ denote the partial order on the finite set $Z^{+,M}_z(V_{N^0},V'_{N^0})$ defined by $x\preceq y$ if $x\in Z^{+,M}_y(V_{N^0},V'_{N^0})$. Then the first part applied to $y\in Z^{+,M}_z(V_{N^0},V'_{N^0})$ shows that $\sum_{x\preceq y}T_x^{V'_{N^0},V_{N^0},M}$ is in the image of S^G_M. A triangular argument now shows that $T_y^{V'_{N^0},V_{N^0},M}$ is in the image of S^G_M for any $y\in Z^{+,M}_z(V_{N^0},V'_{N^0})$, in particular this is true when $y=z$. $\square$

3. Reduction to $\Delta(V')\subset\Delta(V)$

Let V,V' be two irreducible representations of K. We reduce the proof of the inverse Satake theorem for (V,V') to the particular case where their parameters satisfy $\Delta(V')\subset\Delta(V)$. First, we establish some lemmas that are of independent interest.

3.1. **First lemma.** Let $P=MN$ be a parabolic subgroup of G containing B corresponding to $\Delta_P\subset\Delta$. Our first lemma is the computation in a particular case of the generalized Satake transform $S^G_M:\mathcal{H}_G(V,V')\to\mathcal{H}_M(V_{N^0},V'_{N^0})$ (Definition 2.11); it is a generalization of [Her11a, Cor. 2.18].

We fix linear isomorphisms $\iota^{\mathrm{op}},\iota$ as in (2.8) for (V,V'); for $z\in Z^+_G(V,V')$ we recall the elements $T_z^{V',V}\in\mathcal{H}_G(V,V')$, $T_z^{V'_{N^0},V_{N^0}}\in\mathcal{H}_M(V_{N^0},V'_{N^0})$ defined in §2.6, and the subset Δ_z of Δ defined by (2.7).

Lemma 3.1. *Let $z \in Z_G^+(V,V')$. We have $S_M^G(T_z^{V',V}) = T_z^{V'_{N^0},V_{N^0}}$ if $\Delta(V')$ is contained in Δ_P or in Δ_z.*

We will use the lemma only when $P = B, M = Z$.

Proof. Let $z \in Z_G^+(V,V')$. Suppose $m \in M$. Definition 2.11 shows that $(S_M^G T_z^{V',V})(m) = \sum_{x \in N^0 \backslash N} (p' \circ T_z^{V',V})(xm)$, where $p' : V' \twoheadrightarrow V'_{N^0}$ is the quotient map. The description of $T_z^{V',V}$ given in §2.6 shows that the support of $T_z^{V',V}$ is KzK, and the image of $T_z^{V',V}(k_1 z k_2) = k_1 T_z^{V',V}(z) k_2$ is $k_1 V'^{N^0_{z,\mathrm{op}}}$ for $k_1, k_2 \in K$ [HV15, §7.4]. One knows that [HV12, Cor. 3.20]

$$p'(k_1 V'^{N^0_{z,\mathrm{op}}}) \neq 0 \Leftrightarrow k_1 \in P^0 M^0_{V'} P^0_{z,\mathrm{op}}, \tag{3.1}$$

where $P_{V'} = M_{V'} N_{V'}$ is the parabolic subgroup of G corresponding to $\Delta(V')$.

Observe that $\Delta(V') \subset \Delta_P$ implies $M^0_{V'} \subset M^0$ and that $\Delta(V') \subset \Delta_z$ implies $M^0_{V'} \subset M^0_z$, so in either case we know that $P^0 M^0_{V'} P^0_{z,\mathrm{op}} = P^0 P^0_{z,\mathrm{op}}$. If $k_1 \in P^0 P^0_{z,\mathrm{op}}$ then $k_1 z k_2$ lies in $P^0 P^0_{z,\mathrm{op}} z K = P^0 z K$ as $z \in Z^+$ and $z^{-1} P^0_{z,\mathrm{op}} z \subset P^0_{z,\mathrm{op}}$.

Therefore, if $(p' \circ T_z^{V',V})(xm) \neq 0$ for $x \in N$ we deduce that $xm \in P^0 z K \cap P = P^0 z P^0 = N^0(M^0 z M^0)$. It follows that $m \in M^0 z M^0$ and $n \in N^0$. In particular, the support of $S_M^G(T_z^{V',V})$ is contained in $M^0 z M^0$ and $(S_M^G T_z^{V',V})(z) = p' \circ T_z^{V',V}(z)$, which induces the map $\iota : V_{U^0} \to V'_{U^0}$. The lemma follows. □

3.2. **Second lemma.** Our second lemma is the computation of the composite of two particular intertwiners; it is done in [Her11a, Prop. 6.7], [Abe13, Lemma 4.3] when $\mathbf{G}$ is split. Let V'' be a third irreducible representation of K; we fix linear isomorphisms as in (2.8) for (V,V') and (V',V'') and by composition for (V,V''). For $z \in Z_G^+(V,V')$ and $z' \in Z_G^+(V',V'')$, the product $z'z$ lies in $Z_G^+(V,V'')$ (Remark 2.8) and we have the elements $T_z^{V',V} \in \mathcal{H}_G(V,V')$, $T_{z'}^{V'',V'} \in \mathcal{H}_G(V',V'')$ and $T_{z'z}^{V'',V} \in \mathcal{H}_G(V,V'')$ (§2.6).

Lemma 3.2. *Let $z \in Z_G^+(V,V')$ and $z' \in Z_G^+(V',V'')$. We have $T_{z'}^{V'',V'} * T_z^{V',V} = T_{z'z}^{V'',V}$ if $\Delta(V')$ is contained in Δ_z or in $\Delta_{z'}$.*

Proof. By the formula for the convolution product in §2.6, we are lead to analyse the elements $(x,g) \in K \times G$ such that $T_{z'}^{V'',V'}(xz') \circ T_z^{V',V}(z'^{-1}x^{-1}g) \neq 0$. We follow the arguments of the proof of Lemma 3.1. The non-vanishing of $T_z^{V',V}(z'^{-1}x^{-1}g)$ implies $z'^{-1}x^{-1}g = k_1 z k_2$ with $k_1, k_2 \in K$; the homomorphism $T_{z'}^{V'',V'}(xz') = x T_{z'}^{V'',V'}(z')$ factors through the quotient map $p_{z'} : V' \twoheadrightarrow V'_{N^0_{z'}}$ (see §2.6). The image of $T_z^{V',V}(z'^{-1}x^{-1}g)$ is $k_1 V'^{N^0_{z,\mathrm{op}}}$ and by (3.1), $p_{z'}(k_1 V'^{N^0_{z,\mathrm{op}}}) \neq 0$ if and only if $k_1 \in P^0_{z'} M^0_{V'} P^0_{z,\mathrm{op}}$.

We know that $P^0_{z'} M^0_{V'} P^0_{z,\mathrm{op}} = P^0_{z'} P^0_{z,\mathrm{op}}$, since $\Delta(V') \subset \Delta_z$ or $\Delta(V') \subset \Delta_{z'}$. The non-vanishing of $T_{z'}^{V'',V'}(xz') \circ T_z^{V',V}(z'^{-1}x^{-1}g)$ implies $z'^{-1}x^{-1}g = k_1 z k_2 \in P^0_{z'} z K$. As $z' P^0_{z'} z'^{-1} \subset P^0_{z'}$ we deduce $KgK = Kz'zK$. We suppose $g = z'z$ and we analyze the elements $x \in K$ such that $T_{z'}^{V'',V'}(xz') \circ T_z^{V',V}(z'^{-1}x^{-1}z'z) \neq 0$. We have $z'^{-1}x^{-1}z'z \in P^0_{z'} z K$ and $x \in K$, or equivalently $x \in z'zKz^{-1}z'^{-1}z' P^0_{z'} z'^{-1} \cap K = (z'zKz^{-1}z'^{-1} \cap K) z' P^0_{z'} z'^{-1}$. The group $z'Kz'^{-1} \cap K$ contains $z' P^0_{z'} z'^{-1}$ and we claim that it contains also $z'zKz^{-1}z'^{-1} \cap K$. The formula for the convolution product given in §2.6 and this claim imply the lemma. The claim is proved in Lemma 3.3 below. □

We now check the claim used in the proof of Lemma 3.2.

Lemma 3.3. *Let $z, z' \in Z^+$. Then $z'zK(z'z)^{-1} \cap K$ is contained in $z'Kz'^{-1} \cap K$.*

Proof. For $z \in Z^+$ consider the bounded subset $\Omega_z = \{x_0, zx_0\}$ of the apartment of S, so $zKz^{-1} \cap K$ is the pointwise stabilizer of Ω_z in the kernel of the Kottwitz homomorphism [Vig16, Def. 3.14]. For $\alpha \in \Phi$ let $r_{\Omega_z}(\alpha) = \max\{0, -\langle \alpha, \nu(z) \rangle\}$. By Bruhat-Tits theory (following [Vig16, §3.6], noting that the description of the pointwise stabilizer in equation [Vig16, (42)] is valid not just for points x but for bounded subsets of the apartment of S) we then know that $zKz^{-1} \cap K$ is generated by the groups $U_{\alpha + r_{\Omega_z}(\alpha)} \subset U_\alpha$ for $\alpha \in \Phi$ and the cosets $s_\beta Z^0 \subset \mathcal{N}^0$ for $\beta \in \Phi$ such that $\langle \beta, \nu(z) \rangle = 0$. The lemma follows by noting that $r_{\Omega_{zz'}}(\alpha) \geq r_{\Omega_{z'}}(\alpha)$ and that $\langle \beta, \nu(zz') \rangle = 0$ implies $\langle \beta, \nu(z') \rangle = 0$ for any roots $\alpha, \beta \in \Phi$. ☐

3.3. Third lemma.

Lemma 3.4. *Let $z \in Z^+$ and $x = z \prod_{\alpha \in \Delta} a_\alpha^{n(\alpha)}$ with $n(\alpha) \in \mathbb{N}$. If $\langle \alpha, v(z) \rangle$ is large enough for those $\alpha \in \Delta$ with $n(\alpha) > 0$, then $x \in Z^+$.*

Proof. Recall that $v = -\nu$ and that Z^+ is the monoid of $z \in Z$ such that the integers $\langle \beta_a, \nu(z) \rangle$ are ≤ 0 for all $\beta \in \Delta$. We have $\nu(a_\alpha) = \alpha_a^\vee$ (Definition 2.1) and $\langle \beta_a, \nu(x) \rangle = \langle \beta_a, \nu(z) \rangle + \sum_{\alpha \in \Delta} n(\alpha) \langle \beta_a, \alpha_a^\vee \rangle$ for all $\beta \in \Delta$. We have $\langle \beta_a, \alpha_a^\vee \rangle \leq 0$ if $\alpha \neq \beta$ and $\langle \alpha_a, \alpha_a^\vee \rangle = 2$. The integer $\langle \beta_a, \nu(z) \rangle$ is ≤ 0 as $z \in Z^+$. If $n(\beta) = 0$ then $\langle \beta_a, \nu(x) \rangle \leq 0$. If $n(\beta) > 0$ and $\langle \beta_a, \nu(z) \rangle + 2n(\beta) \leq 0$ then $\langle \beta_a, \nu(x) \rangle \leq 0$. ☐

Later we will use it in the following form.

Lemma 3.5. *Suppose $z \in Z$, $J \subset \Delta$, and $n(\alpha) \in \mathbb{N}$ for $\alpha \in J$. Then there exists $y \in Z^+ \cap M'_J$ such that $yz \prod_{\alpha \in J} a_\alpha^{m(\alpha)}$ lies in Z^+ for all $m(\alpha) \in \mathbb{N}, m(\alpha) \leq n(\alpha)$.*

Proof. We can find $y \in Z^+ \cap M'_J$ with $\langle \alpha_a, v(y) \rangle \geq 2n(\alpha) - \langle \alpha_a, v(z) \rangle$ for all $\alpha \in J$. Then we have $\langle \alpha_a, v(yz) \rangle \geq 2m(\alpha)$ for $m(\alpha) \leq n(\alpha)$. The proof of Lemma 3.4 implies $yz \prod_{\alpha \in J} a_\alpha^{m(\alpha)}$ lies in Z^+ for all $m(\alpha) \in \mathbb{N}, m(\alpha) \leq n(\alpha)$. ☐

3.4. Reduction to $\Delta(V') \subset \Delta(V)$.

We are ready to prove that (a special case of) the inverse Satake theorem for a pair (V, V') with parameters satisfying $\Delta(V') \subset \Delta(V)$ implies the inverse Satake transform for a general pair. Note that when $\Delta(V') \subset \Delta(V)$, then $\Delta'(V') \subset \Delta'(V)$.

Theorem 3.6. *Assume $\Delta(V') \subset \Delta(V)$. For $z \in Z_G^+(V, V')$, we have $S^G(\varphi_z) = \tau_z$, where*

$$\varphi_z = \sum_{x \in Z_z^+(V,V')} T_x \quad \text{and} \quad Z_z^+(V, V') = Z^+ \cap z \prod_{\alpha \in \Delta'(V')} a_\alpha^{\mathbb{N}}.$$

Proposition 3.7. *Theorem 3.6 implies the inverse Satake theorem (Theorem 2.12).*

Proof. The proof is divided into several parts.

A) Let (V, V') be an arbitrary pair of irreducible representations of K. We introduce:

(i) The irreducible representation V'' of K with parameters $\psi_{V''} = \psi_{V'}$ and $\Delta(V'') = \Delta(V) \cap \Delta(V')$. Such a representation exists [HV12, Thm. 3.8], $Z_G^+(V, V') \subset Z_G^+(V, V'')$ (Remark 2.8) and $Z_G^+(V', V'') = Z_G^+(V'', V')$ (Example 2.6).

(ii) A central element z' of Z (hence normalizing any character ψ of Z^0) lying in Z^+ (hence in Z_ψ^+ for any ψ) and such that $\Delta_{z'} \cap (\Delta(V) \cup \Delta(V')) = \Delta(V)$. Hence $z' \in Z_G^+(V'', V')$ by (2.9).

Let $z \in Z_G^+(V,V')$ and let $\varphi_z^{V',V} = \sum_{x\in Z_z^+(V,V')} T_x^{V',V}$ as in Theorem 2.12. We reduce the computation of $S^G(\varphi_z^{V',V})$ to the single computation of $S^G(T_{z'}^{V',V''})$ using Theorem 3.6 for (V,V''). As $z \in Z_G^+(V,V'')$ and $\Delta(V'') \subset \Delta(V)$, Theorem 3.6 implies

$$S^G(\varphi_z^{V'',V}) = \tau_z^{V''_{U^0},V_{U^0}}, \tag{3.2}$$

where $\varphi_z^{V'',V} = \sum_x T_x^{V'',V}$ for $x \in Z^+ \cap z\prod_{\alpha\in J} a_\alpha^{\mathbb{N}}$ with

$$J := \Delta(V) \cap \Delta(V') \cap \Delta'_{\psi_{V'}} = \Delta(V'') \cap \Delta'_{\psi_{V''}}.$$

Such an x is contained in $Z_G^+(V,V')$ by Lemma 2.13 and hence in $Z_G^+(V,V'')$. Also, the sets $\Delta(V'')$ and $\Delta(V)$ are contained in $\Delta_{z'}$, and $z' \in Z_G^+(V'',V') \cap Z_{\psi_V}^+$. Lemma 3.2 applied twice gives

$$T_{z'}^{V',V''} * T_x^{V'',V} = T_{z'x}^{V',V}, \quad T_x^{V',V} * T_{z'}^{V,V} = T_{xz'}^{V',V},$$

and Lemma 3.1 applied to $M = Z$, $V = V'$ and $z' \in Z_{\psi_V}^+$ gives

$$S^G(T_{z'}^{V,V}) = \tau_{z'}^{V_{U^0},V_{U^0}}.$$

Since z' is central in Z, we can permute z' and x on the right-hand side, hence $T_{z'x}^{V',V} = T_{xz'}^{V',V}$. We deduce

$$S^G(T_{z'}^{V',V''})S^G(T_x^{V'',V}) = S^G(T_x^{V',V})\tau_{z'}^{V_{U^0},V_{U^0}}. \tag{3.3}$$

Taking the sum of (3.3) for $x \in Z^+ \cap z\prod_{\alpha\in J} a_\alpha^{\mathbb{N}}$, we get

$$S^G(T_{z'}^{V',V''})S^G(\varphi_z^{V'',V}) = S^G(\varphi_z^{V',V})\tau_{z'}^{V_{U^0},V_{U^0}}. \tag{3.4}$$

We used only Lemmas 3.1 and 3.2 to get (3.4). Using (3.2) in (3.4) and taking the right convolution by $\tau_{(z')^{-1}}^{V_{U^0},V_{U^0}}$, we obtain

$$S^G(\varphi_z^{V',V}) = S^G(T_{z'}^{V',V''})\tau_z^{V''_{U^0},V_{U^0}}\tau_{(z')^{-1}}^{V_{U^0},V_{U^0}} = S^G(T_{z'}^{V',V''})\tau_{z(z')^{-1}}^{V''_{U^0},V_{U^0}}. \tag{3.5}$$

The computation of $S^G(\varphi_z^{V',V})$ is reduced to the computation of $S^G(T_{z'}^{V',V''})$.

B) We cannot directly apply Theorem 3.6 to compute $S^G(T_{z'}^{V',V''})$ because $\Delta(V')$ is not contained in $\Delta(V'')$. But we show that the computation of $S^G(T_{z'}^{V',V''})$ reduces to the computation of $S^G(T_{z'^2}^{V',V'})$ using Lemmas 3.1 and 3.2.

As $\Delta(V'') \subset \Delta_{z'}$, Lemma 3.1 applied to $M = Z$, V', V'' and $z' \in Z_G^+(V',V'')$ gives

$$S^G(T_{z'}^{V'',V'}) = \tau_{z'}^{V''_{U^0},V'_{U^0}}, \tag{3.6}$$

and Lemma 3.2 applied to $z' \in Z_G^+(V',V'')$ and $z' \in Z_G^+(V'',V')$ gives

$$T_{z'}^{V',V''} * T_{z'}^{V'',V'} = T_{z'^2}^{V',V'}.$$

Applying the Satake transform, using (3.6) and taking a right convolution by $\tau_{(z')^{-1}}^{V'_{U^0},V''_{U^0}}$ we get

$$S^G(T_{z'}^{V',V''}) = S^G(T_{z'^2}^{V',V'})\tau_{(z')^{-1}}^{V'_{U^0},V''_{U^0}}.$$

Plugging this value of $S^G(T_{z'}^{V',V''})$ into (3.5) and using that z' is central in Z we get

$$S^G(\varphi_z^{V',V}) = S^G(T_{z'^2}^{V',V'})\tau_{(z')^{-1}}^{V'_{U^0},V''_{U^0}}\tau_{z(z')^{-1}}^{V''_{U^0},V_{U^0}} = S^G(T_{z'^2}^{V',V'})\tau_{(z')^{-2}z}^{V'_{U^0},V_{U^0}}. \tag{3.7}$$

C) We now compute $S^G(T_{z'^2}^{V',V'})$. Applying Theorem 3.6 to $V = V'$ and to $z'^2 \in Z_G^+(V',V')$ gives

$$S^G(\varphi_{z'^2}^{V',V'}) = \tau_{z'^2}^{V'_{U^0},V'_{U^0}}$$

for $\varphi_{z'^2}^{V',V'} = \sum_{x\in Z_{z'^2}^+(V',V')} T_x^{V',V'}$ where $Z_{z'^2}^+(V',V') = Z^+ \cap z'^2 \prod_{\alpha\in\Delta'(V')} a_\alpha^{\mathbb{N}}$.

But we want to compute $S^G(T_{z'^2}^{V',V'})$. We can choose any element z' that satisfies **A**) (ii). We choose such a z' with the property that $z'^2 \prod_{\alpha\in\Delta'(V')\setminus\Delta'(V)} a_\alpha^{\epsilon(\alpha)}$ lies in Z^+ for all $\epsilon(\alpha) \in \{0,1\}$ (this is possible by Lemma 3.4). For such a z' and $\alpha \in \Delta'(V') \setminus \Delta'(V)$, we have $z'^2 a_\alpha \in Z_{\psi_{V'}}^+$ (recall from Definition 2.1 that $a_\alpha \in Z_{\psi_{V'}}$ as $\psi_{V'}$ is trivial on $Z^0 \cap M'_\alpha$). Theorem 3.6 applied to $V = V'$ and $z'^2 a_\alpha \in Z_{\psi_{V'}}^+$ gives

$$S^G(\varphi_{z'^2 a_\alpha}^{V',V'}) = \tau_{z'^2 a_\alpha}^{V'_{U^0},V'_{U^0}} = \tau_{z'^2}^{V'_{U^0},V'_{U^0}}\tau_\alpha^{V'_{U^0},V'_{U^0}}.$$

We see that $\varphi_{z'^2}^{V',V'} - \varphi_{z'^2 a_\alpha}^{V',V'}$ is the sum of $T_x^{V',V'}$ for $x \in Z^+ \cap z'^2 \prod_{\beta\in\Delta'(V')-\{\alpha\}} a_\beta^{\mathbb{N}}$ and

$$S^G(\varphi_{z'^2}^{V',V'} - \varphi_{z'^2 a_\alpha}^{V',V'}) = \tau_{z'^2}^{V'_{U^0},V'_{U^0}}(1 - \tau_\alpha^{V'_{U^0},V'_{U^0}}).$$

By iteration we obtain that

$$\tau_{z'^2}^{V'_{U^0},V'_{U^0}} \prod_{\alpha\in\Delta'(V')\setminus\Delta'(V)} (1 - \tau_\alpha^{V'_{U^0},V'_{U^0}})$$

is the sum of $S^G(T_x^{V',V'})$ for $x \in Z^+ \cap z'^2 \prod_{\beta\in\Delta'(V')\cap\Delta'(V)} a_\beta^{\mathbb{N}}$. But z'^2 is the only element $z'^2 \prod_{\beta\in\Delta'(V')\cap\Delta'(V)} a_\beta^{n(\beta)}$ with $n(\beta) \in \mathbb{N}$ such that

$$\langle \alpha_a, \nu(z'^2)\rangle + \sum_{\beta\in\Delta'(V')\cap\Delta'(V)} n(\beta)\langle\alpha_a, \beta_a^\vee\rangle \le 0 \quad \forall\alpha\in\Delta.$$

The reason is that all the $\beta \in \Delta'(V') \cap \Delta'(V)$ are contained in $\Delta(V)$ hence in $\Delta_{z'}$, and that the matrix $(d_\alpha\langle\alpha_a, \beta_a^\vee\rangle)_{\alpha,\beta\in\Delta'(V')\cap\Delta'(V)}$ is positive definite for some $d_\alpha > 0$. We deduce:

$$S^G(T_{z'^2}^{V',V'}) = \tau_{z'^2}^{V'_{U^0},V'_{U^0}} \prod_{\alpha\in\Delta'(V')\setminus\Delta'(V)} (1 - \tau_\alpha^{V'_{U^0},V'_{U^0}}). \tag{3.8}$$

D) Plugging the value of $S^G(T_{z'^2}^{V',V'})$ given by (3.8) into (3.7) we get

$$S^G(\varphi_z^{V',V}) = \tau_{z'^2}^{V'_{U^0},V'_{U^0}} \prod_{\alpha\in\Delta'(V')\setminus\Delta'(V)} (1 - \tau_\alpha^{V'_{U^0},V'_{U^0}})\tau_{(z')^{-2}z}^{V'_{U^0},V_{U^0}}. \tag{3.9}$$

As z' is central in Z, the first term on the right-hand side commutes with the product and using $\tau_{z'^2}^{V'_{U^0},V'_{U^0}}\tau_{(z')^{-2}z}^{V'_{U^0},V_{U^0}} = \tau_z^{V'_{U^0},V_{U^0}}$, the element z'^2 disappears from the formula (3.9). As

$\tau_\alpha^{V'_{U^0},V'_{U^0}}\tau_z^{V'_{U^0},V_{U^0}} = \tau_z^{V'_{U^0},V_{U^0}}\tau_\alpha^{V_{U^0},V_{U^0}}$ for $\alpha \in \Delta'_{\psi_V} = \Delta'_{\psi_{V'}}$ (Remark 2.7), we obtain the formula of Theorem 2.12:

$$S^G(\varphi_z^{V',V}) = \tau_z^{V'_{U^0},V_{U^0}} \prod_{\alpha\in\Delta'(V')\setminus\Delta'(V)} (1-\tau_\alpha^{V_{U^0},V_{U^0}}). \tag{3.10}$$

E) Choose a system of representatives X for $Z_G^+(V,V')/Z^0$ in $Z_G^+(V,V')$ such that $x \in X$, $xa_\alpha \in Z_G^+(V,V')$ implies that $xa_\alpha \in X$. In particular, the $T_x^{V',V}$ for $x\in X$ form a basis of $\mathcal{H}_G(V,V')$. Recalling that $\varphi_z^{V',V} = \sum_{x\in Z_z^+(V,V')} T_x^{V',V}$ and that $Z_z^+(V,V') = Z^+\cap z\prod_{\alpha\in\Delta'(V)\cap\Delta'(V')} a_\alpha^{\mathbb{N}}$, Lemma 2.13 implies that the expansion of the $\varphi_z^{V',V}$ in terms of the basis $T_x^{V',V}$ $(z,x\in X)$ is triangular. Therefore the $\varphi_z^{V',V}\in\mathcal{H}_G(V,V')$ for $z\in X$ form a basis of $\mathcal{H}_G(V,V')$. As S^G is injective, this implies that the elements on the right-hand side of the formula (3.10) form a basis of the image of S^G. ☐

4. Pro-p Iwahori Hecke ring

The inverse Satake theorem for a pair (V,V') of irreducible representations of K with parameters satisfying $\Delta(V')\subset\Delta(V)$ (Theorem 3.6) relies on the theory of the pro-p Iwahori Hecke ring of G [Vig16] and on the results presented in this chapter.

4.1. **Bruhat order on the Iwahori Weyl group.** The Iwahori subgroups of G are the conjugates of the Iwahori subgroup $K(1)B^0_{\mathrm{op}}$; their pro-p Sylow subgroups are the pro-p Iwahori subgroups of G, and are the conjugates of the pro-p Iwahori subgroup

$$I = K(1)U^0_{\mathrm{op}}.$$

We have $K(1)B^0_{\mathrm{op}} = IZ^0$ and $I = U^0_{\mathrm{op}}Z(1)(U\cap I)$ (in any order) with the notation of §2.1. The map $n\mapsto IZ^0nIZ^0$ induces a bijection from the Iwahori Weyl group $W=\mathcal{N}/IZ^0$ onto the set $IZ^0\backslash G/IZ^0$ of double cosets of G modulo the Iwahori group IZ^0, and the map $n\mapsto InI$ induces a bijection from the pro-p Iwahori Weyl group $W(1)=\mathcal{N}/Z(1)$ onto the set $I\backslash G/I$ of double cosets of G modulo the pro-p Iwahori group I; the group $W(1)$ is an extension of W by $Z_k = Z^0/Z(1)$. The action of $\mathcal{N}$ on the apartment x_0+V_{ad} factors through W. We identify x_0+V_{ad} with V_{ad} by sending x_0 to $0\in V_{\mathrm{ad}}$. The Iwahori Weyl group W contains the group $W^{\mathrm{aff}} = (\mathcal{N}\cap G')/(Z^0\cap G')$ identified with the affine Weyl group of Φ_a via the action of $\mathcal{N}$ on V_{ad}. The quotient map $W\twoheadrightarrow W_0=\mathcal{N}/Z$ splits as it induces an isomorphism from $\mathcal{N}^0/Z^0$ onto W_0, and the kernel $\Lambda = Z/Z^0$ of $W\to W_0$ is commutative and finitely generated. The homomorphism $\nu: Z\to V_{\mathrm{ad}}$ factors through Λ and induces an isomorphism from $\Lambda\cap W^{\mathrm{aff}}$ onto the coroot lattice $\nu(Z\cap G') = \oplus_{\alpha\in\Delta}\mathbb{Z}\alpha_a^\vee$ of Φ_a (defined in (2.4)). The lattice $\nu(Z)$ contains the coroot lattice and is contained in the lattice of coweights

$$P(\Phi_a^\vee) = \{x\in V_{\mathrm{ad}} \mid \langle\alpha_a, x\rangle\in\mathbb{Z} \text{ for all } \alpha\in\Delta\}.$$

The Iwahori group $K(1)P^0_{\mathrm{op}} = IZ^0$ is the fixator of the fundamental antidominant alcove $\mathfrak{C}^-$ of vertex 0 contained in the antidominant closed Weyl chamber $\mathfrak{D}^-$ (defined in (2.5)). For $\alpha\in\Phi, n\in\mathbb{Z}$, the reflection $s_{\alpha_a-n}: x\mapsto x-(\langle\alpha_a,x\rangle-n)\alpha_a^\vee$ of V_{ad} with respect to a wall $\langle\alpha_a,x\rangle = n$ of V_{ad} is conjugate in W^{aff} to a reflection with respect to a wall of $\mathfrak{C}^-$; let $\mathfrak{S}$ (resp. S^{aff}) denote the set of reflections with respect to the walls $\mathrm{Ker}(\alpha_a-n)$ of V_{ad} (resp. of $\mathfrak{C}^-$). Let Ω be the W-normalizer of S^{aff}. The Iwahori Weyl group admits two semidirect product decompositions

$$W = \Lambda\rtimes W_0 = W^{\mathrm{aff}}\rtimes\Omega.$$

The image ${}_1W^{\mathrm{aff}}$ of $\mathcal{N}\cap G'$ in $W(1)$ is a normal subgroup and is an extension of W^{aff} by a subgroup Z_k^{aff} of Z_k. The inverse image $W^{\mathrm{aff}}(1)$ of W^{aff} in $W(1)$ is ${}_1W^{\mathrm{aff}}Z_k$. Denoting by $\mathfrak{S}(1)$ (resp. $S^{\mathrm{aff}}(1)$, resp. $\Omega(1)$) the inverse image of $\mathfrak{S}$ (resp. S^{aff}, resp. Ω) in $W(1)$, we have

$$W(1) = {}_1W^{\mathrm{aff}}\Omega(1), \quad {}_1W^{\mathrm{aff}}\cap\Omega(1) = Z_k^{\mathrm{aff}}, \tag{4.1}$$

$\mathfrak{S}(1) = {}_1\mathfrak{S}Z_k,\ S^{\mathrm{aff}}(1) = {}_1S^{\mathrm{aff}}Z_k$ where ${}_1W^{\mathrm{aff}}\cap\mathfrak{S}(1) = {}_1\mathfrak{S},\ {}_1W^{\mathrm{aff}}\cap S^{\mathrm{aff}}(1) = {}_1S^{\mathrm{aff}}$.

Definition 4.1. Let $\lambda_\alpha\in\Lambda$ be the image of $a_\alpha\in Z\cap M'_\alpha$ (Definition 2.1).

Note that λ_α is independent of any choices. By Definition 2.1, $\nu(\lambda_\alpha)=\nu(a_\alpha)=\alpha_a^\vee$, and

$$\Lambda\cap W^{\mathrm{aff}} = \prod_{\alpha\in\Delta}\lambda_\alpha^{\mathbb{Z}}. \tag{4.2}$$

The length ℓ of the Coxeter system $(W^{\mathrm{aff}},S^{\mathrm{aff}})$ extends to a length on W (by $\ell(wu)=\ell(w)$ for $w\in W^{\mathrm{aff}}$, $u\in\Omega$) and further inflates to a length on $W(1)$, still denoted by ℓ. For $\tilde w,\tilde u\in W(1)$ lifting $w\in W^{\mathrm{aff}}, u\in\Omega$, we have $\ell(\tilde w\tilde u)=\ell(wu)=\ell(w)$. There is a useful formula for the length of λw where $\lambda\in\Lambda, w\in W_0$ [Vig16, Cor. 5.10] (the signs are different because S^{aff} is the set of reflections with respect to the walls of the dominant alcove $\mathfrak{C}^+=-\mathfrak{C}^-$ in loc. cit.):

$$\ell(\lambda w) = \sum_{\alpha_a\in\Phi_a^+\cap w(\Phi_a^+)}|\langle\alpha_a,\nu(\lambda)\rangle| + \sum_{\alpha_a\in\Phi_a^+\cap w(\Phi_a^-)}|\langle\alpha_a,\nu(\lambda)\rangle+1| \tag{4.3}$$

$$= \ell(\lambda)-\ell(w)+2|\{\alpha\in\Phi_a^+\cap w(\Phi_a^-),\ \langle\alpha_a,\nu(\lambda)\rangle\geq 0\}|. \tag{4.4}$$

In particular, for $\lambda\in\Lambda^+=Z^+/Z^0$ we have $\ell(\lambda)=-\langle 2\rho,\nu(\lambda)\rangle$, where 2ρ is the sum of positive roots of Φ_a, and $\ell(w\lambda)=\ell(\lambda)+\ell(w)$.

