# MAT327H1F

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Recall : $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous if

1) $\epsilon$–$\delta$ definition (basic def)

If $\forall\, x \in \mathbb{R}^m\ \forall\, \epsilon > 0\ \exists\, \delta > 0$ st $\forall\, y\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

Notation : open ball $B_{\delta,m}(x) = \{y \in \mathbb{R}^m : |y - x| < \delta\}$.

Recall : A subset $U \subset \mathbb{R}^m$ is ***open*** if $\forall\, x \in U, \exists\, \delta > 0,\ s.t.\ B_{\delta,m}(x) \subset U$.

Ex : A line is not open.

Claim : $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous $\Leftrightarrow \forall\, U \subset \mathbb{R}^n$ open, $f^{-1}(U) \subset \mathbb{R}^m$ open.

Recall :

$f^{-1}(\bigcup_\alpha X_\alpha) = \bigcup_\alpha f^{-1}(X_\alpha)$
$f^{-1}(\bigcap_\alpha X_\alpha) = \bigcap_\alpha f^{-1}(X_\alpha)$
$f^{-1}(X^c) \ = \ f^{-1}(X)^c$

Proof :

"$\Rightarrow$"
Given $U \subset \mathbb{R}^n$ open, want to show $f^{-1}(U) \subset \mathbb{R}^m$ open. Pick $x \in f^{-1}(U)$, since $U$ is open $\exists\, \epsilon > 0$, $B_{\epsilon,n}(f(x)) \subset U$. Since $f$ is continuous, $\exists\, \delta > 0\ s.t.\ f(B_{\delta,m}(x)) \subset B_{\epsilon,n}(f(x))$. $f(B_{\delta,m}(x)) \subset U \Leftrightarrow B_{\delta,m}(x) \subset f^{-1}(U) \Rightarrow f^{-1}(U)$ open.
"$\Leftarrow$"
Given any $x \in \mathbb{R}^m$, $\epsilon > 0$. Need to show $\exists\, \delta > 0$, $s.t.\ f(B_{\delta,m}(x)) \subset B_{\epsilon,n}(f(x))$. We know that $f^{-1}(B_{\epsilon,n}(f(x)))$ open $b/c$ $B_{\epsilon,n}$ is open.
$x \in f^{-1}(B_{\epsilon,n}(f(x)))$ because $f(x) \in B_{\epsilon,n}(f(x)) \Rightarrow \exists\, \delta > 0$, s.t. $B_{\delta,m}(x) \subset f^{-1}(B_{\epsilon,n}(f(x))) \Leftrightarrow f(B_{\delta,m}(x)) \subset B_{\epsilon,n}(f(x))$ so $f$ is continuous.

Some properties of open sets

1) $\mathbb{R}^n$, $\emptyset$ are open sets.
2) arbitrary unions of open sets are open
3) finite intersections of open sets are open.

Ex $\bigcap_{n=1}^\infty B_{1/n,n}(0) = \{0\}$ not open. Hence, third condition is necessary to be finite.