Definition 4.2. The Bruhat partial order $\leq$ of $(W^{\mathrm{aff}},S^{\mathrm{aff}})$ inflates to a partial order $\leq$ on W and to a preorder $\leq$ on $W(1)$.

- $w_1u_1\leq w_2u_2 \Leftrightarrow w_1\leq w_2, u_1=u_2$ for $w_1,w_2\in W^{\mathrm{aff}}, u_1,u_2\in\Omega$ [Vig06, Appendix].
- $\tilde w_1\leq\tilde w_2\Leftrightarrow w_1\leq w_2$ for $\tilde w_1,\tilde w_2\in W(1)$ with images $w_1,w_2\in W$ [Vig06, Appendix].

There is the partial order $\preceq$ on V_{ad} determined by $-\Delta_a^\vee$ (the basis of Φ_a corresponding to the anti-dominant closed Weyl chamber $\mathfrak{D}^-$ (2.5)): $x_1\preceq x_2$ if and only if $x_1-x_2\in\sum_{\alpha\in\Delta}\mathbb{N}\alpha_a^\vee$. The next proposition compares the "Bruhat order" $\leq$ on $\Lambda^+=Z^+/Z^0$ and the partial order $\preceq$ on $\nu(\Lambda^+)$.

Proposition 4.3. *Let $\lambda_1,\lambda_2\in\Lambda^+$. Then*

$$\lambda_1\leq\lambda_2 \ \Leftrightarrow\ \lambda_1\in\lambda_2\prod_{\alpha\in\Delta}\lambda_\alpha^{\mathbb{N}} \ \Leftrightarrow\ (\nu(\lambda_1)\preceq\nu(\lambda_2),\ \lambda_1\in\lambda_2W^{\mathrm{aff}}).$$

The latter equivalence is clear because $\nu(\lambda_\alpha)=\alpha_a^\vee$ and by (4.2). The first one follows from the next two lemmas [Rap05] (we thank Xuhua He for drawing our attention to them).

Lemma 4.4. *Let $\alpha\in\Delta$ and $\lambda\in\Lambda^+$ such that $\lambda\lambda_\alpha\in\Lambda^+$. Then*

$$\lambda\lambda_\alpha<\lambda s_\alpha<\lambda.$$

Proof. [Rap05, Remark 3.9]. Recall $\nu(\lambda_\alpha)=\alpha_a^\vee$ (Definition 4.1). We have $\langle 2\rho,\alpha_a^\vee\rangle=2$ where 2ρ is the sum of positive roots $\alpha_a\in\Phi_a^+$ [Bou02, VI.1.11, Prop. 29 (iii)]. We deduce

$$\ell(\lambda)=\langle 2\rho,v(\lambda)\rangle=\langle 2\rho,v(\lambda\lambda_\alpha)\rangle-\langle 2\rho,v(\lambda_\alpha)\rangle=\langle 2\rho,v(\lambda\lambda_\alpha)\rangle+\langle 2\rho,\alpha_a^\vee\rangle=\ell(\lambda\lambda_\alpha)+2.$$

Also, $\ell(\lambda s_\alpha) = \ell(\lambda) - 1$, as $\langle \alpha_a, \nu(\lambda)\rangle \leq -2$, since $\lambda\lambda_\alpha \in \Lambda^+$. We have that $s_\alpha \lambda_\alpha = s_{\alpha_a+1}$ is an affine reflection in $\mathfrak{S}$. Also, $\lambda\lambda_\alpha = (\lambda s_\alpha)(s_\alpha\lambda_\alpha)$, $\ell(\lambda s_\alpha) = \ell(\lambda) - 1$ and $\ell(\lambda\lambda_\alpha) = \ell(\lambda s_\alpha) - 1$. Recalling the Definition 4.2 of the Bruhat order, we get the lemma. □

Half of the first equivalence of Proposition 4.3 follows from this lemma (proof of [Rap05, Prop. 3.5]). Indeed, let $\lambda_1, \lambda_2 \in \Lambda^+$ such that $\lambda_1 \in \lambda_2 \prod_{\alpha\in\Delta} \lambda_\alpha^{n(\alpha)}$ with $n(\alpha) \in \mathbb{N}$. By Lemma 3.5, there exists $\lambda \in \Lambda^+$ such that $\lambda\lambda_2 \prod_{\alpha\in\Delta} \lambda_\alpha^{m(\alpha)}$ lies in Λ^+ for all integers $m(\alpha) \in \mathbb{N}, m(\alpha) \leq n(\alpha)$. There is a chain $(x_i)_{1\leq i\leq n}$ from $x_1 = \lambda\lambda_2$ to $x_n = \lambda\lambda_1$ in Λ^+ such that $x_{i+1} = x_i\lambda_\alpha$ for some $\alpha \in \Delta$. Lemma 4.4 implies $x_{i+1} < x_i$. Hence $\lambda\lambda_1 \leq \lambda\lambda_2$. We have $\ell(\lambda\lambda_i) = \ell(\lambda) + \ell(\lambda_i)$ by the length formula (4.3) and $\lambda\lambda_1 \leq \lambda\lambda_2$ is equivalent to $\lambda_1 \leq \lambda_2$. Therefore if $\lambda_1, \lambda_2 \in \Lambda^+$ are such that $\lambda_1 \in \lambda_2 \prod_{\alpha\in\Delta} \lambda_\alpha^{\mathbb{N}}$ we have $\lambda_1 \leq \lambda_2$.

Lemma 4.5. *Let $\mathcal{P}$ be a W_0-invariant convex subset of V_{ad} and let $x_1, x_2 \in W$ such that $x_1 \leq x_2$. If $x_2(0) \in \mathcal{P}$ then $x_1(0) \in \mathcal{P}$.*

Proof. [Rap05, Lemma 3.3]. We can reduce to $x_1 = s_{\alpha_a+m}x_2$ for a simple affine reflection s_{α_a+m} with $s_{\alpha_a+m}x_2 < x_2$ and $\alpha_a \in \Phi_a, m \in \mathbb{Z}$. In particular $\alpha_a + m$ is positive on the alcove $\mathfrak{C}^-$. Then $\alpha_a + m$ is negative on the alcove $x_2(\mathfrak{C}^-)$. Hence $m \geq 0$ and $\langle \alpha_a, x_2(0)\rangle + m \leq 0$. This implies that $x_1(0) = x_2(0) - (\langle \alpha_a, x_2(0)\rangle + m)\alpha_a^\vee$ lies between $x_2(0)$ and $s_\alpha(x_2(0)) = x_2(0) - \langle \alpha_a, x_2(0)\rangle\alpha_a^\vee$. The lemma is now clear. The lemma is true (with the same argument) for any element in the closure of $\mathfrak{C}^-$ instead of the origin 0. □

The second half of the first equivalence in Proposition 4.3 follows from this lemma. For $w \in W_0$ and $\lambda \in \Lambda^+$, $w(v(\lambda)) \in v(\lambda) - \sum_{\alpha\in\Delta} \mathbb{N}\alpha_a^\vee$ because $v(\lambda)$ lies in the cone $\mathfrak{D}^+ \cap P(\Phi_a^\vee)$ of dominant coweights [Bou02, VI.1.6, Prop. 18]. The convex envelope in V_{ad} of the W_0-conjugate of $\nu(\lambda)$ is a convex W_0-invariant polygon $\mathcal{P}(\lambda)$ contained in $\nu(\lambda) + \sum_{\alpha\in\Delta} \mathbb{R}_{\geq 0}\alpha_a^\vee$. Let $\lambda_1, \lambda_2 \in \Lambda^+$ such that $\lambda_1 \leq \lambda_2$, hence $\lambda_1 \in \lambda_2 \prod_{\alpha\in\Delta} \lambda_\alpha^{\mathbb{Z}}$ by (4.2). By Lemma 4.5, $\nu(\lambda_1) \in \mathcal{P}(\lambda_2)$ hence $\lambda_1 \in \lambda_2 \prod_{\alpha\in\Delta} \lambda_\alpha^{\mathbb{N}}$. This ends the proof of Proposition 4.3.

4.2. **Bases of the pro-p Iwahori Hecke ring.** The pro-p Iwahori Hecke ring of G is a ring isomorphic to $\mathrm{End}_G(\text{c-Ind}_I^G \mathbb{Z})$, where I acts trivially on $\mathbb{Z}$. We see the pro-p Iwahori ring of G as the convolution algebra $\mathcal{H}_\mathbb{Z}$ of functions $\varphi : G \to \mathbb{Z}$ which are compactly supported and constant on the double cosets of G modulo I. The $\mathbb{Z}$-module $\mathcal{H}_\mathbb{Z}$ has several important bases indexed by $w \in W(1)$.

I) A double coset IxI for $x \in \mathcal{N}$ depends only on the image $w \in W(1)$ of x in the pro-p Iwahori Weyl group $W(1) = \mathcal{N}/Z(1)$ and is also denoted by IwI. The characteristic functions $T_w \in \mathcal{H}_\mathbb{Z}$ of IwI for $w \in W(1)$ form a natural basis of the $\mathbb{Z}$-module $\mathcal{H}_\mathbb{Z}$, called the Iwahori-Matsumoto basis. Let R be a commutative ring. We still denote by T_w the element $1 \otimes T_w$ in the R-algebra $\mathcal{H}_R = R \otimes_\mathbb{Z} \mathcal{H}_\mathbb{Z}$. The definition of the other bases of $\mathcal{H}_\mathbb{Z}$ is more elaborate.

The relations verified by the basis elements $T_w \in \mathcal{H}_\mathbb{Z}$ for $w \in W(1)$ are:

- The braid relations $T_{w_1}T_{w_2} = T_{w_1w_2}$ if $\ell(w_1) + \ell(w_2) = \ell(w_1w_2)$; hence $t \mapsto T_t$ gives an embedding $\mathbb{Z}[Z_k] \hookrightarrow \mathcal{H}_\mathbb{Z}$.
- The quadratic relations $T_{\tilde{s}}^2 = q(s)T_{\tilde{s}^2} + c(\tilde{s})T_{\tilde{s}}$ for $\tilde{s} \in S^{\mathrm{aff}}(1)$ lifting a simple reflection $s \in S^{\mathrm{aff}}$. We have $\tilde{s}^2 \in Z_k, q : \mathfrak{S} \to q^{\mathbb{N}} - \{1\}$ is a W-invariant function (for conjugation), $c : \mathfrak{S}(1) \to \mathbb{Z}[Z_k]$ is a $W(1)$-invariant function (for the conjugation action on Z_k and on $\mathfrak{S}(1)$) satisfying $c(wt) = c(tw) = tc(w)$ for $w \in \mathfrak{S}(1), t \in Z_k$.

Remark 4.6 ([Vig16, §3.8, §4.2]). Let $s \in S^{\mathrm{aff}}$. We denote by H_s the affine hyperplane of V_{ad} fixed by s, $\alpha + r \in \Phi^{\mathrm{aff}}$ an affine root of G [Vig16, 3.5] such that $H_s = \mathrm{Ker}(\alpha + r)$. Let $u \in (U_\alpha \cap \mathfrak{K}_s) \setminus \mathfrak{K}_s(1)$, $m(u)$ the only element in $\mathcal{N} \cap U_{-\alpha} u U_{-\alpha}$ where $\mathfrak{K}_s$ is the parahoric subgroup of G fixing the face of $\mathfrak{C}^-$ contained in H_s. We have $q(s) = |Im(u)I/I|$ and the image of $m(u)$ in $W(1)$ is a lift $\tilde{s}$ of s contained in ${}_1W^{\mathrm{aff}}$. A lift $\tilde{s}$ obtained in this way is called *admissible.*

The quotient of $\mathfrak{K}_s$ by its pro-p radical $\mathfrak{K}_s(1)$ is the group $G_{k,s}$ of rational points of a finite connected reductive k-group with maximal torus Z_k and of semisimple rank 1. Let $G'_{k,s}$ the subgroup of $G_{k,s}$ generated by the unipotent elements, $Z_{k,s} = Z_k \cap G'_{k,s}$. We have $Z_{k,s} \subset Z_k^{\mathrm{aff}}$ and $c(\tilde{s}) \in \mathbb{Z}[Z_{k,s}]$. This implies $c(w) \in \mathbb{Z}[Z_k^{\mathrm{aff}}]$ for $w \in {}_1\mathfrak{S}$.

II) We now give the second basis [Vig16, Lemma 4.12, Prop. 4.13]. There exist unique elements $T^*_w \in \mathcal{H}_{\mathbb{Z}}$ for $w \in W(1)$ such that

- $T^*_{w_1} T^*_{w_2} = T^*_{w_1 w_2}$ if $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$,
- $T^*_u = T_u$ if $u \in \Omega(1)$ (i.e. $\ell(u) = 0$),
- $T^*_{\tilde{s}} = T_{\tilde{s}} - c(\tilde{s})$ if $\tilde{s} \in S^{\mathrm{aff}}(1)$.

They form a basis of $\mathcal{H}_{\mathbb{Z}}$, as the Iwahori-Matsumoto expansion of T^*_w is triangular:

$$T^*_{\tilde{w}} = \sum_{x \in W, x \leq w} h^*_x, \quad h^*_x = c^*(\tilde{w}, \tilde{x}) T_{\tilde{x}}, \tag{4.5}$$

where $\tilde{w}, \tilde{x} \in W(1)$ lift $w, x \in W$, $c^*(\tilde{w}, \tilde{x}) \in \mathbb{Z}[Z_k]$ (h^*_x does not depend on the choice of $\tilde{x}$ lifting x) and $c^*(\tilde{w}, \tilde{w}) = 1$.

Remark 4.7. When the characteristic of R is p (in particular when $R = C$), we have $q(s) = 0$ in R and $T^2_{\tilde{s}} = c(\tilde{s}) T_{\tilde{s}}$, $T^*_{\tilde{s}} T_{\tilde{s}} = T_{\tilde{s}} T^*_{\tilde{s}} = 0$ for $\tilde{s} \in S^{\mathrm{aff}}(1)$; for an admissible lift $\tilde{s} \in {}_1S^{\mathrm{aff}}$,

$$c(\tilde{s}) = -|Z_{k,s}|^{-1} \sum_{t \in Z_{k,s}} T_t. \tag{4.6}$$

The $\mathbb{Z}$-submodule $\mathcal{H}^{\mathrm{aff}}_{\mathbb{Z}}$ with basis T_w for $w \in {}_1W^{\mathrm{aff}}$ is a subalgebra, T^*_w for $w \in {}_1W^{\mathrm{aff}}$ is also a basis of $\mathcal{H}^{\mathrm{aff}}_{\mathbb{Z}}$, and $c^*(\tilde{w}, \tilde{x}) \in \mathbb{Z}[Z_k^{\mathrm{aff}}]$ for $\tilde{w}, \tilde{x} \in {}_1W^{\mathrm{aff}}$.

For $\tilde{w} \in W(1)$ lifting $w \in W$, we have [Vig16, Prop. 4.13]

$$T_w T^*_{w^{-1}} = q_w,$$

where $w \mapsto q_w : W \to q^{\mathbb{N}}$ is the function defined by [Vig16, Def. 4.14] with properties

- $q_{w_1} q_{w_2} = q_{w_1 w_2}$ if $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$,
- $q_u = 1$ if $u \in \Omega$ (i.e. $\ell(u) = 0$),
- $q_s = q(s)$ for $s \in S^{\mathrm{aff}}$ as in the quadratic relation of $T(\tilde{s})$.

For $w_1, w_2 \in W$, the positive square root

$$q_{w_1, w_2} = (q_{w_1} q_{w_2} q^{-1}_{w_1 w_2})^{1/2}$$

belongs to $q^{\mathbb{N}}$ [Vig16, Lemma 4.19] and $q_{w_1, w_2} = 1$ if and only if $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$ [Vig16, Lemma 4.16]. We inflate q_w and q_{w_1, w_2} to $W(1)$, we put $q_{\tilde{w}} = q_w$ and $q_{\tilde{w}_1, \tilde{w}_2} = q_{w_1, w_2}$ for $\tilde{w}, \tilde{w}_1, \tilde{w}_2 \in W(1)$ lifting w, w_1, w_2.

Remark 4.8. [Vig16, Prop. 4.13(6)]. There is also a unique function $w \mapsto c_w : W^{\mathrm{aff}}(1) \to \mathbb{Z}[Z_k]$ satisfying $c_{w_1} c_{w_2} = c_{w_1 w_2}$ if $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$, $c_{\tilde{s}} = c(\tilde{s})$ for $\tilde{s} \in S^{\mathrm{aff}}(1)$, and $c_t = t$ for $t \in Z_k$.

Remark 4.9. Some properties of $c^*(w,x)$ for $x,w \in W(1), x \leq w$, follow easily from the braid relations for T^*_w and T_x:

(i) For $t \in Z_k$, we have $c^*(tw,x) = tc^*(w,x)$ and $c^*(w,xt)xtx^{-1} = c^*(w,tx)t = c^*(w,x)$ because $T^*_{tw} = T_t T^*_w$ and $c^*(w,x)T_x = c^*(w,xt)T_{xt} = c^*(w,xt)T_{xtx^{-1}}T_x = c^*(w,tx)T_{tx} = c^*(w,tx)T_t T_x$.

(ii) For $v \in \Omega(1)$ we have $c^*(wv,xv) = c^*(w,x)$ because $T^*_w T_v = T^*_{wv}$ and $T_x T_v = T_{xv}$.

III) The other bases of $\mathcal{H}_{\mathbb{Z}}$ are associated to spherical orientations of V_{ad}; they generalize the Bernstein basis of an affine Hecke algebra. The spherical orientations are in one-to-one correspondence with the Weyl chambers of V_{ad} (cf. [Vig16, Def. 5.16]). If $\mathfrak{D}_o$ is the Weyl chamber of a spherical orientation o and $w \in W(1) = \mathcal{N}/Z(1)$ an element of image $w_0 \in W_0 = \mathcal{N}/Z$, we denote by $o \cdot w$ the orientation of Weyl chamber $w_0^{-1}(\mathfrak{D}_o)$. In particular $o \cdot \lambda = o$ when $\lambda \in \Lambda(1) = Z/Z(1)$. There is a basis $E_o(w)$ for $w \in W(1)$ of $\mathcal{H}_{\mathbb{Z}}$ associated to each spherical orientation o [Vig16, §5.3].

The main properties of the elements $E_o(w)$ are:

- Multiplication formula $E_o(w_1)E_{o\cdot w_1}(w_2) = q_{w_1,w_2}E_o(w_1w_2)$ for $w_1, w_2 \in W(1)$.
- Triangular Iwahori-Matsumoto expansion [Vig16, Cor. 5.26]

$$E_o(\tilde{w}) = \sum_{x \in W, x \leq w} h_o(x), \quad h_o(x) = c_o(\tilde{w},\tilde{x})T_{\tilde{x}}, \tag{4.7}$$

  where $\tilde{w}, \tilde{x} \in W(1)$ lift $w, x \in W$, $c_o(\tilde{w},\tilde{x}) \in \mathbb{Z}[Z_k]$ ($h_o(x)$ does not depend on the choice of $\tilde{x}$ lifting x) and $c_o(\tilde{w},\tilde{w}) = 1$.
- $E_o(\lambda) = \begin{cases} T_\lambda & \text{if } \nu(\lambda) \in \mathfrak{D}_o \\ T^*_\lambda & \text{if } \nu(\lambda) \in -\mathfrak{D}_o \end{cases}$ for $\lambda \in \Lambda(1)$.

When R is a ring of characteristic p (in particular $R = C$), in $\mathcal{H}_R$ we have

$$E_o(w_1)E_{o\cdot w_1}(w_2) = \begin{cases} E_o(w_1w_2) & \text{if } \ell(w_1) + \ell(w_2) = \ell(w_1w_2), \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.10. The integral Bernstein basis $(E(w) = E_{o^-}(w))_{w \in W(1)}$ is the basis associated to the spherical orientation o^- corresponding to the antidominant Weyl chamber $\mathfrak{D}^-$ (2.5).

For $x \in \mathcal{N}$ of image $w \in W(1)$ we write also $T(x) = T_w, T^*(x) = T^*_w, E_o(x) = E_o(w)$.

4.3. **Representations of K and Hecke modules.** The submodule $\mathcal{H}_{\mathbb{Z}}(K,I)$ of functions with support in K in the pro-p Iwahori Hecke algebra $\mathcal{H}_{\mathbb{Z}}$ is the submodule of basis T_w for $w \in W_0(1)$; it is a subalgebra of $\mathcal{H}_{\mathbb{Z}}$ canonically isomorphic to the algebra of intertwiners $\mathrm{End}_K(\text{c-Ind}_I^K \mathbb{Z})$.

We may view $\mathcal{H}_{\mathbb{Z}}(K,I)$ as the convolution algebra $\mathcal{H}_{\mathbb{Z}}(G_k, U_{k,\mathrm{op}})$ of functions $G_k \to \mathbb{Z}$ which are constant on the double cosets modulo $U_{k,\mathrm{op}}$. The irreducible representations V of G_k are in one-to-one correspondence with the characters of $\mathcal{H}_C(G_k, U_{k,\mathrm{op}})$ [CL76, Cor. 7.5], [CE04, Thm. 6.10]. The representation V corresponds to the character χ giving the action of $\mathcal{H}_C(G_k, U_{k,\mathrm{op}})$ on the line $V^{U_{k,\mathrm{op}}}$. We consider V as an irreducible representation of K and χ as a character of $\mathcal{H}_C(K,I)$ giving the action of $\mathcal{H}_C(K,I)$ on $V^I = V^{U_{k,\mathrm{op}}}$.

A character χ of $\mathcal{H}_C(K,I)$ is determined by a C-character ψ_χ of Z^0 such that $\psi_\chi(t) = \chi(T(t))$ for $t \in Z^0$ and by the subset $\Delta(\chi)$ of Δ_{ψ_χ} (4.18) defined by

$$\chi(T_{\tilde{s}_\alpha}) = \begin{cases} -1 & \text{if } \alpha \in \Delta_{\psi_\chi} \setminus \Delta(\chi) \\ 0 & \text{if } \alpha \in \Delta(\chi) \text{ or } \alpha \notin \Delta_{\psi_\chi} \end{cases}, \tag{4.8}$$

where $\tilde{s}_\alpha$ is an admissible lift of s_α (Remark 4.6). The pair $(\psi_\chi, \Delta(\chi))$ is called the *parameter of* χ.

- $V = V(U_k) \oplus V^{U_{k,\mathrm{op}}}$ where $V(U_k)$ is the kernel of the quotient map $V \twoheadrightarrow V_{U_k}$ [CE04, Thm. 6.12]. In particular, Z_k acts on the lines $V^{U_{k,\mathrm{op}}}$ and V_{U_k} by the same character ψ_V.
- The stabilizer of $V^{U_{k,\mathrm{op}}}$ in G_k is the parabolic subgroup $P_{\Delta(\chi),k,\mathrm{op}}$ [CL76, Prop. 6.6, Thm. 7.1].
- The stabilizer of $V(U_k)$ in G_k is the parabolic subgroup $P_{\Delta(V),k}$ (see §2.2).

Lemma 4.11. *The parameter* $(\psi_V, \Delta(V))$ *of* V *and the parameter* $(\psi_\chi, \Delta(\chi))$ *of* χ *satisfy* $\psi_V = \psi_\chi^{-1},\ \Delta(V) = \Delta(\chi)$.

Proof. We have $fT(t^{-1}) = tf$ for $t \in Z_k$ hence $\psi_\chi = \psi_V^{-1}$, because

$$fh = \sum_{x \in I\backslash K} h(x)x^{-1}f \quad \text{for } h \in \mathcal{H}_C(K,I),\ f \in V^I.$$

Let w_Δ be the longest element of W_0. The group $U_{k,\mathrm{op}}$ is conjugate to U_k by w_Δ, the stabilizer $P_{\Delta(\chi),k,\mathrm{op}}$ of $V^{U_{k,\mathrm{op}}}$ is the conjugate by w_Δ of the stabilizer of the line V^{U_k}, which is $P_{-w_\Delta(\Delta(V)),k}$ [AHHV17, III.9 Remark 1]. Hence $\Delta(V) = \Delta(\chi)$. □

4.4. **The elements $c_w^x \in \mathbb{Z}[Z_k]$.** Our motivation is to explicitly compute the expansion of T_w^* in the Iwahori-Matsumoto basis in $\mathcal{H}_\mathbb{Z}$ modulo q (Theorem 4.23). We associate to the function $c : \mathfrak{S}(1) \to \mathbb{Z}[Z_k]$ defining the quadratic relation of T_s for $s \in S^{\mathrm{aff}}(1)$, elements

$$c_w^x \in \mathbb{Z}[Z_k] \quad \text{for } x, w \in W(1),\ x \leq w,$$

and we study their properties.

Notation 4.12. The action of $W(1)$ by conjugation on Z_k factors through W and we write $w \cdot c = \tilde{w}c\tilde{w}^{-1}$ for $c \in \mathbb{Z}[Z_k]$ and $\tilde{w} \in W(1)$ lifting $w \in W$. We write also $w_1 \cdot w_2 = w_1 w_2 w_1^{-1}$ for w_1, w_2 in $W(1)$ (or w_1, w_2 in W).

For a sequence $\underline{\tilde{w}} = (\tilde{s}_1, \dots, \tilde{s}_n)$ in $S^{\mathrm{aff}}(1)$ lifting a sequence $\underline{w} = (s_1, \dots, s_n)$ in S^{aff}, write $\tilde{w} := \tilde{s}_1 \cdots \tilde{s}_n, w := s_1 \cdots s_n$ for the products of the terms of the sequences. We take 1 for the "product of the terms" of the empty sequence ( ). The lifts of the sequence $\underline{w}$ in S^{aff} are the sequences $(t_1\tilde{s}_1, \dots, t_n\tilde{s}_n)$ in $S^{\mathrm{aff}}(1)$, where $t_i \in Z_k$.

Definition 4.13. Let $\underline{\tilde{w}} = (\tilde{s}_1, \dots, \tilde{s}_n)$ be a sequence in $S^{\mathrm{aff}}(1)$ and $\underline{\tilde{x}} = (\tilde{s}_{i_1}, \dots, \tilde{s}_{i_r})$ with $1 \leq i_1 < \cdots < i_r \leq n$ a subsequence of $\underline{\tilde{w}}$. We define $c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}}$ as the product of the following elements of $\mathbb{Z}[Z_k]$:

$c(\tilde{s}_1) \cdots c(\tilde{s}_{i_1-1})$
$s_{i_1} \cdot (c(\tilde{s}_{i_1+1}) \cdots c(\tilde{s}_{i_2-1}))$
$s_{i_1}s_{i_2} \cdot (c(\tilde{s}_{i_2+1}) \cdots c(\tilde{s}_{i_3-1}))$
$\cdots$
$s_{i_1} \cdots s_{i_r} \cdot (c(\tilde{s}_{i_r+1}) \cdots c(\tilde{s}_{i_n}))$.

Remark 4.14. Strictly speaking, for the subsequence $\underline{\tilde{x}}$ we need to remember the sequence of integers $i_1 < \cdots < i_r$.

Example 4.15. We have $c_{\underline{\tilde{w}}}^{\underline{\tilde{w}}} = 1$.

When $\tilde{w} = \tilde{s}_1 \cdots \tilde{s}_n$ is a reduced decomposition, we have $c_{\underline{\tilde{w}}}^{(\,)} = c_{\tilde{w}}$ (Remark 4.8).

Take $1 \leq m \leq n$ and cut the sequences $\underline{\tilde{w}}$ and $\underline{\tilde{x}}$ in two: $\underline{\tilde{w}} = \underline{\tilde{w}}_1\underline{\tilde{w}}_2$ and $\underline{\tilde{x}} = \underline{\tilde{x}}_1\underline{\tilde{x}}_2$ with $\underline{\tilde{w}}_1 = (\tilde{s}_1, \ldots, \tilde{s}_m), \underline{\tilde{w}}_2 = (\tilde{s}_{m+1}, \ldots, \tilde{s}_n)$, $\underline{\tilde{x}}_1 = (\tilde{s}_{i_1}, \ldots, \tilde{s}_{i_t}), \underline{\tilde{x}}_2 = (\tilde{s}_{i_{t+1}}, \ldots, \tilde{s}_{i_r})$ where $i_t \leq m < i_{t+1}$. The sequence decompositions $\underline{\tilde{w}} = \underline{\tilde{w}}_1\underline{\tilde{w}}_2$ and $\underline{\tilde{x}} = \underline{\tilde{x}}_1\underline{\tilde{x}}_2$ are called *compatible*. For $i = 1, 2$, the sequence $\underline{\tilde{x}}_i$ is a subsequence of $\underline{\tilde{w}}_i$ and we have $c_{\underline{\tilde{w}}_i}^{\underline{\tilde{x}}_i}$. The terms in the product defining $c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}_1}$ or $x_1 \cdot c_{\underline{\tilde{w}}_2}^{\underline{\tilde{x}}_2}$ appear in the product defining $c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}}$ except the last term $x_1 \cdot (c(\tilde{s}_{i_t+1}) \cdots c(\tilde{s}_m))$ of $c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}_1}$ and the first term $x_1 \cdot (c(\tilde{s}_{m+1}) \cdots c(\tilde{s}_{i_{t+1}-1}))$ of $x_1 \cdot c_{\underline{\tilde{w}}_2}^{\underline{\tilde{x}}_2}$; their product $x_1 \cdot (c(\tilde{s}_{i_t+1}) \cdots c(\tilde{s}_{i_{t+1}-1}))$ appears in $c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}}$. Then, we get a one-to-one correspondence with the terms appearing in the product defining $c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}}$:

$$c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}_1} (x_1 \cdot c_{\underline{\tilde{w}}_2}^{\underline{\tilde{x}}_2}). \tag{4.9}$$

This useful formula allows us to study $c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}}$ by induction on the length n of $\underline{\tilde{w}}$.

Example 4.16. When $\underline{\tilde{x}}_2 = \underline{\tilde{w}}_2$ we have $c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}_1}$.

When $m = n-1$ and $i_r < n$, we have $c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}}(x \cdot c(\tilde{s}_n))$.

By iteration of (4.9) we deduce:

Lemma 4.17. *Let $\underline{\tilde{w}}$ and $\underline{\tilde{x}}$ be two sequences in $S^{\mathrm{aff}}(1)$ such that $\underline{\tilde{x}}$ is a subsequence of $\underline{\tilde{w}}$ and consider compatible sequences decompositions $\underline{\tilde{w}} = \underline{\tilde{w}}_1 \cdots \underline{\tilde{w}}_k$ and $\underline{\tilde{x}} = \underline{\tilde{x}}_1 \cdots \underline{\tilde{x}}_k$. Then*

$$c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} \;=\; c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}_1}\ (x_1 \cdot c_{\underline{\tilde{w}}_2}^{\underline{\tilde{x}}_2})\ (x_1x_2 \cdot c_{\underline{\tilde{w}}_3}^{\underline{\tilde{x}}_3})\ \cdots\ (x_1 \cdots x_{k-1} \cdot c_{\underline{\tilde{w}}_k}^{\underline{\tilde{x}}_k}).$$

The function $c : S^{\mathrm{aff}}(1) \to \mathbb{Z}[Z_k]$ satisfies:

Lemma 4.18. *For $\tilde{s} \in S^{\mathrm{aff}}(1)$ lifting $s \in S^{\mathrm{aff}}$ and $c \in \mathbb{Z}[Z_k]$, we have $s \cdot c(\tilde{s}) = c(\tilde{s})$ and $c(\tilde{s})\, c = c(\tilde{s})\,(s \cdot c)$.*

Proof. The equalities $c(\tilde{s})\, t = c(\tilde{s})\,(s \cdot t)$ for $t \in Z_k$ and $c(\tilde{s})\, c = c(\tilde{s})\,(s \cdot c)$ for $c \in \mathbb{Z}[Z_k]$ are equivalent. Suppose that $\tilde{s}$ is an admissible lift of s (Remark 4.6). Then, the lemma is proved in [Vig16, Prop. 4.4]. The other lifts of s are $\tilde{s}t$ for $t \in Z_k$ and $s \cdot c(\tilde{s}t) = s \cdot (c(\tilde{s})t) = (s \cdot c(\tilde{s}))\,(s \cdot t) = c(\tilde{s})t = c(\tilde{s}t)$. For $t, t' \in Z_k$, we have $c(\tilde{s}t)\, t' = c(\tilde{s})tt' = c(\tilde{s})\,(s \cdot tt') = c(\tilde{s})t\,(s \cdot t') = c(\tilde{s}t)\,(s \cdot t')$. ☐

Lemma 4.19. *Let $\underline{\tilde{w}}$ and $\underline{\tilde{x}}$ be two sequences in $S^{\mathrm{aff}}(1)$ such that $\underline{\tilde{x}}$ is a subsequence of $\underline{\tilde{w}}$ and let $c \in \mathbb{Z}[Z_k]$. Then, $c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}}\,(x \cdot c) = c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}}\,(w \cdot c)$.*

Proof. We cut the sequences $\underline{\tilde{w}}$ and $\underline{\tilde{x}}$ in two (as above with $m = n-1$). Let $\underline{\tilde{w}}_1 = (\tilde{s}_1, \ldots, \tilde{s}_{n-1}), \underline{\tilde{w}}_2 = (\tilde{s}_n)$.

When $i_r = n$, applying Example 4.16 we have $c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}_1}$ where $\underline{\tilde{x}}_1 = (\tilde{s}_{i_1}, \ldots, \tilde{s}_{i_{r-1}})$. By induction on n, $c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}_1}\,(x_1 \cdot c) = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}_1}\,(w_1 \cdot c)$. Hence $c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}}\,(x \cdot c) = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}_1}\,(x_1 s_n \cdot c) = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}_1}\,(w_1 s_n \cdot c) = c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}}\,(w \cdot c)$.

When $i_r \neq n$, applying Example 4.16 (twice), Lemma 4.18, as well as induction on n we have $c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}}\,(x \cdot c) = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}}(x \cdot c(\tilde{s}_n)c) = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}}(x \cdot c(\tilde{s}_n)(s_n \cdot c)) = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}}(x \cdot c(\tilde{s}_n))(x s_n \cdot c) = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}}(x \cdot c(\tilde{s}_n))(w_1 s_n \cdot c) = c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}}\,(w \cdot c)$. ☐

Proposition 4.20. *Let $\underline{\tilde{w}}$ be a sequence in $S^{\mathrm{aff}}(1)$ and $\underline{\tilde{x}}$ a subsequence of $\underline{\tilde{w}}$ such that $\tilde{w} = \tilde{s}_1 \cdots \tilde{s}_n$ and $\tilde{x} = \tilde{s}_{i_1} \cdots \tilde{s}_{i_r}$ are reduced decompositions (i.e. $n = \ell(w), r = \ell(x)$), and $t, u \in Z_k$. Then the product $tu^{-1} c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}}$ depends only on $t\tilde{w}, u\tilde{x} \in W(1)$.*

Proof. We have to prove $tu^{-1}c^{\underline{\tilde{x}}}_{\underline{\tilde{w}}} = t'u'^{-1}c^{\underline{\tilde{x}}'}_{\underline{\tilde{w}}'}$, when $\underline{\tilde{w}}' = (\tilde{s}'_1, \ldots, \tilde{s}'_n)$ is a sequence in $S^{\mathrm{aff}}(1)$, $\underline{\tilde{x}}' = (\tilde{s}'_{j_1}, \ldots, \tilde{s}'_{j_r})$ is a subsequence of $\underline{\tilde{w}}'$ and t', u' are elements in Z_k, satisfying $t\tilde{w} = t'\tilde{w}'$ and $u\tilde{x} = u'\tilde{x}'$. Then w, w' have the same length n, and x, x' have the same length r. The proof is divided into several steps and uses induction on n.

A) Assume $\underline{\tilde{w}} = \underline{\tilde{w}}'$. Then $t = t'$ and we will prove $u^{-1}c^{\underline{\tilde{x}}}_{\underline{\tilde{w}}} = u'^{-1}c^{\underline{\tilde{x}}'}_{\underline{\tilde{w}}}$. By symmetry, we have three cases:

$$(1)\ i_r = j_r = n,\ \ (2)\ i_r < n \text{ and } j_r < n,\ \ (3)\ i_r = n \text{ and } j_r < n.$$

We denote by $\underline{\tilde{w}}^\flat, \underline{w}^\flat$ the sequences obtained by erasing the last term of in the sequences $\underline{\tilde{w}}, \underline{w}$; the products of the terms in $\underline{\tilde{w}}^\flat$ and of $\underline{w}^\flat$ are denoted by $\tilde{w}^\flat$ and $w^\flat$. We examine each case separately, using Example 4.16. We have:

(1) $c^{\underline{\tilde{x}}}_{\underline{\tilde{w}}} = c^{\underline{\tilde{x}}^\flat}_{\underline{\tilde{w}}^\flat}, c^{\underline{\tilde{x}}'^\flat}_{\underline{\tilde{w}}^\flat} = c^{\underline{\tilde{x}}'}_{\underline{\tilde{w}}}$. By induction on n, $u^{-1}c^{\underline{\tilde{x}}}_{\underline{\tilde{w}}} = u'^{-1}c^{\underline{\tilde{x}}'}_{\underline{\tilde{w}}}$.

(2) $c^{\underline{\tilde{x}}}_{\underline{\tilde{w}}} = c^{\underline{\tilde{x}}}_{\underline{\tilde{w}}^\flat}(x \cdot c(\tilde{s}_n))$ and $c^{\underline{\tilde{x}}'}_{\underline{\tilde{w}}} = c^{\underline{\tilde{x}}'}_{\underline{\tilde{w}}^\flat}(x' \cdot c(\tilde{s}_n))$. By induction on n, and noting that $x = x'$, $u^{-1}c^{\underline{\tilde{x}}}_{\underline{\tilde{w}}} = u'^{-1}c^{\underline{\tilde{x}}'}_{\underline{\tilde{w}}}$.

(3) $c^{\underline{\tilde{x}}}_{\underline{\tilde{w}}} = c^{\underline{\tilde{x}}^\flat}_{\underline{\tilde{w}}^\flat}$ and $c^{\underline{\tilde{x}}'}_{\underline{\tilde{w}}} = c^{\underline{\tilde{x}}'}_{\underline{\tilde{w}}^\flat}(x' \cdot c(\tilde{s}_{i_r}))$. Since $s_{i_1} \cdots s_{i_r} = s_{j_1} \cdots s_{j_r}$ are reduced decompositions, by the exchange condition there exists $1 \leq k \leq r$ such that $s_{j_{k+1}} \cdots s_{j_r} s_{i_r} = s_{j_k} \cdots s_{j_r}$ and $x^\flat = s_{i_1} \cdots s_{i_{r-1}} = s_{j_1} \cdots s_{j_{k-1}} s_{j_{k+1}} \cdots s_{j_r}$. Suppressing the k-th term of the sequence $\underline{\tilde{x}}'$ we get $\underline{\tilde{x}}'^\star = (\tilde{s}_{j_1}, \ldots, \tilde{s}_{j_{k-1}}, \tilde{s}_{j_{k+1}}, \ldots, \tilde{s}_{j_r})$ and $\tilde{x}'^\star = \tilde{s}_{j_1} \cdots \tilde{s}_{j_{k-1}} \tilde{s}_{j_{k+1}} \cdots \tilde{s}_{j_r}$ lifting $x^\flat$. Let $u'' \in Z_k$ such that $u\tilde{x}^\flat = u''\tilde{x}'^\star$. By induction on n, $u^{-1}c^{\underline{\tilde{x}}^\flat}_{\underline{\tilde{w}}^\flat} = u''^{-1}c^{\underline{\tilde{x}}'^\star}_{\underline{\tilde{w}}^\flat}$; hence

$$u''^{-1}c^{\underline{\tilde{x}}'^\star}_{\underline{\tilde{w}}^\flat} = u'^{-1}c^{\underline{\tilde{x}}'}_{\underline{\tilde{w}}^\flat}(x' \cdot c(\tilde{s}_{i_r})) \tag{4.10}$$

implies $u^{-1}c^{\underline{\tilde{x}}}_{\underline{\tilde{w}}} = u'^{-1}c^{\underline{\tilde{x}}'}_{\underline{\tilde{w}}}$. We now prove (4.10). Applying Lemma 4.17 to the compatible decompositions $\underline{\tilde{w}}^\flat = \underline{\tilde{w}}_1(\tilde{s}_{j_k})\underline{\tilde{w}}_3$, $\underline{\tilde{x}}'^\star = \underline{\tilde{x}}'_1(\ )\underline{\tilde{x}}'_3$, and $\underline{\tilde{x}}' = \underline{\tilde{x}}'_1(\tilde{s}_{j_k})\underline{\tilde{x}}'_3$ we get $c^{\underline{\tilde{x}}'^\star}_{\underline{\tilde{w}}^\flat} = c^{\underline{\tilde{x}}'_1}_{\underline{\tilde{w}}_1}(x'_1 \cdot c(\tilde{s}_{j_k})c^{\underline{\tilde{x}}'_3}_{\underline{\tilde{w}}_3})$ and $c^{\underline{\tilde{x}}'}_{\underline{\tilde{w}}^\flat} = c^{\underline{\tilde{x}}'_1}_{\underline{\tilde{w}}_1}(x'_1 s_{j_k} \cdot c^{\underline{\tilde{x}}'_3}_{\underline{\tilde{w}}_3})$. We have $c(\tilde{s}_{j_k})c^{\underline{\tilde{x}}'_3}_{\underline{\tilde{w}}_3} = c(\tilde{s}_{j_k})\,(s_{j_k} \cdot c^{\underline{\tilde{x}}'_3}_{\underline{\tilde{w}}_3})$ by Lemma 4.18 so that $c^{\underline{\tilde{x}}'^\star}_{\underline{\tilde{w}}^\flat} = c^{\underline{\tilde{x}}'}_{\underline{\tilde{w}}^\flat}(x'_1 \cdot c(\tilde{s}_{j_k}))$. Hence

$$u''^{-1}(x'_1 \cdot c(\tilde{s}_{j_k})) = u'^{-1}(x' \cdot c(\tilde{s}_{i_r})) \tag{4.11}$$

implies (4.10). We now prove (4.11). We have $u'\tilde{s}_{j_1} \cdots \tilde{s}_{j_r} = u''\tilde{s}_{j_1} \cdots \tilde{s}_{j_{k-1}}\tilde{s}_{j_{k+1}} \cdots \tilde{s}_{j_r}\tilde{s}_{i_r}$. Therefore $u'((\tilde{s}_{j_1} \cdots \tilde{s}_{j_r}) \cdot \tilde{s}_{i_r}^{-1}) = u''((\tilde{s}_{j_1} \cdots \tilde{s}_{j_{k-1}}) \cdot \tilde{s}_{j_k}^{-1})$. Taking the inverse shows $(\tilde{x}' \cdot \tilde{s}_{i_r})u'^{-1} = (\tilde{x}'_1 \cdot \tilde{s}_{j_k})u''^{-1}$ and $u'^{-1}(x' \cdot c(\tilde{s}_{i_r})) = (x' \cdot c(\tilde{s}_{i_r}))u'^{-1} = c((\tilde{x}' \cdot \tilde{s}_{i_r})u'^{-1}) = c((\tilde{x}'_1 \cdot \tilde{s}_{j_k})u''^{-1}) = c(\tilde{x}'_1 \cdot \tilde{s}_{j_k})u''^{-1} = u''^{-1}(x'_1 \cdot c(\tilde{s}_{j_k}))$. This ends the proof of case **A**).

B) Assume $\underline{w} = \underline{w}'$. We will prove that $tu^{-1}c^{\underline{\tilde{x}}}_{\underline{\tilde{w}}} = t'u'^{-1}c^{\underline{\tilde{x}}'}_{\underline{\tilde{w}}'}$ by induction on n. When $n = 1$ this follows from the following identities for $a \in Z_k$: $c^{(\ )}_{(a\tilde{s}_1)} = c(a\tilde{s}_1) = ac(\tilde{s}_1) = ac^{(\ )}_{(\tilde{s}_1)}$ and $c^{(a\tilde{s}_1)}_{(a\tilde{s}_1)} = 1 = c^{(\tilde{s}_1)}_{(\tilde{s}_1)}$. For $n > 1$ we will reduce to case **A**) as follows. Let $\underline{\tilde{x}}'' = (\tilde{s}'_{i_1}, \ldots, \tilde{s}'_{i_r})$. Choose non-trivial decompositions $\underline{\tilde{w}} = \underline{\tilde{w}}_1\underline{\tilde{w}}_2$, $\underline{\tilde{w}}' = \underline{\tilde{w}}'_1\underline{\tilde{w}}'_2$ with $\ell(w_i) = \ell(w'_i) > 0$ for $i = 1, 2$. Then we have compatible decompositions $\underline{\tilde{x}} = \underline{\tilde{x}}_1\underline{\tilde{x}}_2$ and $\underline{\tilde{x}}'' = \underline{\tilde{x}}''_1\underline{\tilde{x}}''_2$. In particular, $w_i = w'_i$, $x_i = x''_i$, and we can choose $t_i, u_i \in Z_k$ such that $\tilde{w}_i = t_i\tilde{w}'_i$, $u_i\tilde{x}_i = \tilde{x}''_i$ for $i = 1, 2$. By induction we have that $u_i^{-1}c_{\underline{\tilde{w}}_i}^{\underline{\tilde{x}}_i} = t_i c_{\underline{\tilde{w}}_i'}^{\underline{\tilde{x}}_i''}$. Hence from (4.9) and Lemma 4.19 we get

$$c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}_1}(x_1\cdot c_{\underline{\tilde{w}}_2}^{\underline{\tilde{x}}_2}) = t_1u_1c_{\underline{\tilde{w}}_1'}^{\underline{\tilde{x}}_1''}(x_1''\cdot t_2u_2c_{\underline{\tilde{w}}_2'}^{\underline{\tilde{x}}_2''}) = t_1(w_1'\cdot t_2)u_1(x_1\cdot u_2)c_{\underline{\tilde{w}}_1'}^{\underline{\tilde{x}}_1''}(x_1''\cdot c_{\underline{\tilde{w}}_2'}^{\underline{\tilde{x}}_2''})$$
$$= t_1(w_1'\cdot t_2)u_1(x_1\cdot u_2)c_{\underline{\tilde{w}}'}^{\underline{\tilde{x}}''}.$$

Hence $tu^{-1}c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = t'u^{-1}u_1(x_1\cdot u_2)c_{\underline{\tilde{w}}'}^{\underline{\tilde{x}}''}$. This equals $t'u'^{-1}c_{\underline{\tilde{w}}'}^{\underline{\tilde{x}}'}$ by case **A**), since $uu_1^{-1}(x_1\cdot u_2)^{-1}\tilde{x}'' = u'\tilde{x}'$.

C) Assume that $\underline{w} = (s,s',s,\dots)$, $\underline{w}' = (s',s,s',\dots)$, where $w = ss's\cdots = s'ss'\cdots = w'$ is a braid relation in W^{aff}. Choose lifts $\tilde{s},\tilde{s}'\in S^{\mathrm{aff}}(1)$ of $s,s'\in S^{\mathrm{aff}}$. Then by part **B**) we may assume without loss of generality that $\underline{\tilde{w}} = (\tilde{s},\tilde{s}',\tilde{s},\dots)$, $\underline{\tilde{w}}' = (\tilde{s}',\tilde{s},\tilde{s}',\dots)$. (Use the same integers $i_1<\cdots<i_r$ for the old and the new $\underline{\tilde{w}}$, and similarly for $\underline{\tilde{w}}'$.) Then the case $r=n$ is obvious because $\underline{\tilde{w}} = \underline{\tilde{x}},\ \underline{\tilde{w}}' = \underline{\tilde{x}}',\ tu^{-1} = t'u'^{-1}$ and $c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = c_{\underline{\tilde{w}}'}^{\underline{\tilde{x}}'} = 1$, so we assume $r<n$. We prove $tu^{-1}c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = t'u'^{-1}c_{\underline{\tilde{w}}'}^{\underline{\tilde{x}}'}$.

As $r<n$ the sequence $\underline{x}' = \underline{x}$ is unique. By symmetry we suppose that the last terms of $\underline{w}$ and $\underline{x}$ are equal.

(1) We reduce to the case where $i_k = n-r+k$ and $j_k = n-1-r+k$ for all $1\le k\le r$. For $\underline{\tilde{y}} = (\tilde{s}_{n-r+1},\dots,\tilde{s}_n)$ and $\tilde{y} = \tilde{s}_{n-r+1}\cdots\tilde{s}_n$, we have $\tilde{x} = \tilde{y}$. By **A**), $c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = c_{\underline{\tilde{w}}}^{\underline{\tilde{y}}}$. As $s_{j_r}' = s_{i_r} = s_n = s_{n-1}'$, we have similarly for $\underline{\tilde{y}}' = (\tilde{s}_{n-r}',\dots,\tilde{s}_{n-1}')$, $\tilde{x}' = \tilde{y}'$ and $c_{\underline{\tilde{w}}'}^{\underline{\tilde{x}}'} = c_{\underline{\tilde{w}}'}^{\underline{\tilde{y}}'}$. We have $u'\tilde{y}' = u\tilde{y}$ and the equalities $tu^{-1}c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = t'u'^{-1}c_{\underline{\tilde{w}}'}^{\underline{\tilde{x}}'}$ and $tu^{-1}c_{\underline{\tilde{w}}}^{\underline{\tilde{y}}} = t'u'^{-1}c_{\underline{\tilde{w}}'}^{\underline{\tilde{y}}'}$ are equivalent.

(2) We assume $i_k = n-r+k$ and $j_k = n-1-r+k$ for $1\le k\le r$. Then $\underline{\tilde{x}} = \underline{\tilde{x}}'$ and $u=u'$ as $\underline{x} = \underline{x}'$. We prove $tc_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = t'c_{\underline{\tilde{w}}'}^{\underline{\tilde{x}}'}$ where $t\tilde{w} = t'\tilde{w}'$. We consider the sequence decompositions $\underline{\tilde{w}} = \underline{\tilde{w}}_1\underline{\tilde{x}},\ \underline{\tilde{w}}' = \underline{\tilde{w}}_1'\underline{\tilde{x}}'(\tilde{s}_n')$. Applying Lemma 4.17, Example 4.15, and Lemma 4.19, we have $c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = c_{\underline{\tilde{w}}_1}^{(\,)}c_{\underline{\tilde{x}}}^{\underline{\tilde{x}}} = c_{\tilde{w}_1},\ c_{\underline{\tilde{w}}'}^{\underline{\tilde{x}}} = c_{\underline{\tilde{w}}_1'}^{(\,)}c_{\underline{\tilde{x}}}^{\underline{\tilde{x}}}(x\cdot c(\tilde{s}_n')) = c_{\tilde{w}_1'}(x\cdot c(\tilde{s}_n')) = c_{\tilde{w}_1'}(w_1'x\cdot c(\tilde{s}_n'))$. We have $w_1'x\cdot c(\tilde{s}_n') = c(\tilde{w}_1'\tilde{x}\cdot\tilde{s}_n') = tt'^{-1}c(\tilde{s}_1)$ because $\tilde{w}_1'\tilde{x}\tilde{s}_n' = \tilde{w}' = tt'^{-1}\tilde{w} = tt'^{-1}\tilde{s}_1\tilde{w}_1'\tilde{x}$. Therefore $t'c_{\underline{\tilde{w}}'}^{\underline{\tilde{x}}} = tc(\tilde{s}_1)c_{\tilde{w}_1'} = tc_{\tilde{w}_1} = tc_{\underline{\tilde{w}}}^{\underline{\tilde{x}}}$.

D) To end the proof we reduce to case **A**) using **B**) and **C**). Since the change of reduced expressions in W is given by iteration of the braid relations, we may assume that there are sequence decompositions $\underline{\tilde{w}} = \underline{\tilde{w}}_1\underline{\tilde{w}}_2\underline{\tilde{w}}_3,\ \underline{w}' = \underline{\tilde{w}}_1'\underline{\tilde{w}}_2'\underline{\tilde{w}}_3'$ where $\underline{w}_2,\underline{w}_2'$ correspond to a braid relation $w_2 = w_2'$ as in **C**) and $\underline{w}_1 = \underline{w}_1', \underline{w}_3 = \underline{w}_3'$. Again by **B**) we may assume without loss of generality that $\underline{\tilde{w}}_1 = \underline{\tilde{w}}_1'$, $\underline{\tilde{w}}_3 = \underline{\tilde{w}}_3'$, and that $\underline{\tilde{w}}_2 = (\tilde{s},\tilde{s}',\tilde{s},\dots)$, $\underline{\tilde{w}}_2' = (\tilde{s}',\tilde{s},\tilde{s}',\dots)$ for some $\tilde{s},\tilde{s}'\in S^{\mathrm{aff}}(1)$. We will reduce to case **A**) by extracting a subsequence $\underline{\tilde{x}}''$ from $\underline{\tilde{w}}'$ such that $b'\tilde{x} = \tilde{x}''$ (for some $b'\in Z_k$) and $tb'^{-1}c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = t'c_{\underline{\tilde{w}}'}^{\underline{\tilde{x}}''}$.

From $t\tilde{w} = t'\tilde{w}'$ we deduce that $t = w_1\cdot a, t' = w_1\cdot a'$ for some $a,a'\in Z_k$ such that $a\tilde{w}_2 = a'\tilde{w}_2'$. We have the compatible decomposition $\underline{\tilde{x}} = \underline{\tilde{x}}_1\,\underline{\tilde{x}}_2\,\underline{\tilde{x}}_3$. Choose a subsequence $\underline{\tilde{x}}_2''$ of $\underline{\tilde{w}}_2'$ such that $b\tilde{x}_2 = \tilde{x}_2''$ (for some $b\in Z_k$), hence $(x_1\cdot b)\tilde{x} = \tilde{x}''$. Then by **C**) we have $ab^{-1}c_{\underline{\tilde{w}}_2}^{\underline{\tilde{x}}_2} = a'c_{\underline{\tilde{w}}_2'}^{\underline{\tilde{x}}_2''}$. The sequence $\underline{\tilde{x}}'' = \underline{\tilde{x}}_1\underline{\tilde{x}}_2''\underline{\tilde{x}}_3$ is a subsequence of $\underline{\tilde{w}}'$. Applying Lemmas 4.17 and 4.19:

$$c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}_1}(x_1\cdot c_{\underline{\tilde{w}}_2}^{\underline{\tilde{x}}_2})(x_1x_2\cdot c_{\underline{\tilde{w}}_3}^{\underline{\tilde{x}}_3}) = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}_1}(w_1\cdot c_{\underline{\tilde{w}}_2}^{\underline{\tilde{x}}_2})(x_1x_2\cdot c_{\underline{\tilde{w}}_3}^{\underline{\tilde{x}}_3}).$$

We deduce that $t(x_1 \cdot b)^{-1} c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}_1}(x_1 \cdot ab^{-1} c_{\underline{\tilde{w}}_2}^{\underline{\tilde{x}}_2})(x_1x_2 \cdot c_{\underline{\tilde{w}}_3}^{\underline{\tilde{x}}_3}) = c_{\underline{\tilde{w}}_1}^{\underline{\tilde{x}}_1}(x_1 \cdot a' c_{\underline{\tilde{w}}'_2}^{\underline{\tilde{x}}''_2})(x_1x''_2 \cdot c_{\underline{\tilde{w}}_3}^{\underline{\tilde{x}}_3}) = t' c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}}$. □

We denote $tu^{-1} c_{\underline{\tilde{w}}}^{\underline{\tilde{x}}} = c_{t\tilde{w}}^{u\tilde{x}}$ in Proposition 4.20. This defines $c_w^x \in \mathbb{Z}[Z_k]$ for $x, w \in W^{\mathrm{aff}}(1)$ and $x \leq w$.

When $x,\ w \in W(1)$ satisfy $x \leq w$ there exists $v \in \Omega(1)$ unique modulo Z_k such that $xv, wv \in W^{\mathrm{aff}}(1)$ with $xv \leq wv$ by definition of the Bruhat order (Definition 4.2). By Lemma 4.19 the element c_{wv}^{xv} does not depend on the choice of v and we can define $c_w^x = c_{wv}^{xv}$.

To summarize:

Definition 4.21. Let $x,\ w \in W(1)$ such that $x \leq w$. We define c_w^x as

$$c_w^x = c_{wv}^{xv} = t\, c_{\underline{wv}}^{\underline{txv}} \in \mathbb{Z}[Z_k]$$

where $v \in \Omega(1)$, $t \in Z_k$, $\underline{txv} = (s_{i_1}, \ldots, s_{i_r})$ is a subsequence of $\underline{wv} = (s_1, \ldots, s_n)$ in $S^{\mathrm{aff}}(1)$ such that $wv = s_1 \cdots s_n$ and $txv = s_{i_1} \cdots s_{i_r}$ are reduced decompositions.

Proposition 4.22. *The elements $c_w^x \in \mathbb{Z}[Z_k]$ for $x, w \in W(1), x \leq w$ satisfy the following properties:*

(i) $c_w^w = 1$.
(ii) $c_{twv}^{uxv} = tu^{-1} c_w^x$ *for* $t, u \in Z_k$, $v \in \Omega(1)$.
(iii) $c_{v\cdot w}^{v\cdot x} = v \cdot c_w^x$ *for* $v \in \Omega(1)$.
(iv) $c_w^x(x \cdot c) = c_w^x(w \cdot c)$ *for* $c \in \mathbb{Z}[Z_k]$.
(v) $c_{w_1w_2}^{x_1x_2} = c_{w_1}^{x_1}(x_1 \cdot c_{w_2}^{x_2})$ *if* $x_i, w_i \in W(1)$, $x_i \leq w_i$, $\ell(x_1x_2) = \ell(x_1) + \ell(x_2), \ell(w_1w_2) = \ell(w_1) + \ell(w_2)$.
(vi) $c_w^x = c_{wv}^{xv}$ *if* $v \in W(1)$, $\ell(xv) = \ell(x) + \ell(v), \ell(wv) = \ell(w) + \ell(v)$.
(vii) $c_w^1 = c_w$ *for* $w \in W^{\mathrm{aff}}(1)$.
(viii) $c_w^x \in c_v^x\, \mathbb{Z}[Z_k]$ *for* $x, v, w \in W(1)$ *such that* $x \leq v \leq w$.

These properties come from the definition of c_w^x and properties of the $c(s)$ $(s \in S^{\mathrm{aff}}(1))$, as well as Example 4.15 and Lemma 4.19. Items (iii)–(v) are first proved for x, w, x_i, w_i in $W^{\mathrm{aff}}(1)$ and then extended to $W(1)$. Item (vi) is a consequence of (v) and (i).

4.5. **The Iwahori-Matsumoto expansion of T_w^* modulo q.** We compute the triangular decomposition of T_w^* modulo q; with the notation of (4.5), we will prove the congruence in $\mathbb{Z}[Z_k]$: for $x, w \in W(1)$ and $x \leq w$,

$$c^*(w, x) \equiv (-1)^{\ell(w)-\ell(x)} c_w^x \mod q. \tag{4.12}$$

For $h, h' \in \mathcal{H}_{\mathbb{Z}}$, we write $h \equiv h' \mod q$ if $h - h' \in q\mathcal{H}_{\mathbb{Z}}$. An equivalent formulation of the congruence is:

Theorem 4.23. *Suppose that $\tilde{w} \in W(1)$ lifts $w \in W$. We have*

$$T_{\tilde{w}}^* \equiv \sum_{x \in W, x \leq w} (-1)^{\ell(w)-\ell(x)} k_x^* \mod q, \quad k_x^* = c_{\tilde{w}}^{\tilde{x}} T_{\tilde{x}} \text{ for any } \tilde{x} \in W(1) \text{ lifting } x.$$

Proof. We assume $w \in W^{\mathrm{aff}}$. We can reduce to this case because $c^*(wv, xv) = c^*(w, x), c_{wv}^{xv} = c_w^x$ for $x, w \in W^{\mathrm{aff}}(1), x \leq w, v \in \Omega(1)$ (Remark 4.9, Proposition 4.22).

One easily checks the theorem when $\ell(w) = 0$ or $\ell(w) = 1$. For $t \in Z_k$, $T_t^* = T_t$ and $c_t^t = 1$. For $s \in S^{\mathrm{aff}}(1)$, $T_s^* = T_s - c(s)$ and $c_s^s = 1, c_s^1 = c(s)$.

In general we prove the theorem by induction on $\ell(w)$. Assume that $\ell(w)\geq 1$ and apply the braid relation to $\tilde w=\tilde w_1\tilde s$ in $W^{\mathrm{aff}}(1)$ lifting $w=w_1 s$ with $\ell(w)=\ell(w_1)+\ell(s)=\ell(w_1)+1$. By induction $T^*_{\tilde w}=T^*_{\tilde w_1}T^*_{\tilde s}$ is congruent modulo q to

$$\sum_{x\leq w_1}(-1)^{\ell(w_1)-\ell(x)}c^{\tilde x}_{\tilde w_1}T_{\tilde x}T^*_{\tilde s}=\sum_{x\leq w_1}(-1)^{\ell(w)-\ell(x)}c^{\tilde x}_{\tilde w_1}T_{\tilde x}c(\tilde s)+\sum_{x\leq w_1}(-1)^{\ell(w_1)-\ell(x)}c^{\tilde x}_{\tilde w_1}T_{\tilde x}T_{\tilde s}.$$

The first sum on the right-hand side equals

$$S_1=\sum_{x\leq w_1}(-1)^{\ell(w)-\ell(x)}c^{\tilde x}_{\tilde w}T_{\tilde x}$$

because $T_{\tilde x}c(\tilde s)=(x\cdot c(\tilde s))T_{\tilde x}$ and $c^{\tilde x}_{\tilde w_1}(x\cdot c(\tilde s))=c^{\tilde x}_{\tilde w}$ by Proposition 4.22. To analyze the second sum S_2 on the right-hand side, as in [AHHV17, IV.9] we divide the set $\{x\in W\mid x\leq w_1\}$ into the disjoint union $X\sqcup Y\sqcup Ys$ where

$$X=\{x\in W\mid x\leq w_1, xs\not\leq w_1\},\ Y=\{x\in W\mid xs<x\leq w_1\}.$$

We examine separately the contribution of X and of $Y\sqcup Ys$. For $x\in X$ we have $x<xs$. The contribution of X in S_2 is

$$S_2(X)=\sum_{x\in X}(-1)^{\ell(w_1)-\ell(x)}c^{\tilde x}_{\tilde w_1}T_{\tilde x}T_{\tilde s}=\sum_{x\in X}(-1)^{\ell(w_1)-\ell(x)}c^{\tilde x}_{\tilde w_1}T_{\tilde x\tilde s}=\sum_{x\in Xs}(-1)^{\ell(w)-\ell(x)}c^{\tilde x\tilde s^{-1}}_{\tilde w\tilde s^{-1}}T_{\tilde x}.$$

For $x\in Xs$ we have $xs<x$ hence $c^{\tilde x\tilde s^{-1}}_{\tilde w\tilde s^{-1}}=c^{\tilde x}_{\tilde w}$ (Proposition 4.22). We have $Xs=\{x\in W\mid x\leq w, x\not\leq w_1\}$ [AHHV17, IV.9 Lemma 2]. Hence,

$$S_1+S_2(X)=\sum_{x\leq w}(-1)^{\ell(w)-\ell(x)}c^{\tilde x}_{\tilde w}T_{\tilde x}.$$

We now show that the contribution of $Y\sqcup Ys$ in S_2 lies in $q\mathcal{H}_{\mathbb{Z}}$ (hence the theorem). The contribution of $Y\sqcup Ys$ is

$$S_2(Y\sqcup Ys)=\sum_{x\in Y}(-1)^{\ell(w_1)-\ell(x)}(c^{\tilde x}_{\tilde w_1}T_{\tilde x}-c^{\tilde x\tilde s}_{\tilde w_1}T_{\tilde x\tilde s})T_{\tilde s}.$$

We have $c^{\tilde x\tilde s}_{\tilde w_1}=c^{\tilde x}_{\tilde w}=c^{\tilde x}_{\tilde w_1}(x\cdot c(\tilde s))=c^{\tilde x}_{\tilde w_1}(xs\cdot c(\tilde s))$ by Proposition 4.22 and Lemma 4.18, as $xs<x<w_1<w=w_1s$. Therefore $c^{\tilde x\tilde s}_{\tilde w_1}T_{\tilde x\tilde s}=c^{\tilde x}_{\tilde w_1}(xs\cdot c(\tilde s))T_{\tilde x\tilde s}=c^{\tilde x}_{\tilde w_1}T_{\tilde x\tilde s}c(\tilde s)$, and

$$c^{\tilde x}_{\tilde w_1}T_{\tilde x}-c^{\tilde x\tilde s}_{\tilde w_1}T_{\tilde x\tilde s}=c^{\tilde x}_{\tilde w_1}T_{\tilde x\tilde s}T_{\tilde s}-c^{\tilde x}_{\tilde w_1}T_{\tilde x\tilde s}c(\tilde s)=c^{\tilde x}_{\tilde w_1}T_{\tilde x\tilde s}(T_{\tilde s}-c(\tilde s))=c^{\tilde x}_{\tilde w_1}T_{\tilde x\tilde s}T^*_{\tilde s}.$$

As $T^*_{\tilde s}T_{\tilde s}=q(s)\tilde s^2$ and q divides $q(s)$ we have $S_2(Y\sqcup Ys)\in q\mathcal{H}_{\mathbb{Z}}$. □

4.6. **The Iwahori-Matsumoto expansion of $E_{o_J}(w)$.** Let $J\subset\Delta$ and $P_J=M_JN_J$ the corresponding parabolic subgroup of G containing B. The group $I\cap M_J$ is a pro-p-Iwahori subgroup of M_J and we can apply to M_J and $I\cap M_J$ the theory of the pro-p Iwahori Hecke algebra given in the preceding sections for G and I. We indicate with an index J the objects associated to M_J instead of G.

On the positive side: the root system Φ_J of M_J is generated by J, the Weyl group $W_{J,0}=(\mathcal{N}\cap M_J)/Z$ of M_J is generated by the s_α for $\alpha\in J$, the Iwahori Weyl group $W_J=(\mathcal{N}\cap M_J)/Z^0$ of M_J is a semidirect product $W_J=\Lambda\rtimes W_{J,0}$, the sets $\mathfrak{S}_J$ and W_J^{aff} are contained in $\mathfrak{S}$ and W^{aff}, and we have the semidirect product $W_J=W_J^{\mathrm{aff}}\rtimes\Omega_J$ where Ω_J is the normalizer of S_J^{aff} in W_J. The pro-p Iwahori Weyl group $W_J(1)=(\mathcal{N}\cap M_J)/Z(1)$ of M_J is the inverse image of W_J in $W(1)$, ${}_1W_J^{\mathrm{aff}}$ is the inverse image of W_J^{aff} in $W(1)$ and $W_J(1)={}_1W_J^{\mathrm{aff}}\,\Omega_J(1)$, where $\Omega_J(1)$ is the inverse image of Ω_J in $W(1)$. The pro-p Iwahori Hecke ring $\mathcal{H}_{J,\mathbb{Z}}$ of M_J admits the bases $(T^J_w)_{w\in W_J(1)}$, $(T^{J,*}_w)_{w\in W_J(1)}$, $(E^J_o(w))_{w\in W_J(1)}$ for spherical orientations o of

$V_{J,\mathrm{ad}}$, and the integral Bernstein basis $(E^J(w))_{w\in W_J(1)}$. We have $q^J(w) = q(w)$ for $w \in \mathfrak{S}_J$ and $c^J(w) = c(w)$ for $w \in \mathfrak{S}_J(1)$ [Vig, Thm. 2.21].

On the negative side: the set S_J^{aff} of simple reflections is not contained in S^{aff}, the length ℓ_J of W_J is not the restriction of ℓ, Ω_J is not contained in Ω, the Bruhat order $\leq_J$ of W_J^{aff} is not the restriction of the Bruhat order $\leq$ of W^{aff}, the functions $w \mapsto q_w^J : W_J \to q^{\mathbb{N}}, (w_1, w_2) \mapsto q^J_{w_1,w_2} : W_J \times W_J \to q^{\mathbb{N}}, w \mapsto c_w^J : W_J(1) \to \mathbb{Z}[Z_k]$ are not the restrictions of the functions $w \mapsto q_w, (w_1, w_2) \mapsto q_{w_1,w_2}, w \mapsto c_w$ for W and $W(1)$. The linear injective map respecting the Iwahori-Matsumoto bases

$$\iota_J : \mathcal{H}_{J,\mathbb{Z}} \to \mathcal{H}_{\mathbb{Z}} \quad T_w^J \to T_w$$

does not respect products.

Definition 4.24. An element $z \in Z$ is called *J-positive* if $\langle \alpha, v(z)\rangle \geq 0$ for all $\alpha \in \Phi^+ \setminus \Phi_J^+$. When $z \in Z$ of image $\lambda \in \Lambda$ is J-positive, $\lambda w \in W_J$ is called *J-positive* for all $w \in W_{J,0}$, and lifts of λw in $W_J(1)$ are also called J-positive.

Remark 4.25. Z^+ is the set of $z \in Z$ which are J-positive for all $J \subset \Delta$.

For $w_1, w_2 \in W_J(1), w_1 \leq_J w_2$, if w_2 is J-positive the same is true for w_1 [Abe19, Lemma 4.1].

Notation 4.26. For $w \in W(1)$ or W, let $n(w) \in \mathcal{N}$ denote an element with image w; when $w \in W$ the image of $n(w)$ in $W(1)$ is a lift $\tilde{n}(w)$ of w. In particular, when $w \in W_0 = \mathcal{N}^0/Z^0 \subset W$ we have $n(w) \in \mathcal{N}^0$. We do not require the lifts $n(w) \in \mathcal{N}^0$ for $w \in W_0$ to satisfy the relations of [AHHV17, IV.6 Proposition]. The advantage is that this allows us to check compatibilities and to avoid some silly mistakes.

We have [Vig15, Thm. 1.4]:

- The $\mathbb{Z}$-submodule of $\mathcal{H}_{J,\mathbb{Z}}$ with basis T_w^J for the J-positive elements $w \in W_J(1)$ is a subalgebra $\mathcal{H}^+_{J,\mathbb{Z}}$ of $\mathcal{H}_{J,\mathbb{Z}}$, called the *$J$-positive subalgebra*.
- $\mathcal{H}_{J,\mathbb{Z}}$ is a localization of $\mathcal{H}^+_{J,\mathbb{Z}}$.
- The restriction of ι_J to $\mathcal{H}^+_{J,\mathbb{Z}}$ respects products.
- Another basis of $\mathcal{H}^+_{J,\mathbb{Z}}$ is $T_w^{J,*}$ for the J-positive elements $w \in W_J(1)$ (by the triangular decomposition (4.5) and Remark 4.25).
- Similarly, for any spherical orientation o of $V_{J,\mathrm{ad}}$, the elements $E_o^J(w)$ for the J-positive elements $w \in W_J(1)$ form a basis of $\mathcal{H}^+_{J,\mathbb{Z}}$ (by the triangular decomposition (4.7) and Remark 4.25).

Let w_J denote the longest element of $W_{J,0}$. For $z \in Z$, the integral Bernstein elements $E^J_{o^+}(z) = E^J_{o_J^+}(z) \in \mathcal{H}_{J,\mathbb{Z}}$ associated to the orientation o_J^+ of $V_{J,\mathrm{ad}}$ of dominant Weyl chamber $\mathfrak{D}_J^+$ and $E_{o_J}(z) \in \mathcal{H}_{\mathbb{Z}}$ associated to the orientation o_J of V_{ad} of Weyl chamber $\mathfrak{D}_{o_J} = w_J(\mathfrak{D}^-)$ satisfy:

Lemma 4.27. *When $z \in Z$ is J-positive, $\iota_J(E^J_{o^+}(z)) = E_{o_J}(z)$.*

Proof. The proof follows the arguments of [Oll14, Lemma 3.8], [Abe19, Lemma 4.6], [Vig15, Prop. 2.19]. Let $z \in Z$. The element $v(z)$ lies in the image by w_J of the dominant Weyl chamber $\mathfrak{D}^+$ of V_{ad} if and only if

$$\langle \alpha, v(z)\rangle \geq 0 \ \text{ for } \alpha \in w_J(\Phi^+) = (\Phi^+ \setminus \Phi_J^+) \cup \Phi_J^-. \tag{4.13}$$

When $v(z) \in w_J(\mathfrak{D}^+) \Leftrightarrow \nu(z) = -v(z) \in w_J(\mathfrak{D}^-)$ we have $\nu_J(z) \in \mathfrak{D}_J^+$ because

$$\langle \alpha, v(z)\rangle \geq 0 \quad \text{for } \alpha \in \Phi_J^- \Leftrightarrow \langle \alpha, \nu_J(z)\rangle \geq 0 \quad \text{for } \alpha \in \Phi_J^+.$$

Thus when $v(z) \in w_J(\mathfrak{D}^+)$ the integral Bernstein elements $E^J_{o+}(z) = E^J_{o_J^+}(z) \in \mathcal{H}_{J,\mathbb{Z}}$ and $E_{o_J}(z) \in \mathcal{H}_{\mathbb{Z}}$ satisfy

$$E^J_{o+}(z) = T^J(z), \quad E_{o_J}(z) = T(z), \quad \iota_J(E^J_{o+}(z)) = E_{o_J}(z). \tag{4.14}$$

On the other hand, let $z, z_1, z_2 \in Z$ such that $z = z_1 z_2^{-1}$ and $\lambda_1, \lambda_2 \in \Lambda$ the images of z_1, z_2. For any orientation o of V_{ad} (resp. $V_{J,\mathrm{ad}}$), we have in $\mathcal{H}_{\mathbb{Z}}$ (resp. $\mathcal{H}_{J,\mathbb{Z}}$)

$$E_o(z_1)q_{\lambda_2} = q_{\lambda_1,\lambda_2^{-1}} E_o(z)E_o(z_2) \quad (\text{resp. } E^J_o(z_1)q^J_{\lambda_2} = q^J_{\lambda_1,\lambda_2^{-1}} E^J_o(z)E^J_o(z_2)). \tag{4.15}$$

This follows from the multiplication formula in §4.2 which gives in $\mathcal{H}_{\mathbb{Z}}$

$$E_o(z_1)E_o(z_2^{-1}) = q_{\lambda_1,\lambda_2^{-1}} E_o(z), \quad E_o(z_2)E_o(z_2^{-1}) = q_{\lambda_2,\lambda_2^{-1}} = q_{\lambda_2}$$

and the analogous formula in $\mathcal{H}_{J,\mathbb{Z}}$. For $z \in Z$ general, we can find z_1, z_2 as above such that $v(z_1), v(z_2)$ lie in $w_J(\mathfrak{D}^+)$. For such elements we obtain from (4.14) and (4.15) that

$$q_{\lambda_1,\lambda_2^{-1}} E_{o_J}(z)T(z_2) = q_{\lambda_2}T(z_1), \quad q^J_{\lambda_1,\lambda_2^{-1}} E^J_{o+}(z)T^J(z_2) = q^J_{\lambda_2}T^J(z_1). \tag{4.16}$$

We now suppose that $z \in Z$ is J-positive. We choose $z_1, z_2 \in Z$ such that $z = z_1 z_2^{-1}$ and $v(z_1), v(z_2) \in w_J(\mathfrak{D}^+)$, in particular z_1, z_2 are J-positive. As $E^J_{o+}(z)$ and $T^J(z_i)$ lie in $\mathcal{H}^+_{J,\mathbb{Z}}$, the algebra homomorphism $\iota_J : \mathcal{H}^+_{J,\mathbb{Z}} \to \mathcal{H}_{\mathbb{Z}}$ applied to the second formula in (4.16) gives

$$q^J_{\lambda_1,\lambda_2^{-1}} \iota_J(E^J_{o+}(z))T(z_2) = q^J_{\lambda_2}T(z_1).$$

In $\mathcal{H}_{\mathbb{Q}}$ where $T(z)$ is invertible we have, using again (4.16),

$$\iota_J(E^J_{o+}(z)) = (q^J_{\lambda_1,\lambda_2^{-1}})^{-1} q^J_{\lambda_2} T(z_1)T(z_2)^{-1} = (q^J_{\lambda_1,\lambda_2^{-1}})^{-1} q^J_{\lambda_2} q_{\lambda_1,\lambda_2^{-1}} q^{-1}_{\lambda_2} E_{o_J}(z).$$

The coefficient of $T(z)$ in the Iwahori-Matsumoto expansion of $\iota_J(E^J_{o+}(z))$ and of $E_{o_J}(z)$ being 1, we deduce $q^J_{\lambda_1,\lambda_2^{-1}}(q^J_{\lambda_2})^{-1} = q_{\lambda_1,\lambda_2^{-1}} q^{-1}_{\lambda_2}$ and $\iota_J(E^J_{o+}(z)) = E_{o_J}(z)$ in $\mathcal{H}_{\mathbb{Q}}$ hence also in $\mathcal{H}_{\mathbb{Z}}$. $\square$

Suppose $z \in Z^+$ with images $\tilde{\lambda} \in \Lambda^+(1), \lambda \in \Lambda^+$. We have $E^J_{o+}(z) = T^{J,*}(z)$ and z is J-positive hence $E_{o_J}(z) = \iota_J(T^{J,*}(z))$. By the triangular Iwahori-Matsumoto expansion of $T^{J,*}(z)$ (4.5),

$$E_{o_J}(z) = \sum_{x \in W_J, x \leq_J w} c^{J,*}(\tilde{\lambda}, \tilde{x})T(\tilde{x}). \tag{4.17}$$

(In particular, by (4.7), $c_{o_J}(\tilde{\lambda}, \tilde{x}) = c^{J,*}(\tilde{\lambda}, \tilde{x})$ for $\tilde{x} \in W_J(1)$ with $\tilde{x} \leq_J \tilde{\lambda}$.) For later use we need the value of $E_{o_{J'}}(zn(w_J w_{J'})^{-1})$ for $J' \subset J \subset \Delta$. The computation will use (4.17) and the following Lemma 4.29 (whose proof uses Lemma 4.28). Recall the surjective map $\Phi \to \Phi_a$ (2.4) respecting positive roots.

Lemma 4.28 ([Oll15, Lemma 2.9 ii]). *Let $w \in W, \lambda \in \Lambda^+$ such that $w \leq \lambda$. Then there exists $\lambda_1 \in \Lambda^+$ such that $\lambda_1 \leq \lambda$ and $w \in W_0\lambda_1 W_0$. In particular, $\nu(\lambda_1) - \nu(\lambda) \in \sum_{\alpha \in \Delta} \mathbb{Q}_{\geq 0}\alpha^\vee$.*

Proof. Since our assumptions on W are more general than in [Oll15] we give a brief sketch of the proof. We have that $w \leq w_\Delta\lambda$, the longest element of $W_0\lambda W_0$. Choose $\lambda_1 \in \Lambda^+$ such that $w \in W_0\lambda_1 W_0$. Since $w_\Delta\lambda$, $w_\Delta\lambda_1$ are the longest elements of their double cosets, the lifting property of Coxeter groups [BB05, Prop. 2.2.7] shows inductively that $w_\Delta\lambda_1 \leq w_\Delta\lambda$, so $\lambda_1 \leq w_\Delta\lambda$. By using the lifting property again we deduce that $\lambda_1 \leq \lambda$. (We repeatedly use that $\ell(w\lambda) = \ell(w) + \ell(\lambda)$ for $w \in W_0$, $\lambda \in \Lambda^+$. This is a consequence of (4.3).) $\square$

Lemma 4.29. *Let $J' \subset J \subset \Delta$ and $\lambda \in \Lambda$ such that $\langle \alpha, v(\lambda)\rangle > 0$ for all $\alpha \in J \setminus J'$.*

(i) *For $\lambda_1 \in \Lambda^+$ such that $v(\lambda) - v(\lambda_1) \in \sum_{\beta \in J'} \mathbb{Q}_{\geq 0}\beta^\vee$, we have $\langle \gamma, v(\lambda_1)\rangle > 0$ for all $\gamma \in \Phi^+_J \setminus \Phi^+_{J'}$.*

(ii) *Suppose $\lambda \in \Lambda^+$ and $x \in W_{J'}$ with $x \leq_{J'} \lambda$. Then $\ell(x) = \ell(xw_{J'}w_J) + \ell(w_{J'}w_J)$.*

Proof. (i) For $\alpha \in J \setminus J'$ and $\beta \in J'$, we have $\langle \alpha, \beta^\vee\rangle \leq 0$ hence $\langle \alpha, v(\lambda)\rangle \leq \langle \alpha, v(\lambda_1)\rangle$. Let $\gamma \in \Phi^+_J \setminus \Phi^+_{J'}$. There exists $\alpha \in J \setminus J'$ such that $\gamma - \alpha$ is a sum of roots in Φ^+. Since $\lambda_1 \in \Lambda^+$, $\langle \gamma - \alpha, v(\lambda_1)\rangle \geq 0$ hence $\langle \alpha, v(\lambda_1)\rangle \leq \langle \gamma, v(\lambda_1)\rangle$ and $\langle \alpha, v(\lambda)\rangle \leq \langle \gamma, v(\lambda_1)\rangle$. Hence $\langle \gamma, v(\lambda_1)\rangle > 0$ for $\gamma \in \Phi^+_J \setminus \Phi^+_{J'}$.

(ii) There exists $\lambda_1 \in \Lambda^{+,J'}$ such that $x \in W_{J',0}\lambda_1 W_{J',0}$ and $v(\lambda) - v(\lambda_1) \in \bigoplus_{\beta \in J'} \mathbb{Q}_{\geq 0}\beta^\vee$ (Lemma 4.28, $v = -\nu$). In particular, $0 \leq \langle \alpha, v(\lambda)\rangle \leq \langle \alpha, v(\lambda_1)\rangle$ for $\alpha \in J \setminus J'$, hence $\lambda_1 \in \Lambda^+$. We write $x = \lambda_x v_x$ with $\lambda_x = v_1 \cdot \lambda_1 \in \Lambda$ and $v_1, v_x \in W_{J',0}$.

As $\Phi^+_J \setminus \Phi^+_{J'}$ is stable by $W_{J',0}$ and $\langle \gamma, v(\lambda_1)\rangle > 0$ for $\gamma \in \Phi^+_J \setminus \Phi^+_{J'}$ by (i) we have

$$\langle \gamma, v(\lambda_x)\rangle > 0 \quad \text{for } \gamma \in \Phi^+_J \setminus \Phi^+_{J'}.$$

By the length formula (4.4), $\ell(xw_{J'}w_J) = \ell(\lambda_x v_x w_{J'}w_J)$ is equal to

$$\ell(xw_{J'}w_J) = \ell(\lambda_x) - \ell(v_x w_{J'}w_J) + 2|\{\alpha \in \Phi^+_a \cap v_x w_{J'}w_J(\Phi^-_a), \langle \alpha_a, v(\lambda_x)\rangle \leq 0\}|.$$

As $v_x \in W_{J',0}$ we have $\ell(v_x w_{J'}w_J) = \ell(w_J) - \ell(v_x w_{J'}) = \ell(w_J) - \ell(w_{J'}) + \ell(v_x) = \ell(v_x) + \ell(w_{J'}w_J)$. Hence $\ell(\lambda_x) - \ell(v_x w_{J'}w_J) = \ell(\lambda_x) - \ell(v_x) - \ell(w_{J'}w_J)$. We have $\Phi^+_a \cap v_x w_{J'}w_J(\Phi^-_a) = \Phi^+_a \cap [(\Phi^-_a \setminus \Phi^-_{a,J}) \cup (\Phi^+_{a,J} \setminus \Phi^+_{a,J'}) \cup v_x(\Phi^-_{a,J'})] = (\Phi^+_{a,J} \setminus \Phi^+_{a,J'}) \cup (\Phi^+_a \cap v_x(\Phi^-_a))$, and $\langle \alpha_a, v(\lambda_x)\rangle > 0$ for $\alpha_a \in \Phi^+_{a,J} \setminus \Phi^+_{a,J'}$. Hence
$\ell(xw_{J'}w_J) + \ell(w_{J'}w_J) = \ell(\lambda_x) - \ell(v_x) + 2|\{\alpha_a \in \Phi^+_a \cap v_x(\Phi^-_a), \langle \alpha_a, v(\lambda_x)\rangle \leq 0\}| = \ell(x)$. $\square$

Proposition 4.30. *For $J' \subset J \subset \Delta$ and $z \in Z^+$ of image $\tilde\lambda \in \Lambda^+(1)$ and $\lambda \in \Lambda^+$ such that $\langle \alpha, v(\lambda)\rangle > 0$ for all $\alpha \in J \setminus J'$,*

$$E_{o_{J'}}(zn(w_J w_{J'})^{-1}) = \sum_{x \in W_{J'}, x \leq_{J'} \lambda} c^{J',*}(\tilde\lambda, \tilde x) T(\tilde x n(w_J w_{J'})^{-1})$$

for any lifts $\tilde x \in W_{J'}(1)$ of $x \in W_{J'}$.

Proof. We have $\ell(\lambda) = \ell(\lambda w_{J'}w_J) + \ell(w_{J'}w_J)$ by Lemma 4.29, and the multiplication formula in §4.2 gives

$$E_{o_{J'}}(z) = E_{o_{J'}}(zn(w_J w_{J'})^{-1}) E_{o_{J'} \cdot w_{J'}w_J}(n(w_J w_{J'})).$$

The orientation $o_{J'} \cdot w_{J'}w_J$ of Weyl chamber $w_J w_{J'}(\mathcal{D}_{o_{J'}}) = w_J(\mathcal{D}^-) = \mathcal{D}_{o_J}$ is o_J and $E_{o_J}(n(w_J w_{J'})) = T(n(w_J w_{J'}))$ [Vig16, Example 5.32], so

$$E_{o_{J'}}(z) = E_{o_{J'}}(zn(w_J w_{J'})^{-1}) T(n(w_J w_{J'})).$$

Applying (4.17) and Lemma 4.29

$$E_{o_{J'}}(zn(w_J w_{J'})^{-1}) T(n(w_J w_{J'})) = \sum_{x \in W_{J'}, x \leq_{J'} \lambda} c^{J',*}(\tilde\lambda, \tilde x) T(\tilde x n(w_J w_{J'})^{-1}) T(n(w_J w_{J'})).$$

In $\mathcal{H}_{\mathbb{Q}}$, the basis element $T(n(w_J w_{J'}))$ is invertible and we deduce

$$E_{o_{J'}}(zn(w_J w_{J'})^{-1}) = \sum_{x \in W_{J'}, x \leq_{J'} \lambda} c^{J',*}(\tilde{\lambda}, \tilde{x}) T(\tilde{x} n(w_J w_{J'})^{-1}).$$

This remains true in $\mathcal{H}_{\mathbb{Z}}$. □

Remark 4.31. Comparing with (4.5), (4.7), Proposition 4.30 implies

$$c_{o_{J'}}(\tilde{\lambda} n(w_J w_{J'})^{-1}, \tilde{x} n(w_J w_{J'})^{-1}) = c^{J',*}(\tilde{\lambda}, \tilde{x})$$

for $J' \subset J \subset \Delta$ and $\tilde{\lambda}, \tilde{x} \in W(1)$ lifting $\lambda \in \Lambda^+, x \in W_{J'}, x \leq_{J'} \lambda$.

4.7. $\psi(c(s))$ **for a simple affine reflection.** Let $\psi : Z^0 \to C^\times$ be a character. It is trivial on $Z^0 \cap M'_{\Delta'_\psi}$ (Definition 2.1) by the following lemma.

Lemma 4.32. *For $J \subset \Delta$, the group $Z^0 \cap M'_J$ is generated by $Z^0 \cap M'_\alpha$ for $\alpha \in J$.*

Proof. Let $\langle \cup_{\alpha \in J} Z^0 \cap M'_\alpha \rangle$ denote the group generated by the $Z^0 \cap M'_\alpha$ for $\alpha \in J$. This group is contained in $Z^0 \cap M'_J$ and $Z^0 \cap M'_J$ is contained in the kernel of ν. The group $Z \cap M'_J$ is generated by $Z \cap M'_\alpha$ for $\alpha \in J$ [AHHV17, II.6 Prop.] and the group $Z \cap M'_\alpha$ is generated by $Z^0 \cap M'_\alpha$ and a_α (Definition 2.1) [AHHV17, §III.16]. The group Z normalizes M'_α and Z^0 hence

$$Z \cap M'_J = \langle \cup_{\alpha \in J} Z^0 \cap M'_\alpha \rangle \prod_{\alpha \in J} a_\alpha^{\mathbb{Z}}.$$

The group Z^0 is contained in the kernel of ν and $\nu(a_\alpha) = \alpha_a^\vee$. The $\alpha_a^\vee$ for $\alpha \in J$ are linearly independent, hence an identity $\sum_{\alpha \in J} n(\alpha)\alpha_a^\vee = 0$ with $n(\alpha) \in \mathbb{Z}$ implies $n(\alpha) = 0$ for all $\alpha \in J$. We get $Z \cap M'_J \cap \operatorname{Ker} \nu = \langle \cup_{\alpha \in J} Z^0 \cap M'_\alpha \rangle$, hence $Z^0 \cap M'_J$ is contained in $\langle \cup_{\alpha \in J} Z^0 \cap M'_\alpha \rangle$. □

As in §2.1, $\overline{Z^0 \cap M'_J}$ denotes the image of $Z^0 \cap M'_J$ in Z_k^{aff}.

Remark 4.33. For $\alpha \in \Delta$, the group $\overline{Z^0 \cap M'_\alpha}$ is different from the group Z_{k,s_α} defined in Remark 4.6. The group $\overline{Z^0 \cap M'_\alpha}$ is generated by Z_{k,s_α} and another group $Z_{k,s_{\alpha_a - 1}}$ such that for an admissible lift $\tilde{s}_{\alpha_a - 1}$ of $s_{\alpha_a - 1}$ the value $c(\tilde{s}_{\alpha_a - 1}) \in \mathcal{H}_C$ is given by a formula like (4.6) for $c(\tilde{s}_\alpha)$ with $Z_{k,s_{\alpha_a - 1}}$ instead of Z_{k,s_α} [AHHV17, IV.24 Claim, IV.25–28]. The group $\overline{Z^0 \cap M'_\alpha}$ is also generated by Z_{k,s_α} and $s_\alpha(Z_{k,s_{\alpha_a - 1}})$ because $\overline{Z^0 \cap M'_\alpha}$ and Z_{k,s_α} are normalized by s_α. The set Δ'_ψ (Definition 2.1) is therefore contained in the set

$$\Delta(\psi) := \{\alpha \in \Delta \mid \psi \text{ is trivial on } Z_{k,s_\alpha}\}. \tag{4.18}$$

Lemma 4.34.

(i) *Let $J \subset \Delta$ and $\tilde{\tau} \in {}_1\mathfrak{S}_J$. Then $c(\tilde{\tau}) \in \mathbb{Z}[\,\overline{Z^0 \cap M'_J}\,]$. When $J \subset \Delta'_\psi$, we have $\psi(c(\tilde{\tau})) = -1$.*

(ii) *Let $\alpha \in \Delta \setminus \Delta'_\psi$. Then $\psi(c(\tilde{s}_\alpha)\, c(\tilde{s}_{\alpha_a - 1})) = \psi(c(\tilde{s}_\alpha)\, (s_\alpha \cdot c(\tilde{s}_{\alpha_a - 1}))) = 0$.*

Proof. (i) This follows from Remark 4.6 applied to the Levi subgroup M_J of G. (Recall that $c^J(w) = c(w)$.)

(ii) By hypothesis ψ is not trivial on the image of $Z^0 \cap M'_\alpha$ in Z_k^{aff}, hence if ψ is trivial on Z_{k,s_α}, then ψ is not trivial on $Z_{k,s_{\alpha_a - 1}}$ and on $s_\alpha(Z_{k,s_{\alpha_a - 1}})$. By formula (4.6) and Remark 4.33, $\psi(c(\tilde{s}_\alpha)) = 0$ (resp. $\psi(c(\tilde{s}_{\alpha_a - 1})) = 0$, resp. $\psi(s_\alpha \cdot c(\tilde{s}_{\alpha_a - 1})) = 0$) if and only if ψ is not trivial on Z_{k,s_α} (resp. $Z_{k,s_{\alpha_a - 1}}$, resp. $s_\alpha(Z_{k,s_{\alpha_a - 1}})$). □

4.8. $\psi(c_w^x)$ **for dominant translations.** Let $\psi : Z^0 \to C^\times$ be a character and $\tilde{x}, \tilde{w} \in W(1)$ lifting $x, w \in \Lambda^+$ such that $\tilde{x} \leq \tilde{w}$. To compute $\psi(c_{\tilde{w}}^{\tilde{x}})$ we need some knowledge of the reduced expressions of the elements of Λ^+. This is obtained in the following lemmas.

Lemma 4.35. *Let $\alpha \in \Delta$, $\lambda \in \Lambda^+$ such that $\lambda_\alpha \lambda \in \Lambda^+$ and let $\lambda = s_1 \cdots s_n u$ with $s_i \in S^{\mathrm{aff}}, u \in \Omega$ be a reduced expression. Then there exist $k_1 < k_2$ such that*

- *$\lambda_\alpha \lambda = s_1 \cdots s_{k_1-1} s_{k_1+1} \cdots s_{k_2-1} s_{k_2+1} \cdots s_n u$ is a reduced expression, and*
- *$\{(s_1 \cdots s_{k_1-1}) \cdot s_{k_1},\ (s_1 \cdots s_{k_1-1} s_{k_1+1} \cdots s_{k_2-1}) \cdot s_{k_2}\} = \{s_\alpha, s_\alpha \lambda_\alpha\}$ or $\{s_\alpha, \lambda_\alpha s_\alpha\}$.*

Proof. As in Lemma 4.4 we have
$$\lambda_\alpha \lambda < s_\alpha \lambda_\alpha \lambda < \lambda$$
because $\ell(s_\alpha \lambda_\alpha \lambda) = \ell(\lambda^{-1} \lambda_\alpha^{-1} s_\alpha) = \ell(\lambda_\alpha \lambda) + 1 = \ell(\lambda) - 1$ (using (4.4)), and we have $s_\alpha \lambda_\alpha = s_{\alpha_a+1} \in \mathfrak{S}$. By the strong exchange condition there exists i such that $s_\alpha \lambda_\alpha s_1 \cdots s_i = s_1 \cdots s_{i-1}$ and there exists j such that either of the following hold:

(1) $j < i$, $s_\alpha s_1 \cdots s_j = s_1 \cdots s_{j-1}$: hence $(s_1 \cdots s_{j-1}) \cdot s_j = s_\alpha$ and $(s_1 \cdots s_{j-1} s_{j+1} \cdots s_{i-1}) \cdot s_i = (s_\alpha s_1 \cdots s_{i-1}) \cdot s_i = s_\alpha \cdot s_\alpha \lambda_\alpha = \lambda_\alpha s_\alpha$; we take $k_1 = j, k_2 = i$.

(2) $j > i$, $s_\alpha s_1 \cdots s_{i-1} s_{i+1} \cdots s_j = s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1}$: hence $(s_1 \cdots s_{i-1}) \cdot s_i = s_\alpha \lambda_\alpha$ and $(s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1}) \cdot s_j = s_\alpha$; we take $k_1 = i, k_2 = j$. ☐

Remark 4.36. We will apply Lemma 4.35 as follows. For a choice of lifts in $W(1)$, we have $c_{\tilde{\lambda}}^{\tilde{\lambda}\tilde{\lambda}_\alpha} = t(s_1 \cdots s_{k_1-1} \cdot c(\tilde{s}_{k_1}))\,(s_1 \cdots s_{k_1-1} s_{k_1+1} \cdots s_{k_2-1} \cdot c(\tilde{s}_{k_2}))$ for some $t \in Z_k$, by definition of c_w^x. Hence, as $\lambda_\alpha s_\alpha = s_{\alpha_a-1}, s_\alpha \lambda_\alpha = s_\alpha s_{\alpha_a-1} s_\alpha$, we have
$$c_{\tilde{\lambda}}^{\tilde{\lambda}\tilde{\lambda}_\alpha} \in c(\tilde{s}_\alpha)\,(s_\alpha \cdot c(\tilde{s}_{\alpha_a-1}))\mathbb{Z}[Z_k] \text{ or } c(\tilde{s}_\alpha) c(\tilde{s}_{\alpha_a-1})\mathbb{Z}[Z_k].$$

By iteration of the lemma, we get:

Lemma 4.37. *Let $\lambda \in \Lambda^+, J \subset \Delta, n(\alpha) \in \mathbb{N}$ for $\alpha \in J$ such that $\lambda \prod_{\alpha \in J} \lambda_\alpha^{m(\alpha)} \in \Lambda^+$ for all $m(\alpha) \in \mathbb{N}, m(\alpha) \leq n(\alpha)$, and let $\lambda = s_1 \cdots s_n u$ with $s_i \in S^{\mathrm{aff}}, u \in \Omega$ be a reduced expression. Then there exist $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ such that*

- *$\lambda \prod_{\alpha \in \Delta} \lambda_\alpha^{n(\alpha)} = s_{i_1} \cdots s_{i_r} u$ is a reduced expression, and*
- *$(s_{i_1} \cdots s_{i_j}) \cdot s_k$ lies in $W_J^{\mathrm{aff}} \subset W^{\mathrm{aff}}$ for any $0 \leq j \leq r$ and $i_j < k < i_{j+1}$.*

Here we let $i_0 = 0$, $i_{r+1} = n + 1$.

Proof. We proceed by induction on $\sum_{\beta \in J} n(\beta)$. Let $\alpha \in J$ such that $n(\alpha) > 0$. Then $\lambda_1 = \lambda \prod_{\beta \in J} \lambda_\beta^{n(\beta)} = \lambda_2 \lambda_\alpha$ and $\lambda_2 \in \Lambda^+$. By the inductive hypothesis, there exist $i_1 < i_2 < \cdots < i_r$ such that $\lambda_2 = s_{i_1} \cdots s_{i_r} u$ is a reduced expression and $(s_{i_1} \cdots s_{i_j}) \cdot s_k$ lies in W_J^{aff} for any $0 \leq j \leq r$ and $i_j < k < i_{j+1}$. From Lemma 4.35 there exist $a < b$ such that $\lambda_1 = s_{i_1} \cdots s_{i_{a-1}} s_{i_{a+1}} \cdots s_{i_{b-1}} s_{i_{b+1}} \cdots s_{i_r} u$ is a reduced expression and $\tau_1 = (s_{i_1} \cdots s_{i_{a-1}}) \cdot s_{i_a},\ \tau_2 = (s_{i_1} \cdots s_{i_{a-1}} s_{i_{a+1}} \cdots s_{i_{b-1}}) \cdot s_{i_b}$ are in W_J^{aff}. We prove that $(i'_1, \dots, i'_{r-2}) = (i_1, \dots, i_{a-1}, i_{a+1}, \dots, i_{b-1}, i_{b+1}, \dots, i_r)$ satisfies the conditions of the lemma. Take $0 \leq j \leq r - 2$ and $i'_j < k < i'_{j+1}$. Then $(s_{i'_1} \cdots s_{i'_j}) \cdot s_k$ lies in W_J^{aff}. Indeed, if $k = i_a$ or i_b this is the condition on a and b. Otherwise, take j' such that $i_{j'} < k < i_{j'+1}$. Then
$$(s_{i'_1} \cdots s_{i'_j}) \cdot s_k = \begin{cases} (s_{i_1} \cdots s_{i_{j'}}) \cdot s_k & \text{if } j' < a \text{ (hence } j = j'), \\ (\tau_1 s_{i_1} \cdots s_{i_{j'}}) \cdot s_k & \text{if } a \leq j' < b \text{ (hence } j = j' - 1), \\ (\tau_2 \tau_1 s_{i_1} \cdots s_{i_{j'}}) \cdot s_k & \text{if } b \leq j' \text{ (hence } j = j' - 2). \end{cases}$$
In any case, this is in W_J^{aff} by the inductive hypothesis and because τ_1, τ_2 are in W_J^{aff}. ☐

Remark 4.38. We will apply Lemma 4.37 as follows. Keep the notation of the lemma, so $i_j < k < i_{j+1}$. Let $\alpha_k \in \Phi$ be a reduced root such that s_k is the reflection in an affine hyperplane of the form $\alpha_k + r = 0$ ($r \in \mathbb{R}$). We have $s_{i_1} \cdots s_{i_j}(\alpha_k) \in \Phi_J$, where $\Phi_J \subset \Phi$ denotes the root subsystem generated by J. Choose lifts $\tilde{s}_{i_1}, \ldots, \tilde{s}_{i_j}, \tilde{s}_k \in {}_1W^{\mathrm{aff}}$ of $s_{i_1}, \ldots, s_{i_j}, s_k$ with $\tilde{s}_k$ admissible. Writing $M'_\beta = \langle U_\beta, U_{-\beta}\rangle$ for any reduced root $\beta \in \Phi$, we have that $\tilde{s}_k$ lies in the image of $\mathcal{N} \cap M'_{\alpha_k}$ in $W(1)$. It follows that $\tilde{s}_{i_1} \cdots \tilde{s}_{i_j} \cdot \tilde{s}_k$ lies in the image of $\mathcal{N} \cap M'_{s_{i_1}\cdots s_{i_j}(\alpha_k)}$ in $W(1)$, so $\tilde{s}_{i_1} \cdots \tilde{s}_{i_j} \cdot \tilde{s}_k \in {}_1W^{\mathrm{aff}}_J \cap \mathfrak{S}(1) = {}_1\mathfrak{S}_J$. Hence by Lemma 4.34 we see that $s_{i_1} \cdots s_{i_j} \cdot c(\tilde{s}_k) = c(\tilde{s}_{i_1} \cdots \tilde{s}_{i_j} \cdot \tilde{s}_k)$ lies in $\mathbb{Z}[\,\overline{Z^0 \cap M'_J}\,]$. Therefore $\psi(s_{i_1} \cdots s_{i_j} \cdot c(\tilde{s}_k)) = -1$ if ψ is trivial on $Z^0 \cap M'_J$.

We are now ready to compute $\psi(c^{\tilde{x}}_{\tilde{w}})$ when $\tilde{x}, \tilde{w}$ are elements of the inverse image $\Lambda^+(1)$ of Λ^+ in $W(1)$.

Theorem 4.39. *Let $\tilde{x}, \tilde{w} \in \Lambda^+(1)$ lifting $x, w \in \Lambda^+$ such that $x \le w$. Then*

$$\psi(c^{\tilde{x}}_{\tilde{w}}) = \begin{cases} (-1)^{\ell(w)-\ell(x)} & \text{if } \tilde{x} \in \tilde{w} \prod_{\alpha \in \Delta'_\psi} a_\alpha^{\mathbb{N}}, \\ 0 & \text{if } x \notin w \prod_{\alpha \in \Delta'_\psi} \lambda_\alpha^{\mathbb{N}}. \end{cases}$$

Proof. We have $x = w \prod_{\alpha \in \Delta} \lambda_\alpha^{n(\alpha)}$ with $n(\alpha) \in \mathbb{N}$ (Proposition 4.3). For $\tilde{\lambda} \in \Lambda^+(1)$, $c^{\tilde{x}\tilde{\lambda}}_{\tilde{w}\tilde{\lambda}} = c^{\tilde{x}}_{\tilde{w}}$ (Proposition 4.22), so by Lemma 3.5 we may assume without loss of generality that $w \prod_{\alpha \in \Delta} \lambda_\alpha^{m(\alpha)} \in \Lambda^+$ for any $0 \le m(\alpha) \le n(\alpha)$.

Assume $n(\alpha) > 0$ for some $\alpha \in \Delta \setminus \Delta'_\psi$. Let $\tilde{w}' = \tilde{x}\tilde{\lambda}_\alpha^{-1}$ for some lift $\tilde{\lambda}_\alpha$ of λ_α, so $\tilde{x} = \tilde{w}'\tilde{\lambda}_\alpha \le \tilde{w}' \le \tilde{w}$. Then $c^{\tilde{x}}_{\tilde{w}} \in c^{\tilde{w}'\tilde{\lambda}_\alpha}_{\tilde{w}'}\mathbb{Z}[Z_k]$ by Proposition 4.22, so $c^{\tilde{x}}_{\tilde{w}} \in c(\tilde{s}_\alpha)\,(s_\alpha \cdot c(\tilde{s}_{\alpha_a-1}))\mathbb{Z}[Z_k]$ or $c(\tilde{s}_\alpha)c(\tilde{s}_{\alpha_a-1})\mathbb{Z}[Z_k]$ by Remark 4.36. Therefore $\psi(c^{\tilde{x}}_{\tilde{w}}) = 0$ by Lemma 4.34.

Assume now $n(\alpha) = 0$ for all $\alpha \in \Delta \setminus \Delta'_\psi$ and that $\tilde{x} \in \tilde{w} \prod_{\alpha \in \Delta'_\psi} a_\alpha^{n(\alpha)}$. Take a reduced expression $\tilde{w} = \tilde{s}_1 \cdots \tilde{s}_n \tilde{u}$ where $\tilde{s}_1, \ldots, \tilde{s}_n \in {}_1S^{\mathrm{aff}}$ are admissible and $\tilde{u} \in \Omega(1)$. Let $J = \Delta'_\psi$. By Lemma 4.37 and Remark 4.38, there exist $i_1 < i_2 < \cdots < i_r$ such that

- $x = w \prod_{\alpha \in \Delta} \lambda_\alpha^{n(\alpha)} = s_{i_1} \cdots s_{i_r} u$ is a reduced expression,
- $\tilde{s}_{i_1} \cdots \tilde{s}_{i_j} \cdot \tilde{s}_k \in {}_1\mathfrak{S}_J$ for any $0 \le j \le r$ and $i_j < k < i_{j+1}$, and
- $s_{i_1} \cdots s_{i_j} \cdot c(\tilde{s}_k) = c(\tilde{s}_{i_1} \cdots \tilde{s}_{i_j} \cdot \tilde{s}_k) \in \mathbb{Z}[\,\overline{Z^0 \cap M'_J}\,]$ and $\psi(s_{i_1} \cdots s_{i_j} \cdot c(\tilde{s}_k)) = -1$ for any $0 \le j \le r$ and $i_j < k < i_{j+1}$.

We have $\tilde{x} = t\tilde{s}_{i_1} \cdots \tilde{s}_{i_r} u$ for some $t \in Z_k$. Taking the product of all $\tilde{s}_{i_1} \cdots \tilde{s}_{i_j} \cdot \tilde{s}_k \in {}_1\mathfrak{S}_J$ we deduce that $(\tilde{w}u^{-1})(t^{-1}\tilde{x}u^{-1})^{-1} = \tilde{w}\tilde{x}^{-1}t \in {}_1W^{\mathrm{aff}}_J$. Since $\tilde{x}^{-1}\tilde{w} = \prod_{\alpha \in J} a_\alpha^{-n(\alpha)} \in {}_1W^{\mathrm{aff}}_J$, it follows by normality that $\tilde{w}\tilde{x}^{-1} \in {}_1W^{\mathrm{aff}}_J$. Thus $t \in Z_k \cap {}_1W^{\mathrm{aff}}_J = Z_k^{\mathrm{aff},J}$, so $\psi(t) = 1$. Therefore, from the definition of $c^{\tilde{x}}_{\tilde{w}}$ we get that $\psi(c^{\tilde{x}}_{\tilde{w}}) = (-1)^{n-r}$. ☐

5. Inverse Satake theorem when $\Delta(V') \subset \Delta(V)$

5.1. **Value of φ_z on a generator.** Let V, V' be two irreducible representations of K with parameters $(\psi_V, \Delta(V)), (\psi_{V'}, \Delta(V'))$ such that $\Delta(V') \subset \Delta(V)$, let $\iota^{\mathrm{op}} : V^{U^0_{\mathrm{op}}} \xrightarrow{\sim} V'^{U^0_{\mathrm{op}}}, \iota : V_{U^0} \xrightarrow{\sim} V'_{U^0}$ be compatible linear isomorphisms (2.8), and let (2.10)

$$z \in Z^+_G(V, V') = \{z \in Z^+ \mid z \cdot \psi_V = \psi_{V'}, \langle \alpha, v(z)\rangle > 0 \text{ for all } \ \alpha \in \Delta(V) \setminus \Delta(V')\}.$$

The Satake transform $S^G : \mathcal{H}_G(V, V') \to \mathcal{H}_Z(V_{U^0}, V'_{U^0})$ is injective (cf. Definition 2.11). After showing that $\tau_z^{V_{U^0}, V'_{U^0}, \iota}$ belongs to the image of S^G we will compute the value of the unique antecedent φ_z on a generator of the representation $\text{c-Ind}_K^G V$ of G (Proposition 5.1). As a generator we take the function $f_v \in \text{c-Ind}_K^G V$ of support K and value at 1 a non-zero element $v \in V^{U^0_{\text{op}}}$. This generator f_v is fixed by the pro-p Iwahori group $I = K(1)U^0_{\text{op}}$ and its image by a G-intertwiner $\text{c-Ind}_K^G V \to \text{c-Ind}_K^G V'$ is also fixed by I. The space $(\text{c-Ind}_K^G V')^I$ of I-invariants of $\text{c-Ind}_K^G V'$ is a right module for the pro-p Iwahori Hecke C-algebra $\mathcal{H}_C$. We will show that $\varphi_z(f_v) = f_{v'} h_z$ where $f_{v'} \in \text{c-Ind}_K^G V'$ has support K and value $v' = \iota^{\text{op}}(v)$ at 1, and $h_z \in \mathcal{H}_C$; then, we will describe h_z using the elements T^*_w and $E_{o_{\Delta(V')}}(w)$ of $\mathcal{H}_C$ for $w \in W(1)$.

Proposition 5.1. *Suppose $z \in Z_G^+(V,V')$. There exists $\varphi_z \in \mathcal{H}_G(V,V')$ such that $S^G(\varphi_z) = \tau_z^{V_{U^0}, V'_{U^0}, \iota}$. The value of φ_z on f_v is $f_{v'} h_z$ where*

$$h_z = E_{o_{\Delta(V')}}(zn(w_{\Delta(V)} w_{\Delta(V')})^{-1}) T^*(n(w_{\Delta(V)} w_{\Delta(V')})).$$

Note that $E_{o_J}(zn(w)^{-1})T^*(n(w))$ does not depend on the choice of the lift $n(w) \in \mathcal{N}$ of $w \in W$ because another choice differs only by multiplication by $t \in Z^0$ and for $n, n' \in \mathcal{N}$, $E_{o_J}(nt^{-1})T^*(tn') = E_{o_J}(n)T(t^{-1})T(t)T^*(n') = E_{o_J}(n)T^*(n')$.

5.2. **Embedding in $\mathfrak{X} = \text{Ind}_B^G(\text{c-Ind}_{Z(1)}^Z 1_C)$.** Proposition 5.1 is essentially the same as Theorem [AHHV17, IV.19 Thm.] which implies the easier part of the change of weight theorem [AHHV17, IV.I Thm. (i)]. (See the end of §5.2 for an explanation why it is essentially the same.) The first step of the proof is to embed the two representations $\text{c-Ind}_K^G V$ and $\text{c-Ind}_K^G V'$ of G in the same representation

$$\mathfrak{X} = \text{Ind}_B^G(\text{c-Ind}_{Z(1)}^Z 1_C).$$

For a C-character ψ of Z^0 let $e_\psi \in \text{c-Ind}_{Z(1)}^Z 1_C$ denote the function of support Z^0 and equal to ψ on Z^0. For $v \in V^{U^0_{\text{op}}} \setminus \{0\}$ of image $\overline{v} \in V_{U^0}$, let $f_v \in \text{c-Ind}_K^G V$ (resp. $e_{\overline{v}} \in \text{c-Ind}_{Z^0}^Z V_{U^0}$) denote the function of support K with $f_v(1) = v$ (resp. of support Z^0 with $e_{\overline{v}}(1) = \overline{v}$). We recall the injective intertwiner [HV12, Def. 2.1]

$$I_V : \text{c-Ind}_K^G V \hookrightarrow \text{Ind}_B^G(\text{c-Ind}_{Z^0}^Z V_{U^0})$$

such that $I_V(f_v)(1) = e_{\overline{v}}$. We have the injective Z-intertwiner

$$j_{\overline{v}} : \text{c-Ind}_{Z^0}^Z V_{U^0} \hookrightarrow \text{c-Ind}_{Z(1)}^Z 1_C$$

sending $e_{\overline{v}}$ to e_{ψ_V}.

Definition 5.2. For $v \in V^{U^0_{\text{op}}} \setminus \{0\}$, let $I_v : \text{c-Ind}_K^G V \hookrightarrow \mathfrak{X}$ be the injective G-equivariant map such that $I_v(f_v)(1) = e_{\psi_V}$.

The intertwiner I_v is the composite of I_V and the injective G-intertwiner

$$\text{Ind}_B^G(j_{\overline{v}}) : \text{Ind}_B^G(\text{c-Ind}_{Z^0}^Z V_{U^0}) \hookrightarrow \mathfrak{X}$$

induced by $j_{\overline{v}}$. For $\varphi \in \mathcal{H}_G(V,V')$, the diagram

$$\begin{tikzcd}
\text{c-Ind}_K^G V \arrow[r, "I_V"] \arrow[d, "\varphi"'] & \text{Ind}_B^G(\text{c-Ind}_{Z^0}^Z V_{U^0}) \arrow[d, "S^G(\varphi)"] \\
\text{c-Ind}_K^G V' \arrow[r, "I_{V'}"'] & \text{Ind}_B^G(\text{c-Ind}_{Z^0}^Z V'_{U^0})
\end{tikzcd}$$

is commutative [HV12, §2]. For $z \in Z$, let $\tau(z)$ be the characteristic function of $zZ(1)$ seen as a Z-intertwiner $\text{c-Ind}_{Z(1)}^{Z} 1_C \to \text{c-Ind}_{Z(1)}^{Z} 1_C$. This makes $\text{c-Ind}_{Z(1)}^{Z} 1_C$ into a left $C[Z/Z(1)]$-module. Let $\overline{v}' = \iota(\overline{v})$. The diagram

$$\begin{tikzcd}
\text{c-Ind}_{Z^0}^{Z} V_{U^0} \arrow[r, "j_{\overline{v}}"] \arrow[d, "\tau_z^{V_{U^0}, V'_{U^0}, \iota}"'] & \text{c-Ind}_{Z(1)}^{Z} 1_C \arrow[d, "\tau(z)"] \\
\text{c-Ind}_{Z^0}^{Z} V'_{U^0} \arrow[r, "j_{\overline{v}'}"'] & \text{c-Ind}_{Z(1)}^{Z} 1_C
\end{tikzcd}$$

is commutative. By functoriality, the diagram

$$\begin{tikzcd}
\operatorname{Ind}_B^G(\text{c-Ind}_{Z^0}^{Z} V_{U^0}) \arrow[r, "\operatorname{Ind}_B^G(j_{\overline{v}})"] \arrow[d, "\tau_z^{V_{U^0}, V'_{U^0}, \iota}"'] & \mathfrak{X} \arrow[d, "\tau(z)"] \\
\operatorname{Ind}_B^G(\text{c-Ind}_{Z^0}^{Z} V'_{U^0}) \arrow[r, "\operatorname{Ind}_B^G(j_{\overline{v}'})"'] & \mathfrak{X}
\end{tikzcd}$$

is also commutative.

Proposition 5.3. *Suppose $z \in Z_G^+(V, V')$. In the $(C[Z/Z(1)], \mathcal{H}_C)$-bimodule $\mathfrak{X}^I$ we have*

$$\tau(z) I_v(f_v) = I_{v'}(f_{v'}) h_z, \;\; h_z = E_{o_{\Delta(V')}}(zn(w_{\Delta(V)} w_{\Delta(V')})^{-1}) T^*(n(w_{\Delta(V)} w_{\Delta(V')})).$$

This proposition implies Proposition 5.1, as we now explain: we see in particular that $\tau(z) I_v(f_v) \in I_{v'}(\text{c-Ind}_K^G V')$, so $\tau(z) I_v(\text{c-Ind}_K^G V) \in I_{v'}(\text{c-Ind}_K^G V')$. Thus there exists a unique $\varphi_z \in \mathcal{H}_G(V, V')$ such that the following diagram commutes:

$$\begin{tikzcd}
\text{c-Ind}_K^G V \arrow[r, "I_v"] \arrow[d, dashed, "\varphi_z"'] & \mathfrak{X} \arrow[d, "\tau(z)"] \\
\text{c-Ind}_K^G V' \arrow[r, "I_{v'}"'] & \mathfrak{X}.
\end{tikzcd}$$

By the above discussion and injectivity of $\operatorname{Ind}_B^G(j_{\overline{v}'})$ we deduce that $\tau_z^{V_{U^0}, V'_{U^0}, \iota} \circ I_V = I_{V'} \circ \varphi_z$. We also have $S^G(\varphi_z) \circ I_V = I_{V'} \circ \varphi_z$. From the discussion of [HV12, §2] it follows that $S^G(\varphi_z) = \tau_z^{V_{U^0}, V'_{U^0}, \iota}$ (both correspond to the map $I_{V'} \circ \varphi_z$ under the adjunction [HV12, (2)], where we take $P = B$ and $W = \text{c-Ind}_{Z^0}^{Z} V'_{U^0}$).

Proposition 5.3 is a variant of [AHHV17, IV.19 Theorem]. In loc. cit. one assumes $\psi_V = \psi_{V'} = \psi$, $\Delta(V) = \Delta(V') \sqcup \{\alpha\}$ and the representation $\mathfrak{X}$ of G is replaced by $\mathfrak{X}_\psi = \operatorname{Ind}_B^G(\text{c-Ind}_{Z^0}^{Z} \psi)$. Identifying $V_{U^0} \simeq \psi_V$, $V'_{U^0} \simeq \psi_{V'}$ via our bases $\overline{v}, \overline{v}'$ we have the embeddings $\operatorname{Ind}_B^G(j_{\overline{v}}) : \mathfrak{X}_{\psi_V} \hookrightarrow \mathfrak{X}$, $\operatorname{Ind}_B^G(j_{\overline{v}'}) : \mathfrak{X}_{\psi_{V'}} \hookrightarrow \mathfrak{X}$. We need to explain why certain arguments of [AHHV17] remain valid or can be adapted to our more general setting.

5.3. **Proof in $\mathfrak{X}^I$.** We start the proof of Proposition 5.3. For $n(w) \in \mathcal{N}^0$ lifting $w \in W_0$, the double coset $Bn(w)I$ does not depend on the choice of $n(w)$; we write $BwI = Bn(w)I$.

Definition 5.4. For a C-character ψ of Z^0, the function $f_{\psi, n(w_\Delta)} \in \mathfrak{X}^I$ has support $Bw_\Delta I$ and its value at $n(w_\Delta)^{-1}$ is e_ψ.

The function $f_{\psi,n(w_\Delta)}$ is the image of the function $f_0 \in \mathfrak{X}^I_\psi$ of [AHHV17, IV.7 Definition] for a fixed choice of $n(w_\Delta)$. As announced earlier, we first show $I_v(f_v) \in f_{\psi_V,n(w_\Delta)}\mathcal{H}_C$.

Lemma 5.5. *We have* $I_v(f_v) = f_{\psi_V,n(w_\Delta)}T(n(w_\Delta)n(w_{\Delta(V)})^{-1})T^*(n(w_{\Delta(V)}))$.

Proof. This is obtained from [AHHV17, IV.9 Proposition] by applying the embedding $\mathfrak{X}_\psi \hookrightarrow \mathfrak{X}$, for a certain choice of $n(w_\Delta)$ and $n(w_{\Delta(V)})$. This is valid for any choice because for $t \in Z^0$, the product $T(nt^{-1})T^*(tn')$ for $n, n' \in \mathcal{N}$ does not depend on t, and neither does $f_{\psi_V,tn(w_\Delta)}T(tn) = tf_{\psi_V,n(w_\Delta)}T(t)T(n)$, recalling

$$fh = \sum_{x\in I\backslash G} h(x)x^{-1}f \quad \text{for } h \in \mathcal{H}_C,\ f \in \mathfrak{X}^I, \tag{5.1}$$

hence $fT(t) = t^{-1}f$. □

Lemma 5.6. *For a C-character ψ of Z^0 and $z \in Z^+$ we have*

$$\tau(z)f_{\psi,n(w_\Delta)} = f_{z\cdot\psi,n(w_\Delta)}T(n(w_\Delta)\cdot z).$$

Proof. When $z \cdot \psi = \psi$ this is obtained from [AHHV17, IV.10 Proposition] by applying the embedding $\mathfrak{X}_\psi \hookrightarrow \mathfrak{X}$. By loc. cit., the support of $f_{z\cdot\psi,n(w_\Delta)}T(n(w_\Delta)\cdot z)$ is $Bw_\Delta I$ and its value at $n(w_\Delta)^{-1}$ is $f_{z\cdot\psi,n(w_\Delta)}(n(w_\Delta)^{-1}(n(w_\Delta)\cdot z^{-1})) = f_{z\cdot\psi,n(w_\Delta)}(z^{-1}n(w_\Delta)^{-1}) = z^{-1}f_{z\cdot\psi,n(w_\Delta)}(n(w_\Delta)^{-1}) = z^{-1}e_{z\cdot\psi} = \tau(z)e_\psi$. Therefore $\tau(z)f_{\psi,n(w_\Delta)} = f_{z\cdot\psi,n(w_\Delta)}T(n(w_\Delta)\cdot z)$. □

Lemmas 5.5 and 5.6 imply

$$\tau(z)I_v(f_v) = f_{z\cdot\psi_V,n(w_\Delta)}T(n(w_\Delta)\cdot z)T(n(w_\Delta)n(w_{\Delta(V)})^{-1})T^*(n(w_{\Delta(V)})).$$

We want to show that the right-hand side is equal to

$$I_{v'}(f_{v'})E_{o_{\Delta(V')}}(zn(w_{\Delta(V)}w_{\Delta(V')})^{-1})T^*(n(w_{\Delta(V)}w_{\Delta(V')})).$$

This is a problem entirely in (the image in $\mathfrak{X}^I$ of) the $\mathcal{H}_C$-module $\mathfrak{X}^I_{\psi_{V'}}$ which is solved implicitly by [AHHV17, IV.19 Theorem] for a special choice of lifts in $\mathcal{N}^0$ of $w_\Delta, w_{\Delta(V)}, w_{\Delta(V')}$ and when $\psi_V = \psi_{V'}, \Delta(V) = \Delta(V') \sqcup \{\alpha\}$. Checking the homogeneity, the choice of the lifts does not matter, but the hypothesis on the parameters of V and of V' forces us to analyze the proof of [AHHV17, IV.19 Theorem]. The sets $\Delta(V)$ and $\Delta(V')$ appear together only when the proof uses [AHHV17, IV.19 Lemma]. But this lemma is valid when $\Delta(V)$ is any subset of Δ containing $\Delta(V')$. With our notation this lemma is:

Lemma 5.7. *For $\Delta(V') \subset J \subset \Delta$ we have*
$I_{v'}(f_{v'}) = f_{z\cdot\psi_V,n(w_\Delta)}T(n(w_\Delta)n(w_J)^{-1})T^*(n(w_J)n(w_Jw_{\Delta(V')})^{-1})T(n(w_Jw_{\Delta(V')}))$.

We now consider the characters. The equality $\psi_V = \psi_{V'}$ appears only when the proof uses [AHHV17, IV.14 Theorem] for $w = 1$, but we can replace it by:

Lemma 5.8. *For a C-character ψ of Z^0, $J \subset \Delta$ and $z \in Z$ we have*

$$f_{z\cdot\psi,n(w_\Delta)}T(n(w_\Delta)n(w_J)^{-1})E_{o_J}(n(w_J)\cdot z) = \begin{cases} \tau(z)f_{\psi,n(w_\Delta)}T(n(w_\Delta)n(w_J)^{-1}) & \text{if } z \in Z^+ \\ 0 & \text{if } z \notin Z^+. \end{cases}$$

Proof. The formula of Lemma 5.6 multiplied on the right by $T(n(w_\Delta)n(w_J)^{-1})$ is

$$\tau(z)f_{\psi,n(w_\Delta)}T(n(w_\Delta)n(w_J)^{-1}) = f_{z\cdot\psi,n(w_\Delta)}T(n(w_\Delta)\cdot z)T(n(w_\Delta)n(w_J)^{-1}).$$

Suppose $z \in Z^+$. In the pro-p Iwahori Hecke algebra,

$$T(n(w_\Delta)\cdot z)T(n(w_\Delta)n(w_J)^{-1}) = T(n(w_\Delta)n(w_J)^{-1})E_{o_J}(n(w_J)\cdot z).$$

This follows from [AHHV17, IV.15] applied to $n(w_J)\cdot z$ instead of λ and to $n(w_\Delta)n(w_J)^{-1}$ instead of n_{w^J} and $n(w_J)^{-1}$ instead of ν_{w_J}. We get the formula of the lemma for $z \in Z^+$.

Suppose now $z \notin Z^+$. As in [AHHV17, IV.15] we take $z_1 \in Z^+$ such that $\langle\alpha, v_Z(z_1)\rangle > 0$ for any $\alpha \in \Phi^+$ and we multiply on the right by $E_{o_J}(n(w_J)\cdot z)$ the formula that we just established for $z_1 \in Z^+$. Using $E_{o_J}(n(w_J)\cdot z_1)E_{o_J}(n(w_J)\cdot z) = 0$ we deduce

$$0 = \tau(z_1)f_{\psi,n(w_\Delta)}T(n(w_\Delta)n(w_J)^{-1})E_{o_J}(n(w_J)\cdot z),$$

and then we multiply on the left by the inverse $\tau(z_1^{-1})$ of $\tau(z_1)$ in $C[Z/Z(1)]$. The result is valid for any ψ and we replace ψ by $z\cdot\psi$ to get the lemma for $z \notin Z^+$. $\square$

By induction on $\ell(w)$ for $w \in W_{J,0}$, Lemma 5.8 is a particular case of a more general result, as explained in [AHHV17, IV.16–18] (again we see that the choice of representatives $n(w)$ for $w \in W_0$ is irrelevant):

Lemma 5.9. *For a C-character ψ of Z^0, $J \subset \Delta$, $z \in Z$ and $w \in W_{J,0}$, we have*

$$\begin{aligned} &f_{z\cdot\psi,n(w_\Delta)}T(n(w_\Delta)n(w_J)^{-1})T^*(n(w))E_{o_J}(n(w)^{-1}n(w_J)\cdot z) \\ &= \begin{cases} \tau(z)f_{\psi,n(w_\Delta)}T(n(w_\Delta)n(w_J)^{-1})T^*(n(w)) & \text{if } z \in Z^+ \\ 0 & \text{if } z \notin Z^+. \end{cases}\end{aligned}$$

Now applying the proof of [AHHV17, IV.19 Theorem] we get Proposition 5.3. (Note that we still get $\ell(zn(w_{\Delta(V)}w_{\Delta(V')})^{-1}) = \ell(n(w_{\Delta(V)}w_{\Delta(V')})\cdot z) - \ell(n(w_{\Delta(V)}w_{\Delta(V')}))$, as $z \in Z_G^+(V,V')$.) This ends the proof of Proposition 5.1.

5.4. **Expansion of φ_z in the basis (T_x) of $\mathcal{H}_G(V,V')$.** We now give the expansion in the basis $(T_z^{V,V',\iota})_{z\in Z_G^+(V,V')/Z^0}$ of $\mathcal{H}_G(V,V')$ (Proposition 2.5) of the function φ_z given in Proposition 5.1 by its value on a generator f_v of c-$\mathrm{Ind}_K^G V$:

$$\varphi_z(f_v) = f_{v'}E_{o_{\Delta(V')}}(zn(w_{\Delta(V)}w_{\Delta(V')})^{-1})T^*(n(w_{\Delta(V)}w_{\Delta(V')})). \tag{5.2}$$

Recall that $Z_z^+(V,V') = Z^+\cap z\prod_{\alpha\in\Delta'(V')} a_\alpha^{\mathbb{N}}$ is finite and contained in $Z_G^+(V,V')$ (Lemma 2.13).

Proposition 5.10. *Let $z \in Z_G^+(V,V')$. The function $\varphi_z \in \mathcal{H}_G(V,V')$ is equal to*

$$\sum_{x\in Z_z^+(V,V')} T_x^{V,V',\iota}.$$

Clearly Propositions 5.1 and 5.10 imply Theorem 3.6.

Proof. Two elements $\varphi_1,\varphi_2 \in \mathcal{H}_G(V,V')$ such that $\varphi_1(f_v)|_{Z^+} = \varphi_2(f_v)|_{Z^+}$ are equal. This follows from two properties:

(i) a basis of $\mathcal{H}_G(V,V')$ is $T_{z'}^{V,V',\iota}$ for z' running through a system of representatives of $Z_G^+(V,V')/Z^0$. So $\varphi_1 = \sum_{z'} a_1(z')T_{z'}^{V,V',\iota}$ for some $a_1(z') \in C$.

(ii) $\varphi_1(f_v)(z') = a_1(z')v'$ for $z' \in Z_G^+(V,V')$ because of the lemma below.

Lemma 5.11. *For $z' \in Z_G^+(V,V')$ the function $T_{z'}^{V,V',\iota}(f_v) \in \text{c-Ind}_K^G V'$ vanishes outside $Kz'K$ and is equal to v' at z'.*

Proof. For $y \in G$, the value of $T_{z'}^{V,V',\iota}(f_v)$ at y,

$$T_{z'}^{V,V',\iota}(f_v)(y) = \sum_{g \in Kz'K/K} T_{z'}^{V,V',\iota}(g)(f_v(g^{-1}y))$$

is 0 if $Kz'^{-1}Ky \cap K = \varnothing$ (hence $T_{z'}^{V,V',\iota}(f_v)$ vanishes outside $Kz'K$) and $T_{z'}^{V,V',\iota}(f_v)(z') = T_{z'}^{V,V',\iota}(z')(f_v(1)) = \iota^{\mathrm{op}}(v) = v'$. □

Therefore it is enough to prove that $\varphi_z(f_v)|_{Z^+} = \sum_{x \in Z_z^+(V,V')} T_x^{V,V',\iota}(f_v)|_{Z^+}$, or equivalently,

$$\varphi_z(f_v)(x) = \begin{cases} v' & x \in Z_z^+(V,V'), \\ 0 & x \in Z^+ \setminus Z^0 Z_z^+(V,V'). \end{cases} \tag{5.3}$$

We now write $J' = \Delta(V')$ and $J = \Delta(V)$. We prove (5.3) in two steps. In the first step we prove (5.3) assuming two claims which are proved in the second step.

A) By the congruence modulo q of the Iwahori-Matsumoto expansion of $E_{o_{J'}}(zn(w_J w_{J'})^{-1})$ (Propositions 4.23 and 4.30), we have

$$f_{v'} E_{o_{J'}}(zn(w_J w_{J'})^{-1}) = \sum_{x \in W_{J'}, x \leq_{J'} \lambda} (-1)^{\ell_{J'}(\lambda) - \ell_{J'}(x)} \psi_{V'}^{-1}(c_{\tilde{\lambda}}^{\tilde{x},J'})\, f_{v'} T(\tilde{x} n(w_J w_{J'})^{-1}),$$

where $\tilde{\lambda}$ is the image of z in $\Lambda^+(1)$ and λ the image of z in Λ^+. We used that $f_{v'}c = \psi_{V'}^{-1}(c) f_{v'}$ for $c \in \mathbb{Z}[Z_k]$, as $f_{v'}T(t) = t^{-1} f_{v'} = \psi_{V'}(t^{-1}) f_{v'}$ for $t \in Z_k$ (5.1). We claim that

$$f_{v'} T(\tilde{x} n(w_J w_{J'})^{-1}) T^*(n(w_J w_{J'}))|_{Z^+} \neq 0 \implies x \in \Lambda^+. \tag{5.4}$$

Now for $x \in \Lambda^+$ we have $x \leq_{J'} \lambda$ if and only if $x \in \Lambda^+ \cap \lambda \prod_{\alpha \in J'} \lambda_\alpha^{\mathbb{N}}$ (Proposition 4.3), and we know the value of $\psi_{V'}^{-1}(c_{\tilde{\lambda}}^{\tilde{x},J'})$ (Theorem 4.39). Obviously $\Delta'_{\psi_{V'}} = \Delta'_{\psi_{V'}^{-1}}$ and $J' \cap \Delta'_{\psi_{V'}} = \Delta'(V')$ hence $x \in \Lambda^+ \cap \lambda \prod_{\alpha \in \Delta'(V')} \lambda_\alpha^{\mathbb{N}}$ (Proposition 4.3) if $\psi_{V'}^{-1}(c_{\tilde{\lambda}}^{\tilde{x},J'}) \neq 0$. Together with (5.2) we obtain

$$\varphi_z(f_v)|_{Z^+} = \sum_{\tilde{x} \in \Lambda^+(1) \cap \tilde{\lambda} \prod_{\alpha \in \Delta'(V')} a_\alpha^{\mathbb{N}}} f_{v'} T(\tilde{x} n(w_J w_{J'})^{-1}) T^*(n(w_J w_{J'}))|_{Z^+}.$$

We claim also that

$$f_{v'} T(\tilde{x} n(w_J w_{J'})^{-1}) T^*(n(w_J w_{J'}))|_{Z^+} = f_{v'} T(\tilde{x} n(w_J w_{J'})^{-1}) T(n(w_J w_{J'}))|_{Z^+}. \tag{5.5}$$

Assuming the claim, the braid relations and $\ell(x) = \ell(x w_{J'} w_J) + \ell(w_J w_{J'})$ (Lemma 4.29) imply

$$\varphi_z(f_v)|_{Z^+} = \sum_{\tilde{x} \in \Lambda^+(1) \cap \tilde{\lambda} \prod_{\alpha \in \Delta'(V')} a_\alpha^{\mathbb{N}}} f_{v'} T(\tilde{x})|_{Z^+}.$$

We finally compute $f_{v'} T(\tilde{x})|_{Z^+}$.

Lemma 5.12. *For $z \in Z$, the function $f_{v'} T(z) \in (\text{c-Ind}_K^G V')^I$ vanishes on Z^+ if $z \notin Z^+$, and $f_{v'} T(z)$ is the function of support KzI with value v' at z if $z \in Z^+$.*

Proof. The map $z \mapsto KzI : Z \to K\backslash G/I$ factors to a bijective map $\Lambda \xrightarrow{\sim} K\backslash G/I$. We have $KzI \cap Z^+ = zZ^0$ if $z \in Z^+$ and $KzI \cap Z^+ = \varnothing$ if $z \in Z \setminus Z^+$ and

$$(f_{v'}T(z))(z) = \sum_{x \in I\backslash IzI} f_{v'}(zx^{-1}).$$

The support of $f_{v'}T(z)$ is contained in KzI hence $f_{v'}T(z) \in (\text{c-Ind}_K^G V')^I$ vanishes on Z^+ if $z \notin Z^+$. In the displayed formula $f_{v'}(zx^{-1}) \neq 0$ implies $zx^{-1} \in K \cap zIz^{-1}I$. Consider the Iwahori decomposition $I = U_{\mathrm{op}}^0(I \cap B)$. If $z \in Z^+$, we have $U_{\mathrm{op}}^0 \subset zU_{\mathrm{op}}^0 z^{-1} \subset U_{\mathrm{op}}$ and $z(I \cap B)z^{-1} \subset I \cap B$. By intersecting with K we get $U_{\mathrm{op}}^0 = K \cap zU_{\mathrm{op}}^0 z^{-1}$. Hence $K \cap zIz^{-1}I = K \cap zU_{\mathrm{op}}^0 z^{-1} I = I$, so $(f_{v'}T(z))(z) = f_{v'}(1) = v'$. $\square$

B) We prove the two claims (5.4) and (5.5). There are weak braid relations in $\mathcal{H}_C$ valid for any pair of elements in $W(1)$.

Lemma 5.13. *For $w_1, w_2 \in W(1)$ there exists $w_2' \in W(1)$ with $w_2' \leq w_2$ and $T_{w_1}T_{w_2} \in C[Z_k]T_{w_1 w_2'}$.*

Proof. This is done by induction on $\ell(w_2)$. When $\tilde{s} \in S^{\mathrm{aff}}(1)$ we have $T_{w_1}T_{\tilde{s}} = T_{w_1\tilde{s}}$ if $w_1 < w_1\tilde{s}$ and $T_{w_1}T_{\tilde{s}} = T_{w_1\tilde{s}^{-1}}T_{\tilde{s}}^2 = T_{w_1\tilde{s}^{-1}}c(\tilde{s})T_{\tilde{s}} = (w_1 \cdot c(\tilde{s}))T_{w_1}$ if $w_1\tilde{s} < w_1$. $\square$

As an application, for $\tilde{w}_1, \tilde{w}_2 \in W(1)$ lifting $w_1, w_2 \in W$, the triangular Iwahori-Matsumoto expansion of $T_{\tilde{w}_2}^*$ and the weak braid relations imply

$$T_{\tilde{w}_1}(T_{\tilde{w}_2}^* - T_{\tilde{w}_2}) \in \sum_{y \in W, y < w_2} C[Z_k]T_{\tilde{w}_1}T_{\tilde{y}} \subset \sum_{y \in W, y < w_2} C[Z_k]T_{\tilde{w}_1\tilde{y}},$$

where $\tilde{y} \in W(1)$ lifts y. We use this result as follows: $f_{v'}T_{\tilde{w}_1}T_{\tilde{w}_2}^*|_{Z^+} = f_{v'}T_{\tilde{w}_1}T_{\tilde{w}_2}|_{Z^+}$ if $f_{v'}T_{\tilde{w}_1\tilde{y}}|_{Z^+} = 0$ for all $y \in W$ with $y < w_2$. The two claims (5.4) and (5.5) follow from:

Lemma 5.14. *Suppose $\tilde{w}_1 \in W(1)$ lifts $w_1 = xw_{J'}w_J$ with $x \in W_{J'}, x \leq_{J'} \lambda$, $\lambda \in \Lambda^+$, and $\tilde{y} \in W(1)$ lifts $y \in W_{J,0}$ with $y \leq w_J w_{J'}$. Then $f_{v'}T_{\tilde{w}_1\tilde{y}}$ vanishes on Z^+ except if $x \in \Lambda^+$ and $y = w_J w_{J'}$.*

Proof. Let $\lambda_x \in \Lambda$ and $v_x \in W_{J',0}$ such that $x = \lambda_x v_x$. We have $\langle \gamma, v(\lambda_x) \rangle > 0$ for $\gamma \in \Phi_J^+ \backslash \Phi_{J'}^+$ by the proof of Lemma 4.29(ii).

We have $w_1 y = \lambda_x v_x w_{J'} w_J y$ where $v_x w_{J'} w_J y \in W_{J,0}$, the support of $f_{v'}T_{\tilde{w}_1\tilde{y}}$ is contained in $Kn(\lambda_x)n(v_x w_{J'} w_J y)I = K(n(v_x w_{J'} w_J y)^{-1} \cdot n(\lambda_x))I$ and recalling the bijection $\Lambda \to K\backslash G/I$, we have $Z \cap K(n(v_x w_{J'} w_J y)^{-1} \cdot n(\lambda_x))I = Z^0(n(v_x w_{J'} w_J y)^{-1} \cdot n(\lambda_x))$. We have $\langle (v_x w_{J'} w_J y)^{-1}(\gamma), v((v_x w_{J'} w_J y)^{-1} \cdot \lambda_x) \rangle = \langle \gamma, v(\lambda_x) \rangle$. If $v_x w_{J'} w_J y \notin W_{J',0}$ there exists $\gamma \in \Phi_J^+ \setminus \Phi_{J'}^+$ with $(v_x w_{J'} w_J y)^{-1}(\gamma) < 0$, hence $f_{v'}T_{\tilde{w}_1\tilde{y}}$ vanishes on Z^+. Hence we may assume that $v_x w_{J'} w_J y \in W_{J',0}$.

We recall:

Lemma 5.15 ([Bou02, IV.1, Exercise 3])**.** *Let $J \subset \Delta$. Every coset $wW_{J,0}$ in W_0 has a unique representative d of minimal length. We have $\ell(du) = \ell(d) + \ell(u)$ for all $u \in W_{J,0}$. An element $d \in W_0$ is the representative of minimal length in $dW_{J,0}$ if and only if $d(J) \subset \Phi^+$.*

The element $w_J w_{J'}$ is the representative of minimal length of the coset $w_J W_{J',0}$. Since $v_x w_{J'} w_J y \in W_{J',0}$, we have $y \in w_J W_{J',0}$, so $y = w_J w_{J'}$, as $y \leq w_J w_{J'}$ by assumption.

We deduce that $f_{v'}T_{\tilde{w}_1\tilde{y}}$ vanishes on Z^+ if $y \neq w_J w_{J'}$.

Assume $y = w_J w_{J'}$. Then $\tilde{x} = \tilde{w}_1 \tilde{y}$ lifts $x = \lambda_x v_x$. If $f_{v'} T_{\tilde{x}}$ does not vanish on Z^+, then by above we have $v_x^{-1} \cdot \lambda_x \in \Lambda^+$. If $v_x^{-1} \cdot \lambda_x \in \Lambda^+$ then $\ell(x) = \ell(v_x(v_x^{-1} \cdot \lambda_x)) = \ell(v_x) + \ell(v_x^{-1} \cdot \lambda_x)$, and by the braid relations $f_{v'} T_{\tilde{x}} = f_{v'} T_{\tilde{v}_x} T_{v_x^{-1} \cdot \lambda_x}$.

The element $f_{v'} \in (\text{c-Ind}_K^G V')^I$ generates a subrepresentation of K isomorphic to V'. The parameter of the character of $\mathcal{H}_C(K, I)$ acting on $C f_{v'}$ is $(\psi_{V'}^{-1}, J')$ (Lemma 4.11). By (4.8), $f_{v'} T_{\tilde{v}_x} = 0$ for $v_x \in W_{J',0} - \{1\}$. We deduce that $f_{v'} T_{\tilde{x}} = 0$, except if $x \in \Lambda^+$ and $y = w_J w_{J'}$. □

This ends the proof of (5.3) hence of Proposition 5.10. □

6. A simple proof of the change of weight theorem for certain G

In this section, we give a simple proof of the change of weight theorem (Theorem 2.2) when $\mathbf{G}$ is split. For GL_n (and more generally for any split group, see §6.6) this gives a more elementary proof than the one in [Her11a] and [Abe13], avoiding the Lusztig-Kato theorem.

Since $\mathbf{G}$ is split, $\mathbf{Z}$ is equal to $\mathbf{S}$ and v_Z gives an isomorphism $X_*(\mathbf{S}) \simeq S/S^0 = \Lambda$, and Bruhat-Tits theory gives a Chevalley group scheme $\mathcal{G}$ with generic fiber $\mathbf{G}$ and such that $\mathcal{G}(\mathcal{O}) = K$ is the special maximal compact open subgroup of G fixing x_0 [Tit79, 3.4.2]. We have $\mathcal{G}(k) = G_k$, the root system Φ of (G, S) identifies canonically with the root system of (G_k, S_k).

Lemma 6.1. *Assume that $\mathbf{G}$ is F-split. For $\alpha \in \Delta$, we have $Z \cap M'_\alpha = \alpha^\vee(F^\times)$, $Z^0 \cap M'_\alpha = \alpha^\vee(\mathcal{O}^\times)$, and $Z_k \cap M'_{\alpha,k} = \alpha^\vee(k^\times)$.*

Proof. Note that $\mathbf{M}_\alpha^{\mathrm{der}}$ is a semisimple group of rank 1 and that $M'_\alpha \subset M_\alpha^{\mathrm{der}}$. Hence the first two equalities are reduced to the case where $\mathbf{G}$ is semisimple of rank 1 and hence isomorphic to SL_2 or PGL_2 [Spr09, Thm. 7.2.4]. In either case the first two equalities are easily verified by hand, noting that $\mathbf{Z} \cong \mathbb{G}_m$ and so the parahoric Z^0 is the maximal compact $\mathcal{O}^\times \subset F^\times$. For the third equality, the same proof as for the first one works, but now one works over k instead of F. □

By the lemma, for a character $\psi \colon Z_k \to C^\times$, which is also regarded as a character of Z^0 by the quotient map $Z^0 \twoheadrightarrow Z_k$, ψ is trivial on $Z_k \cap M'_{\alpha,k}$ if and only if ψ is trivial on $Z^0 \cap M'_\alpha$. Hence $\Delta(V) = \Delta'(V)$ for any irreducible representation V of K.

In this section we prove Theorem 2.3. We will first focus on the case when the center of $\mathbf{G}$ is a torus (i.e. smooth and connected) and the derived subgroup of $\mathbf{G}$ is simply connected. In fact, just as in the first proof of Proposition 2.17 we prove a stronger version which we now state. Fix α, V, V' as in Theorem 2.3.

Theorem 6.2. *Suppose that $\mathbf{G}$ is a split group whose center is a torus and whose derived subgroup is simply-connected. Let $z \in Z^+$ such that $\langle \alpha, v_Z(z) \rangle > 0$, i.e. $z \in Z_G^+(V, V')$. Then there exist G-equivariant homomorphisms $\varphi : \text{c-Ind}_K^G V \to \text{c-Ind}_K^G V'$ and $\varphi' : \text{c-Ind}_K^G V' \to \text{c-Ind}_K^G V$ satisfying*

$$S^G(\varphi) = \tau_z^{V'_{U^0}, V_{U^0}}, \quad S^G(\varphi') = \tau_z^{V_{U^0}, V'_{U^0}} - \tau_{z a_\alpha}^{V_{U^0}, V'_{U^0}}.$$

If moreover $\langle \beta, v_Z(z) \rangle = 0$ for $\beta \in \Delta(V')$, then $\varphi = T_z^{V',V}$ and $\varphi' = T_z^{V,V'}$.

Remark 6.3. Recall that we fixed an isomorphism of vector spaces $\iota \colon V_{U^0} \simeq V'_{U^0}$ (2.8). This is also an isomorphism of representations of Z^0 because $\psi_V = \psi_{V'}$. We have isomorphisms $\mathcal{H}_Z(V_{U^0}, V'_{U^0}) \simeq \mathcal{H}_Z(V'_{U^0}, V_{U^0}) \simeq \mathcal{H}_Z(V'_{U^0}, V'_{U^0}) = \mathcal{H}_Z(V'_{U^0}) \simeq \mathcal{H}_Z(V_{U^0}, V_{U^0}) = \mathcal{H}_Z(V_{U^0})$ and for $x \in Z$, $\tau_x^{V_{U^0}}, \tau_x^{V'_{U^0}}$ correspond to each other under the isomorphism $\mathcal{H}_Z(V_{U^0}) \simeq \mathcal{H}_Z(V'_{U^0})$, and we will just denote them by τ_x. We remark that since $Z = S$ is commutative, $\mathcal{H}_G(V_{U^0})$ is commutative.

The basic idea of the proof is the following. We construct many G-representations π that contain the weight V but not the weight V'. This implies that $\chi \otimes \text{c-Ind}_K^G V \not\simeq \chi \otimes \text{c-Ind}_K^G V'$ for any homomorphism $\chi : \mathcal{H}_G(V) \simeq \mathcal{H}_G(V') \to C$ that occurs in $\mathrm{Hom}_K(V, \pi)$. This in turn implies that $\chi(T_z^{V,V'} * T_z^{V',V}) = 0$ for such χ. When z is as in Theorem 6.2 and chosen minimally, i.e. $\langle \alpha, v_Z(z) \rangle = 1$ and $\langle \beta, v_Z(z) \rangle = 0$ for $\beta \in \Delta \setminus \{\alpha\}$, then it turns out that $S^G(T_z^{V,V'} * T_z^{V',V})$ is so constrained that it is forced to be equal to $\tau_{z^2} - \tau_{z^2 a_\alpha}$. By Lemma 3.1 we have $S^G(T_z^{V',V}) = \tau_z^{V'_{U^0}, V_{U^0}}$, and we deduce that $S^G(T_z^{V,V'}) = \tau_z^{V_{U^0}, V'_{U^0}} - \tau_{z a_\alpha}^{V_{U^0}, V'_{U^0}}$. Using properties of S^G it is then not difficult to deduce the theorem.

6.1. **The case of** GL_2**.** To warm up, in this section we illustrate the proof strategy by showing that $S^G(T_z^{V,V'} * T_z^{V',V}) = \tau_{z^2} - \tau_{z^2 a_\alpha}$ when $\mathbf{G} = \mathrm{GL}_2$, V is the trivial representation 1_K of K, V' is the Steinberg representation St_K of K, and $z = \mathrm{diag}(\varpi, 1)$ where ϖ is a uniformizer. We note that $\tau_\alpha = \tau_{\mathrm{diag}(\varpi^{-1}, \varpi)}$, so $\tau_{z^2 a_\alpha} = \tau_{\mathrm{diag}(\varpi, \varpi)}$. The Satake homomorphism S^G satisfies (see [Her11a, proof of Prop. 6.3] or Lemma 2.9):

- $S^G(T_z^{V',V})(z') \neq 0$ implies $v_Z(z') \in v_Z(z) + \mathbb{R}_{\leq 0} \Delta^\vee$.
- The coefficient of $\tau_z^{V'_{U^0}, V_{U^0}}$ in $S^G(T_z^{V',V})$ is 1.

This also holds after switching V and V'. This means that $S^G(T_z^{V',V}) \in \tau_z^{V'_{U^0}, V_{U^0}} + \sum_{n<0} C \tau^{V'_{U^0}, V_{U^0}}_{\mathrm{diag}(\varpi^{n+1}, \varpi^{-n})}$, similarly after switching V and V', and $S^G(T_z^{V,V'}) \circ S^G(T_z^{V',V}) \in \tau_{z^2} + \sum_{n<0} C \tau_{\mathrm{diag}(\varpi^{n+2}, \varpi^{-n})}$. The support of $S^G(f) \in \mathcal{H}_Z(1_{Z^0})$ is contained in Z^+ for any $f \in \mathcal{H}_G(1_K)$. For $n < 0$, if $\mathrm{diag}(\varpi^{n+2}, \varpi^{-n}) \in Z^+$ then $n = -1$, so

$$S^G(T_z^{V,V'} \circ T_z^{V',V}) = \tau_z^2 + c \tau_{\mathrm{diag}(\varpi, \varpi)}$$

for some $c \in C$. Let $\chi_1 : \mathcal{H}_Z(1_{Z^0}) \to C$ be the character such that $\chi_1(\tau_z) = \chi_1(\tau_{\mathrm{diag}(\varpi,\varpi)}) = 1$. We also denote by χ_1 the character $\chi_1 \circ S^G$ of $\mathcal{H}_G(1_K) \simeq \mathcal{H}_G(\mathrm{St}_K)$. The algebra $\mathcal{H}_G(1_K)$ acts on the line $\mathrm{Hom}_G(\text{c-Ind}_K^G 1_K, 1_G)$ by the character χ_1 because the embedding $1_G \hookrightarrow \mathrm{Ind}_B^G 1_Z$ implies

$$\mathrm{Hom}_K(1_K, 1_G) \hookrightarrow \mathrm{Hom}_K(1_K, \mathrm{Ind}_B^G 1_Z) = \mathrm{Hom}_K(1_K, \mathrm{Ind}_{B^0}^K 1_Z) \simeq \mathrm{Hom}_{Z^0}(1_K|_{Z^0}, 1_Z|_{Z^0}),$$

and the isomorphism $\mathrm{Hom}_K(1_K, 1_G) \to \mathrm{Hom}_{Z^0}(1_K|_{Z^0}, 1_Z|_{Z^0})$ is $\mathcal{H}_G(1_K)$-equivariant via S^G [Her11a, Lemma 2.14]. Hence 1_G is a quotient of $\chi_1 \otimes \text{c-Ind}_K^G 1_K$ and

$$\chi_1 \otimes \text{c-Ind}_K^G 1_K \not\simeq \chi_1 \otimes \text{c-Ind}_K^G \mathrm{St}_K .$$

(If these are isomorphic to each other, then we have a non-zero homomorphism $\text{c-Ind}_K^G \mathrm{St}_K \to \chi_1 \otimes \text{c-Ind}_K^G \mathrm{St}_K \simeq \chi_1 \otimes \text{c-Ind}_K^G 1_K \to 1_G$ which gives $\mathrm{St}_K \to 1_G|_K$ by Frobenius reciprocity. This is a contradiction.) For a character $\chi : \mathcal{H}_Z(1_{Z^0}) \to C$ such that $\chi(\tau_z^2 + c\tau_{\mathrm{diag}(\varpi,\varpi)}) \neq 0$, we have $\chi \otimes \text{c-Ind}_K^G V \simeq \chi \otimes \text{c-Ind}_K^G V'$. Therefore $\chi_1(\tau_z^2 + c\tau_{\mathrm{diag}(\varpi,\varpi)}) = 0$, hence $c = -1$ as desired.

6.2. **Reducibility and change of weight.** Until the end of §6.5, fix $\mathbf{G}, \alpha, V, V'$ as in Theorem 6.2.

Let $\chi\colon \mathcal{H}_Z(V_{U^0}) \to C$ be a character. Since $Z^0 \subset Z$ is normal, $\text{c-Ind}_{Z^0}^Z V_{U^0}$ is a free $\mathcal{H}_Z(V_{U^0})$-module of rank 1. The character $\chi \otimes_{\mathcal{H}_Z(V_{U^0})} \text{c-Ind}_{Z^0}^Z V_{U^0}$ of Z is $z \mapsto \chi(\tau_{z^{-1}})$ because $\tau_{z^{-1}} = z$ as endomorphisms of $\text{c-Ind}_{Z^0}^Z V_{U^0}$; its restriction to Z^0 is ψ_V because $\tau_{z^{-1}} = \psi_V(z)\tau_1 = \psi_V(z)$ in $\mathcal{H}_Z(V_{U^0})$ for $z \in Z^0$. Since ψ_V is trivial on $Z^0 \cap M'_\alpha$, τ_α is well-defined.

Assume that $\chi(\tau_\alpha) = 1$. The character $z \mapsto \chi(\tau_{z^{-1}})$ of Z is trivial on $Z \cap M'_\alpha = \alpha^\vee(F^\times)$, hence we can extend it to a character of M_α that is trivial on $U \cap M_\alpha$ ([Abe13, Proposition 3.3], [AHHV17, II.7 Corollary 1]). We denote this extended character by σ_χ.

Lemma 6.4 ([AHHV17, III.18 Proposition])**.** *Assume that $\chi\colon \mathcal{H}_Z(V_{U^0}) \to C$ satisfies $\chi(\tau_\alpha) = 1$. Then $\operatorname{Hom}_K(V, \operatorname{Ind}_{P_\alpha}^G \sigma_\chi) \neq 0$ and $\operatorname{Hom}_K(V', \operatorname{Ind}_{P_\alpha}^G \sigma_\chi) = 0$.*

Proof. By Frobenius reciprocity, the Iwasawa decomposition $G = P_\alpha K$ and using $P_\alpha^0 = M_\alpha^0 N_\alpha^0$ we have

$$\operatorname{Hom}_K(V_1, \operatorname{Ind}_{P_\alpha}^G \sigma_\chi) = \operatorname{Hom}_K(V_1, \operatorname{Ind}_{P_\alpha^0}^K \sigma_\chi) \simeq \operatorname{Hom}_{M_\alpha^0}((V_1)_{N_\alpha^0}, \sigma_\chi)$$

for any irreducible representation V_1 of K. The parameter of $V_{N_\alpha^0}$ is $(\psi_V, \{\alpha\})$, the parameter of $V'_{N_\alpha^0}$ is $(\psi_V, \varnothing)$ [AHHV17, III.10 Lemma]. On the other hand, the parameter of the character $\sigma_\chi|_{M_\alpha^0}$ is $(\psi_V, \{\alpha\})$ [AHHV17, III.10 Remark]. $\square$

Lemma 6.5. *Assume that $\chi\colon \mathcal{H}_Z(V_{U^0}) \to C$ satisfies $\chi(\tau_\alpha) = 1$. Then*

$$\chi \otimes_{\mathcal{H}_G(V)} \text{c-Ind}_K^G V \not\simeq \chi \otimes_{\mathcal{H}_G(V)} \text{c-Ind}_K^G V'.$$

Proof. By definition of σ_χ we have an M_α-equivariant map $\sigma_\chi \hookrightarrow \operatorname{Ind}_{B \cap M_\alpha}^{M_\alpha}(\chi \otimes_{\mathcal{H}_Z(V_{U^0})} \text{c-Ind}_{Z^0}^Z V_{U^0})$. By exactness of parabolic induction we get

$$\begin{aligned}\operatorname{Hom}_K(V, \operatorname{Ind}_{P_\alpha}^G \sigma_\chi) &\hookrightarrow \operatorname{Hom}_K(V, \operatorname{Ind}_B^G(\chi \otimes_{\mathcal{H}_Z(V_{U^0})} \text{c-Ind}_{Z^0}^Z V_{U^0})) \\ &\simeq \operatorname{Hom}_{Z^0}(V_{U^0}, \chi \otimes_{\mathcal{H}_Z(V_{U^0})} \text{c-Ind}_{Z^0}^Z V_{U^0}),\end{aligned}$$

and this map is $\mathcal{H}_G(V)$-linear with respect to S^G. The latter space is one-dimensional and the Hecke algebra $\mathcal{H}_Z(V_{U^0})$ acts on this line by the character χ. Hence a non-trivial homomorphism $\text{c-Ind}_K^G V \to \operatorname{Ind}_{P_\alpha}^G \sigma_\chi$ (which exists by Lemma 6.4) factors through $\text{c-Ind}_K^G V \twoheadrightarrow \chi \otimes_{\mathcal{H}_G(V)} \text{c-Ind}_K^G V$. If $\chi \otimes_{\mathcal{H}_G(V)} \text{c-Ind}_K^G V$ were isomorphic to $\chi \otimes_{\mathcal{H}_G(V)} \text{c-Ind}_K^G V'$, we would have a non-zero homomorphism $\text{c-Ind}_K^G V' \twoheadrightarrow \chi \otimes_{\mathcal{H}_G(V)} \text{c-Ind}_K^G V' \to \operatorname{Ind}_{P_\alpha}^G \sigma_\chi$ contradicting $\operatorname{Hom}_K(V', \operatorname{Ind}_{P_\alpha}^G \sigma_\chi) = 0$ (Lemma 6.4). $\square$

6.3. **Proof of Theorem 6.2 (minuscule case).** The hypothesis that the center of $\mathbf{G}$ is a torus is equivalent to $\mathbb{Z}\Phi$ being a direct summand of $X^*(\mathbf{S})$, for example by [Mil, (154)]. Hence, for each $\alpha \in \Delta$ we have a fundamental coweight $\mu_\alpha \in X_*(\mathbf{S})$. Namely we have $\langle \alpha, \mu_\alpha \rangle = 1$ and $\langle \beta, \mu_\alpha \rangle = 0$ for any $\beta \in \Delta \setminus \{\alpha\}$. In this section we consider $z \in Z$ such that $v_Z(z) = \mu_\alpha$.

The element $\tau_\alpha - 1 \in \mathcal{H}_Z(V_{U^0})$ is irreducible, since the derived subgroup of $\mathbf{G}$ is simply connected [Abe13, Remark 2.5 and Lemma 4.17] (alternatively, one can argue as in Lemma A.12). Put $f = S^G(T_z^{V,V'} * T_z^{V',V})$ in $\mathcal{H}_Z(V_{U^0})$. Lemma 6.5 implies that $\chi(f) = 0$ for any character $\chi\colon \mathcal{H}_Z(V_{U^0}) \to C$ such that $\chi(\tau_\alpha) = 1$. By the Nullstellensatz, we see that f is contained in the radical of the ideal $(\tau_\alpha - 1)$, hence as $\tau_\alpha - 1$ is irreducible and $\mathcal{H}_Z(V_{U^0})$ is a UFD, we deduce that $f = f'(1 - \tau_\alpha)$ for some $f' \in \mathcal{H}_Z(V_{U^0})$. We will prove that $f' = \tau_{z^2}$.

Consider any $z' \in \operatorname{supp} f'$. We claim that both z' and $z'a_\alpha$ lie in Z^+ and that $v_Z(z') \in 2v_Z(z) + \mathbb{R}_{\le 0}\Delta^\vee$. To see this, pick $r, s \ge 0$ maximal such that $z'a_\alpha^i \in \operatorname{supp} f'$ for $-r \le i \le s$. Then $z'a_\alpha^{-r}$, $z'a_\alpha^{s+1} \in \operatorname{supp} f$, so they both lie in Z^+. By convexity of the dominant region we deduce that z', $z'a_\alpha \in Z^+$. Similarly, as recalled in §6.1, we know that $v_Z(z'a_\alpha^i) \in 2v_Z(z) + \mathbb{R}_{\le 0}\Delta^\vee$ for $i \in \{-r, s+1\}$, hence by convexity we have $v_Z(z') \in 2v_Z(z) + \mathbb{R}_{\le 0}\Delta^\vee$.

There exist $n_\beta \in \mathbb{R}_{\ge 0}$ for $\beta \in \Delta$ such that $v_Z(z') = 2\mu_\alpha - \sum_{\beta\in\Delta} n_\beta \beta^\vee$. Recalling $v_Z(a_\alpha) = -\alpha^\vee$, we have $v_Z(z'a_\alpha) = 2\mu_\alpha - \alpha^\vee - \sum_{\beta\in\Delta} n_\beta\beta^\vee$. Let $\gamma \in \Delta$. If $\gamma \ne \alpha$, then $\sum_{\beta\in\Delta} n_\beta\langle\gamma,\beta^\vee\rangle = -\langle\gamma, v_Z(z')\rangle \le 0$. If $\gamma = \alpha$, then $\sum_{\beta\in\Delta} n_\beta\langle\gamma,\beta^\vee\rangle = 2 - \langle\alpha,\alpha^\vee\rangle - \langle\alpha, v_Z(z'a_\alpha)\rangle = -\langle\alpha, v_Z(z'a_\alpha)\rangle \le 0$. Hence $\sum_{\beta\in\Delta} n_\beta\langle\gamma,\beta^\vee\rangle \le 0$ for any $\gamma \in \Delta$. Since $(d_\gamma\langle\gamma,\beta^\vee\rangle)_{\beta,\gamma\in\Delta}$ is positive definite for some $d_\gamma > 0$, we have $n_\beta = 0$ for any $\beta \in \Delta$. We deduce that $z' \in z^2Z^0$ (as Z^0 is the kernel of v_Z). So $f' \in C^\times\tau_{z^2}$. Since the coefficient of τ_{z^2} in f is 1, we get $f = S^G(T_z^{V,V'} * T_z^{V',V}) = \tau_{z^2} - \tau_{z^2a_\alpha}$.

By Lemma 3.1 we have $S^G(T_z^{V',V}) = \tau_z^{V'_{U^0},V_{U^0}}$, hence we deduce that $S^G(T_z^{V,V'}) = \tau_z^{V_{U^0},V'_{U^0}} - \tau_{za_\alpha}^{V_{U^0},V'_{U^0}}$. This completes the proof of Theorem 6.2 when $v_Z(z) = \mu_\alpha$.

6.4. **Proof of Theorem 6.2 (general case).** We consider now $z \in Z^+$ such that $\langle\alpha, v_Z(z)\rangle > 0$. Take $z_0 \in Z$ such that $v_Z(z_0) = \mu_\alpha$. Then $zz_0^{-1} \in Z^+$ and from (2.2) we deduce the existence of $\theta \in \mathcal{H}_G(V')$ such that $S^G(\theta) = \tau_{zz_0^{-1}}$. Letting $\varphi = \theta * T_{z_0}^{V',V}$ and $\varphi' = T_{z_0}^{V,V'} * \theta$, we see from §6.3 that $S^G(\varphi) = \tau_z$ and $S^G(\varphi') = \tau_z - \tau_{za_\alpha}$.

In the special case that $\langle\beta, v_Z(z)\rangle = 0$ for $\beta \in \Delta(V')$, we have $\Delta(V') \subset \Delta_z \subset \Delta_{zz_0^{-1}}$, so Lemma 3.1 shows that $\theta = T_{zz_0^{-1}}^{V',V'}$. From Lemma 3.2 we then deduce that $\varphi = T_z^{V',V}$ and $\varphi' = T_z^{V,V'}$.

6.5. **A corollary.**

Corollary 6.6. *Suppose that V is an irreducible representation of K and that $z \in Z^+$ satisfies $\langle\alpha, v_Z(z)\rangle \ne 1$ for all $\alpha \in \Delta(V)$. Then the image of $T_z \in \mathcal{H}_G(V)$ under the Satake transform S^G is given by*

$$S^G(T_z) = \tau_z \prod_{\alpha\in\Delta(V)\setminus\Delta_z} (1 - \tau_\alpha).$$

Proof. We induct on $\#(\Delta(V) \setminus \Delta_z)$. If $\Delta(V) \subset \Delta_z$, then $S^G(T_z) = \tau_z$ by Lemma 3.1 and we are done. Otherwise we choose $\alpha \in \Delta(V) \setminus \Delta_z$ and take z_0 such that $v_Z(z_0) = \mu_\alpha$. Then $zz_0^{-2} \in Z^+$, as $\langle\alpha, v_Z(z)\rangle \ge 2$ by assumption. Define V' by the parameter $(\psi_V, \Delta(V) \setminus \{\alpha\})$. Applying Lemma 3.2 twice (using that $\Delta(V') \subset \Delta_{z_0}$) we get that $T_z^{V,V} = T_{z_0}^{V,V'} * T_{zz_0^{-2}}^{V',V'} * T_{z_0}^{V',V}$. As $\Delta(V') \setminus \Delta_{zz_0^{-2}}$ is a proper subset of $\Delta(V) \setminus \Delta_z$ we get by induction that $S^G(T_{zz_0^{-2}}^{V',V'}) = \tau_{zz_0^{-2}} \prod_{\Delta(V')\setminus\Delta_z}(1 - \tau_\beta)$. On the other hand, by Theorem 6.2 we have $S^G(T_{z_0}^{V',V}) = \tau_{z_0}$ and $S^G(T_{z_0}^{V,V'}) = \tau_{z_0}(1 - \tau_\alpha)$. By combining these formulas we get the corollary. □

Remark 6.7. It is not hard to deduce the corollary from Theorem 2.12, noting that $z\prod_{\beta\in X} a_\beta \in Z^+$ for any subset $X \subset \Delta(V) \setminus \Delta_z$.

6.6. **The general split case.** We now use two reduction steps to extend the above proof of Theorem 6.2 to the case of general split groups $\mathbf{G}$.

(1) We remove first the assumption on the center. Suppose that $\mathbf{G}$ is split with simply-connected derived subgroup.

Let $\mathbf{G}_1$ be the quotient of $\mathbf{G} \times \mathbf{Z}$ by the normal subgroup $\{(z, z^{-1}) : z \in \mathbf{Z_G}\}$, where $\mathbf{Z_G}$ is the center of $\mathbf{G}$, as in [DL76, 5.18]. Then the natural map $\mathbf{G} \to \mathbf{G}_1$ is a closed embedding that induces an isomorphism on derived subgroups. The natural map $\mathbf{Z} \to \mathbf{G}_1$ to the second coordinate induces an isomorphism $\mathbf{Z} \xrightarrow{\sim} \mathbf{Z_{G_1}}$. In particular, $\mathbf{G}_1$ is as in Theorem 6.2. It follows that $\mathbf{Z}_1 := \mathbf{Z} \cdot \mathbf{Z_{G_1}} = (\mathbf{Z} \times \mathbf{Z})/\{(z, z^{-1}) : z \in \mathbf{Z_G}\}$ is a minimal Levi (i.e. maximal F-torus) of $\mathbf{G}_1$. Let K_1 be the hyperspecial parahoric subgroup of G_1 fixing the special point x_0. Then we have $K = K_1 \cap G$, see Lemma A.15. We have (as in [Abe13, §3.2]):

Lemma 6.8. *The following hold:*

(i) *The restriction to K of any irreducible representation of K_1 is irreducible. Conversely, any irreducible representation V of K extends to K_1.*

(ii) *Let V_1, V_1' be irreducible representations of K_1 and V, V' their restrictions to K. Then the restriction map $\varphi_1 \mapsto \varphi_1|_G$ gives an isomorphism between $\{\varphi_1 \in \mathcal{H}_{G_1}(V_1, V_1') \mid \operatorname{supp}\varphi_1 \subset K_1 Z K_1\}$ and $\mathcal{H}_G(V, V')$. We have $S^G(\varphi_1|_G) = S^{G_1}(\varphi_1)|_Z$ for any $\varphi_1 \in \mathcal{H}_{G_1}(V_1, V_1')$ with $\operatorname{supp}\varphi_1 \subset K_1 Z K_1$. Moreover, we have $T_z^{V_1', V_1}|_G = T_z^{V', V}$ for any $z \in Z_G^+(V, V')$.*

Given α, V, V', $z \in Z_G^+(V, V')$ as in Theorem 6.2 we choose extensions V_1, V_1' of V, V' to K_1-representations and let φ_1, φ_1' denote the Hecke operators provided by Theorem 6.2 for G_1, V_1, V_1', z. Then, as the supports of τ_z, $\tau_z - \tau_{za_\alpha}$ are contained in $Z(Z_1 \cap K_1)$, we deduce from the lemma that the supports of φ_1, φ_1' are contained in $K_1 Z K_1$. Hence we can take $\varphi = \varphi_1|_G$, $\varphi' = \varphi_1'|_G$. Similarly, Corollary 6.6 continues to hold for $\mathbf{G}$.

(2) To remove the assumption on the derived subgroup, we use a z-extension. (See [CT08, §3] for more on z-extensions.) Suppose that $\mathbf{G}$ is any split reductive group. Choose a split z-extension $r\colon \widetilde{\mathbf{G}} \to \mathbf{G}$, i.e. an F-split group $\widetilde{\mathbf{G}}$ with simply connected derived subgroup which is a central extension of $\mathbf{G}$ and the kernel of r is an (F-split) torus. In particular, part (1) above applies to $\widetilde{\mathbf{G}}$. Set $\widetilde{\mathbf{Z}} = r^{-1}(\mathbf{Z})$; it is a maximal torus of $\widetilde{\mathbf{G}}$. Let $\widetilde{K} \subset \widetilde{G}$ be the special (maximal compact open) parahoric subgroup fixing x_0; the map $\widetilde{K} \to K$ is surjective [Abe13, Lemma 2.1], [HV15, §3.5].

Lemma 6.9. *Let V_1, V_2 be irreducible representations of K and denote by $\widetilde{V}_1, \widetilde{V}_2$ their inflations to $\widetilde{K}$. Then there exist algebra homomorphisms $\Theta_G : \mathcal{H}_{\widetilde{G}}(\widetilde{V}_1, \widetilde{V}_2) \to \mathcal{H}_G(V_1, V_2)$ and $\Theta_Z : \mathcal{H}_{\widetilde{Z}}((\widetilde{V}_1)_{U^0}, (\widetilde{V}_2)_{U^0}) \to \mathcal{H}_Z((V_1)_{U^0}, (V_2)_{U^0})$ such that*

(i) *$S^G \circ \Theta_G = \Theta_Z \circ S^{\widetilde{G}}$;*

(ii) *for $\widetilde{z} \in \widetilde{Z}^+$, $\Theta_G(T_{\widetilde{z}}^{\widetilde{V}_2, \widetilde{V}_1}) = T_z^{V_2, V_1}$ and $\Theta_Z(\tau_{\widetilde{z}}^{(\widetilde{V}_2)_{U^0}, (\widetilde{V}_1)_{U^0}}) = \tau_z^{(V_2)_{U^0}, (V_1)_{U^0}}$, where $z = r(\widetilde{z})$.*

To construct the algebra homomorphism Θ_G, we identify the category of representations of G with the category of representations of $\widetilde{G}$ trivial on the kernel of the surjective homomorphism $r : \widetilde{G} \to G$, and we note that Frobenius reciprocity (applied twice) induces a natural isomorphism $\operatorname{Hom}_G(\text{c-Ind}_K^G V, \sigma) \simeq \operatorname{Hom}_{\widetilde{G}}(\text{c-Ind}_{\widetilde{K}}^{\widetilde{G}} \widetilde{V}, \sigma)$ for representations σ of G (for any irreducible K-representation V with inflation $\widetilde{V}$). In particular we get a $\widetilde{G}$-linear map $j_V : \text{c-Ind}_{\widetilde{K}}^{\widetilde{G}} \widetilde{V} \to \text{c-Ind}_K^G V$ corresponding to the identity map. By Yoneda's lemma the above adjunction gives for any $\varphi \in \mathcal{H}_{\widetilde{G}}(\widetilde{V}_1, \widetilde{V}_2)$ a unique $\Theta_G(\varphi) \in \mathcal{H}_G(V_1, V_2)$ such that $j_{V_2} \circ \varphi = \Theta_G(\varphi) \circ j_{V_1}$. We leave the details of the end of the proof of the lemma to the reader.

The lemma shows that Theorem 6.2 holds even for G since it holds for $\widetilde{G}$: as $r : \widetilde{Z} \to Z$ is surjective, we can choose $\widetilde{z}$ with $r(\widetilde{z}) = z$. Suppose $\widetilde{\varphi}, \widetilde{\varphi}'$ are the Hecke operators provided by Theorem 6.2 for $\widetilde{G}$, $\widetilde{V}$, $\widetilde{V}'$, $\widetilde{z}$. Then we can take $\varphi = \Theta_G(\widetilde{\varphi})$, $\varphi' = \Theta_G(\widetilde{\varphi}')$. Similarly, Corollary 6.6 continues to hold for $\mathbf{G}$.

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Appendix A. A simple proof of the change of weight theorem for quasi-split groups

N. Abe and F. Herzig

The purpose of the appendix is to show that the simple proof of §6 extends to quasi-split groups.

Suppose that $\mathbf{G}$ is a quasi-split connected reductive group over F. As in §2.4, recall that if $\mathbf{H}$ is any connected reductive F-group, then H' denotes the subgroup of H generated by the unipotent radicals of all minimal parabolics. By Kneser–Tits (see e.g. [AHHV17, II.3 Prop.]) we know that $H' = H^{\mathrm{der}}$ if $\mathbf{H}^{\mathrm{der}}$ is simply connected with no anisotropic factors. (Note that the second condition is automatic if $\mathbf{H}$ is quasi-split.) Similarly we define H' for $\mathbf{H}$ connected reductive over k and know that $H' = H^{\mathrm{der}}$ if $\mathbf{H}^{\mathrm{der}}$ is simply connected.

We also recall that all special parahoric subgroups K in this paper are associated to special points in the apartment of S. We let $\mathrm{red} : K \twoheadrightarrow G_k$ denote the natural reduction map whose kernel is the pro-p radical (i.e. largest normal pro-p subgroup) of K.

Theorem A.1. *There exists a special parahoric subgroup K of G such that the following holds.*

Suppose that V, V' are irreducible representations of K and $\alpha \in \Delta$ such that $\psi_V = \psi_{V'}$ and $\Delta(V) = \Delta(V') \sqcup \{\alpha\}$, and let $z \in Z^+$ such that $\langle \alpha, v_Z(z) \rangle > 0$. Then there exist G-equivariant homomorphisms $\varphi : \text{c-Ind}_K^G V \to \text{c-Ind}_K^G V'$ and $\varphi' : \text{c-Ind}_K^G V' \to \text{c-Ind}_K^G V$ satisfying

$$S^G(\varphi) = \tau_z, \quad S^G(\varphi') = \tau_z - \tau_{za_\alpha}.$$

If moreover $\langle \beta, v_Z(z) \rangle = 0$ for $\beta \in \Delta(V')$, then $\varphi = T_z^{V',V}$ and $\varphi' = T_z^{V,V'}$.

Any choice of K works, provided the adjoint quotient $\mathbf{G}_{\mathrm{ad}}$ of $\mathbf{G}$ does not have a simple factor isomorphic to $\mathrm{Res}_{E/F} \mathrm{PU}(m+1, m)$ for some E/F finite separable and $m \geq 1$.

Remark A.2. This is enough to establish Theorems 1–3 of [AHHV17] for quasi-split $\mathbf{G}$, avoiding [AHHV17, §IV], since the proofs given there only require one choice of K.

Remark A.3. There exist quasi-split groups $\mathbf{G}$ and special parahoric subgroups K for which the conclusion of Theorem A.1 fails. We claim that it suffices to show that $\psi_V(Z^0 \cap M'_\alpha) \neq 1$ for some $\mathbf{G}$, K, V, α as in Theorem A.1. Under this condition, Theorem 2.12 tells us that the image $S^G(\mathcal{H}_G(V',V))$ has C-basis τ_z, where z runs through a system of representatives of $Z_G^+(V',V)/Z^0$ in $Z_G^+(V',V)$. If Theorem A.1 were true, then for $z \in Z_G^+(V',V)$ the element $\tau_z - \tau_{za_\alpha}$ would lie in $S^G(\mathcal{H}_G(V',V))$, so $za_\alpha \in Z_G^+(V',V)$. However, for large n we have $za_\alpha^n \notin Z^+$.

For example, if $\mathbf{G} = \mathrm{SU}(2,1)$ defined by a ramified separable quadratic extension of F, then we can choose K such that $\mathbf{G}_k \cong \mathrm{PGL}_2$ and if $\#k$ is odd, then $\mathrm{red}(Z^0 \cap M'_\alpha) = Z_k$ strictly contains $Z_k \cap M'_{\alpha,k}$ (where $\Delta = \{\alpha\}$). Or, suppose that $\mathbf{G} = \mathrm{SU}(2,1)$ defined by the unramified separable quadratic extension. Then for any non-hyperspecial K we have $\mathbf{G}_k \cong \mathrm{U}(1,1)$, and then $\mathrm{red}(Z^0 \cap M'_\alpha) = Z_k$ strictly contains $Z_k \cap M'_{\alpha,k}$ (where $\Delta = \{\alpha\}$). In either case we can therefore choose V such that $\psi_V(Z^0 \cap M'_\alpha) \neq 1$.

A.1. **On special parahoric subgroups.**

Proposition A.4. *There exists a special parahoric subgroup K of G such that for any $\alpha \in \Delta$ the image of $M'_\alpha \cap K$ in G_k is equal to $M'_{\alpha,k}$. Any choice of K works, provided the adjoint group $\mathbf{G}_{\mathrm{ad}}$ does not have a simple factor isomorphic to $\mathrm{Res}_{E/F}\,\mathrm{PU}(m+1,m)$ for some E/F finite separable and $m \geq 1$.*

Proof. Step 1: We show that for any quasi-split $\mathbf{G}$ such that $\mathbf{G}^{\mathrm{der}}$ simply connected we can choose a special parahoric subgroup K such that $\mathrm{red}(G' \cap K) = G'_k$.

Since $\mathbf{G}$, and hence $\mathbf{M}_\alpha$, have simply-connected derived subgroups and $\mathbf{G}$ is quasi-split, we know that $G' = G^{\mathrm{der}}$ and $M'_\alpha = M_\alpha^{\mathrm{der}}$. Note that the pro-$p$ radical of $G' \cap K = G^{\mathrm{der}} \cap K$ is normal in K and hence contained in the pro-p radical of K. Hence we obtain a commutative diagram with injective horizontal arrows as follows:

$$\begin{tikzcd} G^{\mathrm{der}} \cap K \arrow[r, hook] \arrow[d, two heads] & K \arrow[d, two heads] \\ (G^{\mathrm{der}})_k \arrow[r, hook] & G_k \end{tikzcd}$$

Note that the bottom map induces an isomorphism $(G^{\mathrm{der}})'_k \xrightarrow{\sim} G'_k$ (since U and U_{op} are contained in G^{der}). It thus suffices to show that the inclusion $(G^{\mathrm{der}})'_k \subset (G^{\mathrm{der}})_k$ is an equality, and hence it's enough to show that $(\mathbf{G}^{\mathrm{der}})_k$ is semisimple and simply connected (for a suitable choice of K).

Note in the following that our choice of special K is given by a subset $X \subset \Delta_{\mathrm{loc}}$ of the relative local Dynkin diagram of $\mathbf{G}$ [Tit79, 1.11], or equivalently of $\mathbf{G}^{\mathrm{der}}$, consisting of one special vertex in each component of Δ_{loc}. (We write Δ_{loc}, $\Delta_{1,\mathrm{loc}}$ instead of Δ, Δ_1 in [Tit79] in order to avoid confusion.)

We first determine for which K we have that $(\mathbf{G}^{\mathrm{der}})_k$ is semisimple. The absolute rank of $(\mathbf{G}^{\mathrm{der}})_k$ is the relative rank of $\mathbf{G}^{\mathrm{der}}$ over the maximal unramified extension, i.e., it's $|\Delta_{1,\mathrm{loc}}|$ minus the number of components of $\Delta_{1,\mathrm{loc}}$. On the other hand, the absolute semisimple rank of $(\mathbf{G}^{\mathrm{der}})_k$ equals the number of absolute simple roots of $(\mathbf{G}^{\mathrm{der}})_k$, i.e., the cardinality of $\Delta_{1,\mathrm{loc}} - \cup_{v \in X} O(v)$ in the notation of Tits, by [Tit79, 3.5.2]. It thus suffices to show that for any $v \in X$, $O(v)$ contains precisely one point of each component of $\Delta_{1,\mathrm{loc}}$ (it always contains at least one).

Looking at the tables in [Tit79] and keeping in mind the reduction steps to the absolutely almost simple case in [Tit79, 1.12], we see that any choice of X works, as long as it does not contain any non-hyperspecial vertices in type ${}^2A'_{2m}$ (in which case we can take the hyperspecial ones). In other words, we can always choose a special parahoric K such that $(\mathbf{G}^{\mathrm{der}})_k$ is semisimple, and any K works in case the adjoint group $\mathbf{G}_{\mathrm{ad}}$ does not have a simple factor isomorphic to $\mathrm{Res}_{E/F}\,\mathbf{H}$, where $\mathbf{H} \cong \mathrm{PU}(m+1, m)$ is unramified and E/F is finite separable.

Next we recall from [Tit79, §3.5] that, since $\mathbf{G}^{\mathrm{der}}$ is semisimple and simply connected, the residual group $(\mathbf{G}^{\mathrm{der}})_k$ has simply connected derived subgroup, provided we let K correspond to a subset X satisfying the condition in the last sentence of [Tit79, §3.5], i.e. $\cup_{v\in X} O(v)$ contains a "good special vertex" out of each connected component of $\Delta_{1,\mathrm{loc}}$. Note that by Tits' tables this is always possible (in fact even if $\mathbf{G}$ isn't quasi-split). Now note from Tits' tables that when $\mathbf{G}$ is quasi-split, his condition on X is always satisfied, except when $\mathbf{G}_{\mathrm{ad}}$ has a factor of type ${}^2A^{(1)}_{2m,m}$ and the special vertex at the long end is chosen. (In other words, $\mathbf{G}_{\mathrm{ad}}$ has a simple factor isomorphic to $\mathrm{Res}_{E/F}\,\mathbf{H}$, where $\mathbf{H} \cong \mathrm{PU}(m+1, m)$ is ramified and E/F is finite separable.) In this case we choose the special vertex at the other end.

By combining the above, we see that we can always choose a special parahoric K such that $(\mathbf{G}^{\mathrm{der}})_k$ is semisimple simply connected (and hence $\mathrm{red}(G' \cap K) = G'_k$), and any K works in case the adjoint group $\mathbf{G}_{\mathrm{ad}}$ does not have a simple factor isomorphic to $\mathrm{Res}_{E/F}\,\mathrm{PU}(m+1, m)$ and E/F is finite separable (or equivalently when the root system Φ is reduced).

Step 2: We prove the proposition in the case where $\mathbf{G}^{\mathrm{der}}$ *simply connected.*

From Step 1 we know that $\mathrm{red}(M'_\alpha \cap K) = M'_\alpha$ for $\alpha \in \Delta$, provided $\mathbf{M}_{\alpha,\mathrm{ad}}$ isn't isomorphic to $\mathrm{Res}_{E/F}\,\mathrm{PU}(2, 1)$ for some E/F. By considering indices of quasi-split groups, for example in [Tit66], it follows that there is at most one exceptional α in each component of Δ, namely the exceptional α are precisely the multipliable simple roots in components of Δ of type BC_r.

Suppose first that $\mathbf{G}^{\mathrm{der}}$ is almost simple, and suppose that there is an exceptional $\alpha \in \Delta$, i.e. $\mathbf{M}_{\alpha,\mathrm{ad}} \cong \mathrm{Res}_{E/F}\,\mathrm{PU}(2, 1)$ for some E/F. Then the choice of a special point for $\mathbf{M}_\alpha$ coming from Step 1 corresponds to a choice of α-wall H_α in the reduced building of G. (By α-wall we just mean an affine hyperplane parallel to $\ker(\alpha)$.) Now choose arbitrary β-walls H_β for $\beta \in \Delta - \{\alpha\}$. Then the special parahoric subgroup defined by the special point $\cap_{\beta\in\Delta} H_\beta$ works for this proposition.

In general the reduced apartment of G (for S) is a product of reduced apartments for all the almost simple factors of G^{der}, and we obtain a desired special point by taking a product of special points that work for the almost simple factors (previous paragraph).

Step 3: We deduce the proposition in general.

Suppose that $\mathbf{G}$ is any quasi-split group. Pick a z-extension $\pi : \widetilde{\mathbf{G}} \to \mathbf{G}$ of $\mathbf{G}$. Then $\widetilde{\mathbf{G}}$ and $\mathbf{G}$ have the same reduced building, and by Step 1 above we can choose a special point x corresponding to a special parahoric $\widetilde{K}$ of $\widetilde{G}$ such that $\mathrm{red}(\widetilde{G}' \cap \widetilde{K}) = \widetilde{G}'_k$. We will show that $\mathrm{red}(G' \cap K) = G'_k$. The argument showing that $\mathrm{red}(\widetilde{M}'_\alpha \cap \widetilde{K}) = \widetilde{M}'_{\alpha,k}$ implies $\mathrm{red}(M'_\alpha \cap K) = M'_{\alpha,k}$ is completely analogous.

We have $\pi(\widetilde{G}') = G'$. If K denotes the special parahoric of G corresponding to x, then we also have $\pi(\widetilde{K}) = K$ (see part (d) of the proof of [HR08, Proposition 3]). We claim that $\pi(\widetilde{G}' \cap \widetilde{K}) = G' \cap K$. Suppose that $g \in G' \cap K$ and pick $\widetilde{g} \in \widetilde{G}'$ such that $\pi(\widetilde{g}) = g$. Then $\widetilde{g}$ fixes the special point x and it is in the kernel of the Kottwitz homomorphism (since $\widetilde{G}'$ is contained in that kernel). Hence $\widetilde{g} \in \widetilde{K}$, proving the claim. Similarly we see that $\pi(\widetilde{U} \cap \widetilde{K}) = U \cap K$ and $\pi(\widetilde{U}_{\mathrm{op}} \cap \widetilde{K}) = U_{\mathrm{op}} \cap K$.

Now note that the image under π of the pro-p radical of $\widetilde{K}$ is contained in the pro-p radical of K. Hence we get a commutative diagram

$$\begin{tikzcd} \widetilde{K} \arrow[r, two heads, "\pi"] \arrow[d, "\mathrm{red}"'] & K \arrow[d, "\mathrm{red}"] \\ \widetilde{G}_k \arrow[r, two heads, "\overline{\pi}"] & G_k \end{tikzcd}$$

and by the previous paragraph we see that $\overline{\pi}(\widetilde{G}'_k) = G'_k$. It follows that $\mathrm{red}(G' \cap K) = \mathrm{red}(\pi(\widetilde{G}' \cap \widetilde{K})) = \overline{\pi}(\mathrm{red}(\widetilde{G}' \cap \widetilde{K})) = \overline{\pi}(\widetilde{G}'_k) = G'_k$. □

Remark A.5. Surely the map $(G^{\mathrm{der}})_k \to G_k$ in Step 1 of the proof arises from a closed immersion $(\mathbf{G}^{\mathrm{der}})_k \to \mathbf{G}_k$ of algebraic groups, but we do not know a reference.

Corollary A.6. *For any K for which Proposition A.4 holds, we have that* $\mathrm{red}(Z^0 \cap M'_\alpha) = Z_k \cap M'_{\alpha,k}$ *for any* $\alpha \in \Delta$.

Proof. Choose K as in Proposition A.4. Let $K(1) := \ker(K \to G_k)$. Then $Z^0K(1) = \mathrm{red}^{-1}(Z_k)$ and we deduce by the proposition that $Z_k \cap M'_{\alpha,k} = \mathrm{red}(Z^0K(1) \cap M'_\alpha) = \mathrm{red}(Z^0 \cap M'_\alpha)$, noting that we have an Iwahori decomposition $M_\alpha \cap K(1) = (Z \cap K(1))(U_\alpha \cap K(1))(U_{-\alpha} \cap K(1))$ and that U_α, $U_{-\alpha}$ are contained in M'_α. □

A.2. **Setup for the proof of Theorem A.1.** *In Sections A.2–A.4 we will assume that* $\mathbf{G}^{\mathrm{der}}$ *is simply connected and* $\mathbf{G}/\mathbf{G}^{\mathrm{der}}$ *is coflasque.* In Section A.5 we will reduce the general case to that one by using a suitable z-extension.

We recall that an F-torus $\mathbf{T}$ is said to be *coflasque* if we have $H^1(F', X^*(\mathbf{T})) = 0$ for all finite separable extensions F'/F [CT08, §0.8]. Note that any induced torus is coflasque. We remark that if $\mathbf{T}$ is coflasque, then $H^1(F'', X^*(\mathbf{T})) = 0$ for any separable algebraic extension F''/F (because by inflation-restriction it equals $H^1(F'' \cap F(\mathbf{T}), X^*(\mathbf{T}))$, where $F(\mathbf{T})$ is the splitting field of $\mathbf{T}$).

We now observe that our assumptions on $\mathbf{G}$ imply that $\mathbf{Z}$ is a coflasque torus since (i) $\mathbf{Z} \cap \mathbf{G}^{\mathrm{der}}$ is an induced torus because $\mathbf{G}^{\mathrm{der}}$ is simply connected and $\mathbf{G}$ is quasi-split, and (ii) any extension of a coflasque torus by an induced torus is split (by Shapiro's lemma).

Let $\Gamma_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ with inertia subgroup I_F and σ a topological generator of Γ_F/I_F. Let L denote the fixed field of I_F, i.e. the maximal unramified extension of F. Let Φ^{abs} (resp. Δ^{abs}) denote the set of absolute (resp. absolute simple) roots.

Lemma A.7. *Under the above assumptions, we have:*

(i) *the group $X_*(\mathbf{Z})_{I_F}$ is torsion-free;*
(ii) *the group $\Lambda = Z/Z^0$ is a finite free $\mathbb{Z}$-module;*
(iii) *any special parahoric K of G is maximal compact.*

Proof. We first show that if Γ is a profinite group acting smoothly on a finite free $\mathbb{Z}$-module X, then the finite groups $H^1(\Gamma, X)$ and $\mathrm{Hom}_\Gamma(X, \mathbb{Z})_{\mathrm{tor}}$ are dual. By inflation-restriction, as X is torsion-free, we reduce to the case where Γ is finite (replacing Γ with the finite quotient that acts faithfully on X). As $H^1(\Gamma, X) = \hat{H}^1(\Gamma, X)$ and $\mathrm{Hom}_\Gamma(X, \mathbb{Z})_{\mathrm{tor}} = \hat{H}^{-1}(\Gamma, \mathrm{Hom}(X, \mathbb{Z}))$, we conclude by [NSW00, Prop. 3.1.2].

For our coflasque torus $\mathbf{Z}$ we conclude that $(X_*(\mathbf{Z})_{I_F})_{\mathrm{tor}} = 0$, as it is dual to $H^1(I_F, X^*(\mathbf{Z}))$. Hence $\Lambda \cong X_*(\mathbf{Z})_{I_F}^\sigma$ [HR10, Cor. 11.1.2] is a finite free $\mathbb{Z}$-module. This implies that any K is maximal compact [HR10, Prop. 11.1.4]. □

By [Kot97, §7.2] we have a σ-equivariant commutative diagram

$$\begin{tikzcd} \mathbf{Z}(L) \arrow[r, two heads, "w_{\mathbf{Z}}"] \arrow[dr, "v_{\mathbf{Z}}"'] & X_*(\mathbf{Z})_{I_F} \arrow[d, "q_{\mathbf{Z}}"] \\ & \mathrm{Hom}(X^*(\mathbf{Z})^{I_F}, \mathbb{Z}), \end{tikzcd} \tag{A.1}$$

where $q_{\mathbf{Z}}([\lambda])(\mu) = \langle \lambda, \mu \rangle$ and $v_{\mathbf{Z}}(z)(\mu) = \mathrm{ord}_F(\mu(z))$ (where the valuation ord_F is normalized so that $\mathrm{ord}_F(F^\times) = \mathbb{Z}$). By Lemma A.7(i) and [Kot97, §7.2], $q_{\mathbf{Z}}$ is an isomorphism. Since the composite map $j : X_*(\mathbf{Z})^{I_F} \hookrightarrow X_*(\mathbf{Z}) \twoheadrightarrow X_*(\mathbf{Z})_{I_F}$ becomes an isomorphism after $\otimes\mathbb{Q}$, we get a σ-equivariant isomorphism $(q_{\mathbf{Z}} \circ j) \otimes \mathbb{R} : (X_*(\mathbf{Z}) \otimes \mathbb{R})^{I_F} \xrightarrow{\sim} \mathrm{Hom}(X^*(\mathbf{Z})^{I_F}, \mathbb{R})$. Let $\omega : \mathrm{Hom}(X^*(\mathbf{Z})^{I_F}, \mathbb{Z}) \hookrightarrow (X_*(\mathbf{Z}) \otimes \mathbb{R})^{I_F}$ denote the restriction of the inverse of $(q_{\mathbf{Z}} \circ j) \otimes \mathbb{R}$ to the lattice $\mathrm{Hom}(X^*(\mathbf{Z})^{I_F}, \mathbb{Z})$.

By taking σ-invariants in diagram (A.1) composed with ω we obtain

$$\begin{tikzcd} Z \arrow[r, two heads, "w_Z"] \arrow[dr, "v_Z"'] & X_*(\mathbf{Z})^{\sigma}_{I_F} \arrow[d, hook, "q_Z"] \\ & (X_*(\mathbf{Z}) \otimes \mathbb{R})^{\Gamma_F} = X_*(\mathbf{S}) \otimes \mathbb{R}, \end{tikzcd}$$

where w_Z is the Kottwitz homomorphism and v_Z is as in §2.1. Explicitly, for $\lambda \in X_*(\mathbf{Z})$,

$$(\omega \circ q_{\mathbf{Z}})([\lambda]) = \frac{1}{\#(I_F \cdot \lambda)} \sum_{\lambda' \in I_F \cdot \lambda} \lambda' \in (X_*(\mathbf{Z}) \otimes \mathbb{R})^{I_F}. \tag{A.2}$$

A root $\alpha \in \Phi$ determines a finite separable extension F_α/F: it is the fixed field of the stabilizer of any lift $\widetilde{\alpha} \in \Phi^{\mathrm{abs}}$. (All lifts are Γ_F-conjugate, so the choice doesn't matter. Cf. [BT84, 4.1.3].) Let $\varepsilon_\alpha = e(F_\alpha/F)$ denote the ramification degree.

Lemma A.8. *The image of $Z \cap M'_\alpha$ in Λ is a direct summand. Its image under v_Z in $X_*(\mathbf{S}) \otimes \mathbb{R}$ is identified with $\mathbb{Z} \cdot \frac{1}{\varepsilon_\alpha} \alpha_0^\vee$, where α_0 is the greatest multiple of α that is contained in Φ.*

Proof. Note that $X_*(\mathbf{Z} \cap \mathbf{M}_\alpha^{\mathrm{der}})$ is a permutation module (a basis is given by all absolute simple coroots that restrict to α), i.e. $\mathbf{Z} \cap \mathbf{M}_\alpha^{\mathrm{der}}$ is an induced torus. Similarly, $(\mathbf{Z} \cap \mathbf{G}^{\mathrm{der}})/(\mathbf{Z} \cap \mathbf{M}_\alpha^{\mathrm{der}})$ and $\mathbf{Z} \cap \mathbf{G}^{\mathrm{der}}$ are induced tori. Therefore, as $\mathbf{Z}/(\mathbf{Z} \cap \mathbf{G}^{\mathrm{der}})$ is coflasque by assumption, we deduce that $\mathbf{Z}/(\mathbf{Z} \cap \mathbf{M}_\alpha^{\mathrm{der}})$ is coflasque and hence that the sequence $1 \to \mathbf{Z} \cap \mathbf{M}_\alpha^{\mathrm{der}} \to \mathbf{Z} \to \mathbf{Z}/(\mathbf{Z} \cap \mathbf{M}_\alpha^{\mathrm{der}}) \to 1$ is split exact. The natural map $j : Z \cap M_\alpha^{\mathrm{der}} \to Z$ is compatible with the induced map $j_* : X_*(\mathbf{Z} \cap \mathbf{M}_\alpha^{\mathrm{der}})^\sigma_{I_F} \to X_*(\mathbf{Z})^\sigma_{I_F}$ with respect to the functorial Kottwitz maps $w_{Z \cap M_\alpha^{\mathrm{der}}}$, w_Z. The map j_* is clearly a split injection of finite free $\mathbb{Z}$-modules.

As $X_*(\mathbf{Z} \cap \mathbf{M}_\alpha^{\mathrm{der}})$ has $\mathbb{Z}$-basis all $\widetilde{\alpha} \in \Phi^{\mathrm{abs}}$ lifting α, the image of $X_*(\mathbf{Z} \cap \mathbf{M}_\alpha^{\mathrm{der}})^\sigma_{I_F}$ in $X_*(\mathbf{Z})^\sigma_{I_F}$ is generated by $[\sum_{\Phi'} \widetilde{\alpha}^\vee] \in X_*(\mathbf{Z})^\sigma_{I_F}$, where $\Phi' \subset \Phi^{\mathrm{abs}}$ is a set of representatives for the I_F-orbits on the set of roots lifting α. Using (A.2) we see that it is identified with $\frac{1}{\varepsilon_\alpha} \sum \widetilde{\alpha}^\vee$ in $X_*(\mathbf{S}) \otimes \mathbb{R}$, where $\widetilde{\alpha} \in \Delta^{\mathrm{abs}}$ now runs through all lifts of α. By the lemma below this is equal to $\frac{1}{\varepsilon_\alpha} \alpha_0^\vee$. $\square$

Lemma A.9. *Let us drop temporarily all assumptions in §A.2 about $\mathbf{G}$, and only assume that it is a quasi-split connected reductive F-group. Suppose that $\alpha \in \Delta$. Then $\alpha_0^\vee = \sum \widetilde{\alpha}^\vee$ in $X_*(\mathbf{Z})$, where the sum is over all lifts $\widetilde{\alpha}$ of α in Φ^{abs}.*

Proof. We may replace $\mathbf{G}$ with $\mathbf{M}_\alpha^{\mathrm{der}}$ and hence assume that $\mathbf{G}$ is semisimple and $\Delta = \{\alpha\}$. Then $\Delta^{\mathrm{abs}} = \{\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_n\}$ for the lifts $\widetilde{\alpha}_i$ of α in Φ^{abs} and the cocharacters $\widetilde{\alpha}_i^\vee$ span $X_*(\mathbf{Z})\otimes\mathbb{Q}$. In particular, as Γ_F acts transitively on Δ^{abs}, we see that $\alpha^\vee = c\sum\widetilde{\alpha}_i^\vee$ for some constant $c\in\mathbb{Q}$. Note that $2\alpha\in\Phi$ if and only if $\widetilde{\alpha}_1+\widetilde{\alpha}_i\in\Phi^{\mathrm{abs}}$ for some $i>1$ if and only if $\langle\widetilde{\alpha}_1,\widetilde{\alpha}_i^\vee\rangle<0$ (hence equal to -1) for some $i>1$.

If $2\alpha\notin\Phi$, then the $\widetilde{\alpha}_i$ are pairwise orthogonal and $\langle\alpha,\alpha^\vee\rangle = 2$ yields $c=1$. Otherwise, since Γ_F acts transitively on Δ^{abs} and the Dynkin diagram has no loops, it follows that $\langle\widetilde{\alpha}_1,\widetilde{\alpha}_i^\vee\rangle = -1$ for a unique $i>1$. Then $\langle\alpha,\alpha^\vee\rangle = 2$ yields $c=2$. ☐

Remark A.10. Lemma A.8, together with [AHHV17, III.16 Notation], shows that $v_Z(a_\alpha) = -\frac{1}{\varepsilon_\alpha}\alpha_0^\vee$. Recall that in §2.4 we also defined integers e_α. By comparing with [AHHV17, IV.11 Example 3] we deduce that $e_\alpha = 2\varepsilon_\alpha$ if $2\alpha\in\Phi$ and $e_\alpha=\varepsilon_\alpha$ otherwise. Alternatively, we can see this by comparing [BT84, 4.2.21] with [Vig16, (39)].

A.3. **Basic case.** *We assume that* $1\to\mathbf{Z_G}\to\mathbf{Z}\to\mathbf{Z}/\mathbf{Z_G}\to 1$ *is a split exact sequence of* F*-tori.* In particular, the center $\mathbf{Z_G}$ of $\mathbf{G}$ is a torus. We continue to assume that $\mathbf{G}^{\mathrm{der}}$ is simply connected and $\mathbf{G}/\mathbf{G}^{\mathrm{der}}$ is coflasque, as in §A.2.

Suppose that K is any special parahoric subgroup for which Proposition A.4 holds.

Fix an F-splitting $\theta:\mathbf{Z}\twoheadrightarrow\mathbf{Z_G}$ of the exact sequence $1\to\mathbf{Z_G}\to\mathbf{Z}\to\mathbf{Z}/\mathbf{Z_G}\to 1$. Since $X^*(\mathbf{Z}/\mathbf{Z_G}) = \oplus_{\Delta^{\mathrm{abs}}}\mathbb{Z}\widetilde{\alpha}$, we have a *canonical* absolute fundamental coweight $\lambda_{\widetilde{\beta}}\in X_*(\mathbf{Z})$ for any $\widetilde{\beta}\in\Delta^{\mathrm{abs}}$, normalized by demanding that it be orthogonal to $\theta^*X^*(\mathbf{Z_G})$. These are permuted by the action of Γ_F. Thus for any simple root $\beta\in\Delta$ we obtain a *canonical* relative fundamental coweight $\lambda_\beta\in X_*(\mathbf{S}) = X_*(\mathbf{Z})^{\Gamma_F}$ by taking the sum of $\lambda_{\widetilde{\beta}}\in X_*(\mathbf{Z})$ for all lifts $\widetilde{\beta}\in\Delta^{\mathrm{abs}}$ of β. (It is the unique fundamental coweight for β that is orthogonal to $\theta^*X^*(\mathbf{Z_G})$.)

Lemma A.11. *We have* $\Lambda = \mathbb{Z}\frac{1}{\varepsilon_\alpha}\lambda_\alpha\oplus\ker\alpha$ *inside* $X_*(\mathbf{S})\otimes\mathbb{R}$.

Proof. Note that $X_*(\mathbf{Z}) = \bigoplus\mathbb{Z}\lambda_{\widetilde{\beta}}\oplus(\mathbb{Z}\Phi^{\mathrm{abs}})^\perp$, where $\widetilde{\beta}$ runs through Δ^{abs}. It follows that $X_*(\mathbf{Z})_{I_F}^\sigma$ is the direct sum of $\mathbb{Z}[\sum_{\Phi'}\lambda_{\widetilde{\alpha}}]$, where Φ' is as in the proof of Lemma A.8, and a module that is orthogonal to α. As in the proof of Lemma A.8 we see that $[\sum\lambda_{\widetilde{\alpha}}]$ is identified with $\frac{1}{\varepsilon_\alpha}\lambda_\alpha\in X_*(\mathbf{S})\otimes\mathbb{R}$. ☐

As $\alpha\in\Delta(V)$, Corollary A.6 shows that $\psi_V(Z^0\cap M'_\alpha) = 1$. In particular, $\tau_\alpha\in\mathcal{H}_Z(\psi_V)$ is well-defined.

Lemma A.12. *The element* $1-\tau_\alpha$ *of* $\mathcal{H}_Z(\psi_V)$ *is irreducible.*

Proof. As the character $\psi_V: Z^0\to C^\times$ is trivial on $Z^0\cap M'_\alpha$, we can extend it to a character $\eta: Z\to C^\times$ that is trivial on $Z\cap M'_\alpha$. We get an isomorphism $\iota:\mathcal{H}_Z(\psi_V)\xrightarrow{\sim}\mathcal{H}_Z(1) = C[\Lambda]$, defined by $\iota(f)(z) = \eta(z)^{-1}f(z)$ for $z\in Z$. In particular, $\iota(\tau_z) = \eta(z)^{-1}\tau_z$. Thus it suffices to show that $\iota(1-\tau_\alpha) = 1-\tau_{a_\alpha}$ is irreducible in $C[\Lambda]$. By Lemma A.8 and freeness of Λ we can extend $x_1 := a_\alpha$ to a $\mathbb{Z}$-basis $x_1,\dots,x_r$ of Λ. Obviously, $1-x_1$ is irreducible in $C[x_1^{\pm1},\dots,x_r^{\pm1}]$. ☐

Recall that for any $z\in Z^+$ with $\langle\alpha,z\rangle>0$ we have intertwining operators $T_z^{V',V}$: c-$\mathrm{Ind}_K^G V\to$ c-$\mathrm{Ind}_K^G V'$ and $T_z^{V,V'}$: c-$\mathrm{Ind}_K^G V'\to$ c-$\mathrm{Ind}_K^G V$ supported on the double coset KzK.

Proposition A.13. *Suppose $z\in Z$ such that $v_Z(z) = \frac{1}{\varepsilon_\alpha}\lambda_\alpha$. Then $S^G(T_z^{V',V}) = \tau_z$ and $S^G(T_z^{V,V'}) = \tau_z(1-\tau_\alpha)$ in $\mathcal{H}_Z(\psi_V)$.*

Proof. We have that $S^G(T_z^{V',V}) = \tau_z$ by Lemma 3.1 and the coefficient of τ_z in $S^G(T_z^{V,V'})$ is 1. It thus suffices to show that $\psi \in C\tau_{z^2}(1-\tau_{\frac{1}{\varepsilon_\alpha}\alpha_0^\vee})$, where $\psi = S^G(T_z^{V,V'} * T_z^{V',V}) \in \mathcal{H}_Z(\psi_V)$.

Pick any algebra homomorphism $\chi : \mathcal{H}_Z(\psi_V) \to C$. Then as in §6.2 we know that the character $\sigma_\chi := \chi \otimes_{\mathcal{H}_Z(\psi_V)} \text{c-Ind}_{Z^0}^Z \psi_V$ of Z is given by $z \mapsto \chi(\tau_{z^{-1}})$, and that the restriction of σ_χ to Z^0 equals ψ_V. Assume now that $\chi(\tau_\alpha) = 1$. We know that σ_χ is trivial on the image of $Z^0 \cap M'_\alpha$ by above. Moreover, $Z \cap M'_\alpha$ is generated by $Z^0 \cap M'_\alpha$ and a_α, so σ_χ is trivial on $Z \cap M'_\alpha$, as $\sigma_\chi(a_\alpha) = \chi(\tau_\alpha^{-1}) = 1$. As $M_\alpha = \langle Z, U_{\pm\alpha}\rangle$, we have an isomorphism $Z/(Z \cap M'_\alpha) \cong M_\alpha/M'_\alpha$, so σ_χ extends to a smooth character of M_α, which we still denote by σ_χ. By Frobenius reciprocity, the induced representation $\text{Ind}_{P_\alpha}^G \sigma_\chi$ contains V but not V', and the Hecke eigenvalues of V in $\text{Ind}_{P_\alpha}^G \sigma_\chi$ are given by χ via S^G (see Lemma 6.4 and the proof of Lemma 6.5). As in §6.3 we deduce that $\chi(\psi) = 0$.

We saw that $\chi(1-\tau_\alpha) = 0$ implies that $\chi(\psi) = 0$. By the Nullstellensatz we get that ψ is contained in the radical of the ideal $(1-\tau_\alpha)$, hence by Lemma A.12 and the fact that $\mathcal{H}_Z(\psi_V)(\approx C[\Lambda])$ is a UFD, we see that $\psi = \psi'(1-\tau_\alpha)$ for some $\psi' \in \mathcal{H}_Z(\psi_V)$.

As in §6.3, by Lemma 2.9, we now see that if $z' \in Z$ is in the support of ψ', then

$$z' \in Z^+,\ z'a_\alpha \in Z^+; \tag{A.3}$$

$$v_Z(z') \le_{\mathbb{R}} \tfrac{2}{\varepsilon_\alpha}\lambda_\alpha,\ v_Z(z'a_\alpha) \le_{\mathbb{R}} \tfrac{2}{\varepsilon_\alpha}\lambda_\alpha. \tag{A.4}$$

(This follows since for $z' \in \operatorname{supp}\psi$ we have $z' \in Z^+$ and $v_Z(z') \le_{\mathbb{R}} \frac{2}{\varepsilon_\alpha}\lambda_\alpha$.) From (A.4) we can write

$$v_Z(z') = \tfrac{2}{\varepsilon_\alpha}\lambda_\alpha - \textstyle\sum_\Delta n_\beta\beta^\vee \tag{A.5}$$

for some $n_\beta \in \mathbb{R}_{\ge 0}$. Hence by Remark A.10,

$$v_Z(z'a_\alpha) = \tfrac{2}{\varepsilon_\alpha}\lambda_\alpha - \tfrac{1}{\varepsilon_\alpha}\alpha_0^\vee - \textstyle\sum_\Delta n_\beta\beta^\vee. \tag{A.6}$$

For $\gamma \in \Delta - \{\alpha\}$ we pair (A.5) with γ and deduce that $\sum_\Delta n_\beta\langle\gamma,\beta^\vee\rangle \le 0$.

Case 1: $2\alpha \notin \Phi$, *so* $\alpha_0^\vee = \alpha^\vee$. We pair (A.6) with α and deduce that $\sum_\Delta n_\beta\langle\alpha,\beta^\vee\rangle \le 0$. Hence as in §6.3 we get that $n_\beta = 0$ for all $\beta \in \Delta$, so ψ' is a scalar multiple of τ_{z^2}, as required.

Case 2: $2\alpha \in \Phi$, *so* $\alpha_0^\vee = \frac{1}{2}\alpha^\vee$. The above proof goes through, provided we show

$$\langle\alpha, v_Z(z')\rangle \ge \tfrac{1}{\varepsilon_\alpha},\ \langle\alpha, v_Z(z'a_\alpha)\rangle \ge \tfrac{1}{\varepsilon_\alpha} \tag{A.7}$$

for any $z' \in \operatorname{supp}\psi'$. For this it is enough to show that $\langle\alpha, v_Z(z')\rangle \ge \frac{1}{\varepsilon_\alpha}$ for any $z' \in \operatorname{supp}\psi$. As $S^G(T_z^{V',V}) = \tau_z$ by Lemma 3.1 it suffices to show that $\langle\alpha, v_Z(z')\rangle \ge 0$ for any $z' \in \operatorname{supp} S^G(T_z^{V,V'})$. In fact, we will show that $\langle\alpha, v_Z(z')\rangle \ge 0$ for any $z' \in \operatorname{supp} S^G(\varphi)$ and any $\varphi \in \mathcal{H}_G(V_1,V_2)$ (where V_1, V_2 are irreducible representations of K).

By [HV15, §7.9], it suffices to show that $z'^{-1}(U_\alpha \cap K)z'$ is a proper subgroup of $U_\alpha \cap K_+$ for $z' \in Z$ such that $\langle\alpha, z'\rangle < 0$. Using notation as in [HV15, §6] we can write $z'^{-1}(U_\alpha \cap K)z' = U_{\alpha,g(\alpha)-\langle\alpha,z'\rangle}U_{2\alpha,g(2\alpha)-2\langle\alpha,z'\rangle}$ and $U_\alpha \cap K_+ = U_{\alpha,g^*(\alpha)}U_{2\alpha,g^*(2\alpha)}$. Recall that $g^*(\beta) = g(\beta)_+$ if a jump occurs in the $U_{\beta,u}$-filtration (modulo $U_{2\beta}$ if 2β is a root) at $u = g(\beta)$ and $g^*(\beta) = g(\beta)$ otherwise. Also note the set of jumps of the $U_{\beta,u}$-filtration (modulo $U_{2\beta}$) are invariant under shifts by $\langle\beta, z'\rangle$ (as Z acts on the apartment with all its structures). For any fixed $\beta \in \{\alpha, 2\alpha\}$ it follows that $U_{\beta,g(\beta)-\langle\beta,z'\rangle} \subset U_{\beta,g^*(\beta)}$ and if equality holds, then the $U_{\beta,u}$-filtration (modulo $U_{2\beta}$) jumps precisely at the elements $u \in g(\beta) + \langle\beta,z'\rangle\mathbb{Z}$. Thus $z'^{-1}(U_\alpha \cap K)z' \subset U_\alpha \cap K_+$ and if equality holds, then the $U_{\beta,u}$-filtration (modulo $U_{2\beta}$) jumps precisely at the elements $u \in g(\beta) + \langle\beta,z'\rangle\mathbb{Z}$ for $\beta \in \{\alpha, 2\alpha\}$; in particular, $g(2\alpha) = 2g(\alpha)$ from the definition of g.

By [BT84, 4.2.21] the jumps in the $U_{2\alpha,u}$-filtration occur when $u \in \operatorname{ord}_F(F_\alpha^0 - \{0\})$ and in the $U_{\alpha,u}$-filtration (modulo $U_{2\alpha}$) occur when $u \in \frac{1}{2}\operatorname{ord}_F(\ell) + \operatorname{ord}_F(F_\alpha^\times)$. Here, F_α^0 denotes the elements of F_α that are of trace 0 in the separable quadratic extension $F_\alpha/F_{2\alpha}$, $\ell \in F_\alpha$ denotes an element of trace 1 of maximum possible valuation. Note that $F_\alpha^0 - \{0\}$ is principal homogeneous under the $F_{2\alpha}^\times$-action, so the spacing of the jumps in the $U_{2\alpha,u}$-filtration is $\operatorname{ord}_F(F_{2\alpha}^\times)$. The spacing of the jumps in the $U_{\alpha,u}$-filtration (modulo $U_{2\alpha}$) is $\operatorname{ord}_F(F_\alpha^\times)$.

So if equality holds above, then $F_\alpha/F_{2\alpha}$ is ramified and $g(2\alpha) = 2g(\alpha)$. We finish by showing that this is impossible. By the previous paragraph we can pick $\ell' \in F_\alpha^0 - \{0\}$ of the same valuation as ℓ. As $F_\alpha/F_{2\alpha}$ is ramified we can scale ℓ' by an element of $\mathcal{O}_{F_{2\alpha}}^\times$ such that $\operatorname{ord}_F(\ell - \ell') > \operatorname{ord}_F(\ell)$. This contradicts that ℓ has maximum possible valuation among elements of trace 1. (Alternatively, from Tits' tables in [Tit79] the affine root system can only be non-reduced if the adjoint group has a factor isomorphic to $\operatorname{Res}_{E/F} H$, where $H \cong \mathrm{PU}(m+1, m)$ is unramified and E/F is finite separable and in that case the extension $F_\alpha/F_{2\alpha}$ is unramified.) □

We can now deduce Theorem A.1 from Proposition A.13 exactly as in §6.4, replacing μ_α there by $\frac{1}{\varepsilon_\alpha}\lambda_\alpha$. (It is still true, by Lemma A.11, that if $z \in Z^+$ with $\langle \alpha, v_Z(z)\rangle > 0$ and $v_Z(z_0) = \frac{1}{\varepsilon_\alpha}\lambda_\alpha$ then $zz_0^{-1} \in Z^+$.)

A.4. **First reduction step.** We continue to assume that $\mathbf{G}^{\mathrm{der}}$ is simply connected and $\mathbf{G}/\mathbf{G}^{\mathrm{der}}$ is coflasque. We now reduce to the basic case (§A.3).

Proposition A.14. *There exists a quasi-split connected reductive group $\mathbf{G}_1$ containing $\mathbf{G}$ as a closed normal subgroup such that*

(i) $\mathbf{G}_1^{\mathrm{der}} = \mathbf{G}^{\mathrm{der}}$*;*
(ii) *the torus $\mathbf{G}_1/\mathbf{G}_1^{\mathrm{der}}$ is coflasque;*
(iii) $1 \to \mathbf{Z}_{\mathbf{G_1}} \to \mathbf{Z}_1 \to \mathbf{Z}_1/\mathbf{Z}_{\mathbf{G_1}} \to 1$ *is a split exact sequence of F-tori.*

Here, $\mathbf{Z}_1$ denotes the minimal Levi $\mathbf{Z}\cdot\mathbf{Z}_{\mathbf{G_1}} = \mathbf{C}_{\mathbf{G}_1}(\mathbf{Z})$ of $\mathbf{G}_1$.

Proof. We define $\mathbf{G}_1$ and $\mathbf{Z}_1$ exactly as in §6.6(1), so in particular (i) holds. The exact sequence $1 \to \mathbf{Z}_{\mathbf{G_1}} \to \mathbf{Z}_1 \to \mathbf{Z}/\mathbf{Z}_{\mathbf{G}} \to 1$, where the second map is induced by the first projection, has a canonical splitting induced by $\mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$, $z \mapsto (z, z^{-1})$. This implies (iii). Finally, consider the short exact sequence $1 \to \mathbf{G}/\mathbf{G}^{\mathrm{der}} \to \mathbf{G}_1/\mathbf{G}_1^{\mathrm{der}} \to \mathbf{Z}/\mathbf{Z}_{\mathbf{G}} \to 1$. The first term is coflasque by assumption and the last term is induced because it is the maximal torus in the quasi-split adjoint group $\mathbf{G}/\mathbf{Z}_{\mathbf{G}}$. Hence $\mathbf{G}_1/\mathbf{G}_1^{\mathrm{der}}$ is coflasque and (ii) follows. □

Hence the group $\mathbf{G}_1$ is as in §A.3. The reduced buildings of G and G_1 are canonically identified with each other (as the reduced building only depends on the adjoint group), in particular there is a natural bijection between special parahoric subgroups of these two groups. Denote by K_1 any special parahoric subgroup of G_1 and let K denote the corresponding special parahoric subgroup of G.

Lemma A.15. *We have $K = K_1 \cap G$.*

Proof. Consider the commutative diagram given by functoriality of the Kottwitz homomorphism. (Note that the codomains simplify, since $\mathbf{G}^{\mathrm{der}} = \mathbf{G}_1^{\mathrm{der}}$ is simply connected. See [Kot97,

§7.4].)

$$\begin{tikzcd} G \arrow[r, "w_G"] \arrow[d, hook] & X_*(\mathbf{G}/\mathbf{G}^{\mathrm{der}})^{\sigma}_{I_F} \arrow[d] \\ G_1 \arrow[r, "w_{G_1}"] & X_*(\mathbf{G}_1/\mathbf{G}_1^{\mathrm{der}})^{\sigma}_{I_F} \end{tikzcd}$$

We claim that the vertical arrow on the right is injective. The first term in the short exact sequence $1 \to \mathbf{G}/\mathbf{G}^{\mathrm{der}} \to \mathbf{G}_1/\mathbf{G}_1^{\mathrm{der}} \to \mathbf{Z}/\mathbf{Z}_{\mathbf{G}} \to 1$ of F-tori is coflasque, so $X_*(\mathbf{G}/\mathbf{G}^{\mathrm{der}})_{I_F}$ is torsion-free, as noted in the proof of Lemma A.7. Let Γ be a finite quotient of I_F through which it acts on the character groups of the tori in the sequence. Then $H_1(\Gamma, X_*(\mathbf{Z}/\mathbf{Z}_{\mathbf{G}}))$ is torsion, as Γ is finite, so $X_*(\mathbf{G}/\mathbf{G}^{\mathrm{der}})_{I_F} \to X_*(\mathbf{G}_1/\mathbf{G}_1^{\mathrm{der}})_{I_F}$ is injective, which implies the claim.

Since the reduced buildings of G and G_1 are naturally identified and parahoric subgroups are the fixers of facets in the kernel of the Kottwitz homomorphism, it follows that $K = K_1 \cap G$. □

Lemma A.16. *The restriction to K of any irreducible representation of K_1 is irreducible. Conversely, any irreducible representation of K extends to K_1.*

Proof. Note that as $K \lhd K_1$, the pro-p radical of K is normal in K_1, so we get a commutative diagram as follows:

(A.8)
$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, two heads] & K_1 \arrow[d, two heads] \\ G_k \arrow[r, hook] & G_{1,k} \end{tikzcd}$$

Note that $G'_{1,k} \subset G_k \subset G_{1,k}$. It is enough to show that any irreducible representation of $G_{1,k}$ restricts irreducibly to $G'_{1,k}$, and hence to G_k. (Then if V is an irreducible representation of G_k, any irreducible quotient of $\mathrm{Ind}_{G_k}^{G_{1,k}} V$ extends V to $G_{1,k}$.)

We will prove more generally that if $\mathbf{H}$ is any connected reductive group over k and V an irreducible representation of H, then the restriction of V to H' is irreducible. Suppose first that the derived subgroup $\mathbf{H}^{\mathrm{der}}$ is simply connected. Then $H' = H^{\mathrm{der}}$. We know that we can lift V to an irreducible representation of $\mathbf{H}$ with q-restricted highest weight (where $q = \#k$), cf. [Her09, Appendix, (1.3)]. Then its restriction to $\mathbf{H}^{\mathrm{der}}$ is still irreducible with q-restricted highest weight (noting that $\mathbf{H}$ is generated by its center and $\mathbf{H}^{\mathrm{der}}$). Hence V restricted to H^{der} remains irreducible by the result we just cited.

For the general case pick a z-extension $\pi : \widetilde{\mathbf{H}} \twoheadrightarrow \mathbf{H}$, so $\mathbf{R} := \ker \pi$ is an induced torus and $\widetilde{\mathbf{H}}^{\mathrm{der}}$ is simply connected. We have a commutative diagram with exact rows:

$$\begin{tikzcd} 1 \arrow[r] & R \cap \widetilde{H}' \arrow[r] \arrow[d, hook] & \widetilde{H}' \arrow[r] \arrow[d, hook] & H' \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & R \arrow[r] & \widetilde{H} \arrow[r] & H \arrow[r] & 1 \end{tikzcd}$$

By inflation we can consider V as irreducible representation $\widetilde{V}$ of $\widetilde{H}$ that is trivial on R. By above we know the restriction of $\widetilde{V}$ to $\widetilde{H}'$ is irreducible, and hence so is the restriction of V to H'. □

Remark A.17. As in Remark A.5 we expect that the map $G_k \to G_{1,k}$ arises from a closed immersion $\mathbf{G}_k \to \mathbf{G}_{1,k}$.

Lemma A.18. *Proposition A.4 holds for (G, K) if and only if it holds for (G_1, K_1). More precisely, we have* $\mathrm{red}(M'_\alpha \cap K) = M'_{\alpha,k}$ *inside* G_k *if and only if* $\mathrm{red}(M'_{1,\alpha} \cap K_1) = M'_{1,\alpha,k}$ *inside* $G_{1,k}$.

Proof. Fix $\alpha \in \Delta$. We note that $\mathbf{M}_\alpha \lhd \mathbf{M}_{1,\alpha}$ for the Levi subgroups defined by α and that by Lemma A.15 we have $M_\alpha \cap K \lhd M_{1,\alpha} \cap K_1$ for the corresponding special parahoric subgroups. Hence, restricting the top row of diagram (A.8) (applied to Levi subgroups defined by α), we get a commutative diagram

$$\begin{tikzcd} M'_\alpha \cap K \arrow[r, hook] \arrow[d] & M'_{1,\alpha} \cap K_1 \arrow[d] \\ M_{\alpha,k} \arrow[r, hook] & M_{1,\alpha,k} \end{tikzcd}$$

Note that the top row is an isomorphism (by Lemma A.15, as $M'_\alpha = M'_{1,\alpha}$) and that the bottom row induces an isomorphism between the vertical images, as well as between $M'_{\alpha,k}$ and $M'_{1,\alpha,k}$. The lemma follows. $\square$

Choose now any K such that Proposition A.4 holds for (G, K); equivalently, Proposition A.4 holds for (G_1, K_1), by Lemma A.18. From Corollary A.6 and since $\alpha \in \Delta(V)$, we see that $\psi_V(Z^0 \cap M'_\alpha) = 1$. Now we deduce in exactly the same way as in §6.6(1) that Theorem A.1 holds for (G, K), since we know it holds for (G_1, K_1) by §A.3.

A.5. **Second reduction step.** Suppose now that $\mathbf{G}$ is any quasi-split group. We will reduce to the previous case. The following result is proved by Colliot-Thélène [CT08, Prop. 4.1].

Proposition A.19. *The group $\mathbf{G}$ has a (quasi-split) z-extension $\widetilde{\mathbf{G}}$ such that $\widetilde{\mathbf{G}}/\widetilde{\mathbf{G}}^{\mathrm{der}}$ is a coflasque torus.*

Hence the group $\widetilde{\mathbf{G}}$ is as in §A.4. Now choose any special parahoric subgroup $\widetilde{K}$ of $\widetilde{G}$ for which Proposition A.4 holds. Let K denote the corresponding special parahoric subgroup of G. It follows from Step 3 of the proof of Proposition A.4 that Proposition A.4 holds also for (G, K). From Corollary A.6 and since $\alpha \in \Delta(V)$, we see that $\psi_V(Z^0 \cap M'_\alpha) = 1$. Now we deduce in exactly the same way as in §6.6(2) that Theorem A.1 holds for (G, K), since we know it holds for $(\widetilde{G}, \widetilde{K})$ by §A.4.

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