Recall :

A ***topology*** on a set X is a collection τ of subsets of X having the following properties:
(1) $\emptyset$ and X are in τ.
(2) The union of the elements of any subcollection of τ is in τ.
(3) The intersection of the elements of any finite subcollection of τ is in τ.
A set X for which a topology has been specified is called a ***topological space***.

Define $f : X \to Y$ where $X,\ Y$ are topological spaces. then f is ***continuous*** if $f^{-1}(U)$ is open in X for all open $U \subset Y$.

If X is any set and $\tau_1 \subset \tau_2$ are topologies on X, then we say τ_2 is ***finer*** than τ_1. $\tau_1$is ***coarser*** than τ_2. In this case, we also say τ_1 and τ_2 are ***comparable***.

Definition   Given X is a set. A ***basis*** for a topology on X is a collection $\mathcal{B}$ of subsets of X such that

(1) The union of all $B \in \mathcal{B}$ is X.
(2) $B_1 \cap B_2$ is a union of basis elements $\forall\ B_1,\ B_2 \in \mathcal{B}$.

Topology generated by $\mathcal{B}$

$\tau = \{U \subset X |\ U$ is a union of basis elements$\}$

Examples

1) $X = \mathbb{R}$
2) $\mathcal{B} = \{$all intervals $[a,\ b)$ with $a < b$ in $\mathbb{R}\}$
 –Check the basis axioms for this topology.

i) This is easy, since any $x \in \mathbb{R}$ is in $[x,\ x+1)$.
ii) $[a,\ b) \cap [c,\ d)\ =\ [\max(a,\ c),\ \min(b,\ d))$, provided it's not $\emptyset$.
Note : In general the intersection might not be a basis element.

$\mathbb{R}$ with the topology generated by $\mathcal{B}$ is denoted by $\mathbb{R}_l$ (***lower limit topology***).

To compare $\mathbb{R}$, $\mathbb{R}_l$

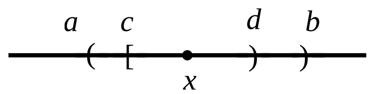
Theorem 3

Given basis $\mathcal{B}_1,\ \mathcal{B}_2$ on a set X generating topologies $\tau_1,\ \tau_2$, then $\tau_1 \subset \tau_2 \Leftrightarrow \forall\ B_1 \subset \mathcal{B}_1,\ \forall\ x \in B_1\ \exists\ B_2 \subset \mathcal{B}_2\ s.t.\ x \in B_2 \subset B_1$.
$\Leftrightarrow$ basis elements of $\mathcal{B}_1$ is union of basis elements of $\mathcal{B}_2$.

Proof : (left as an exercise)

Apply this theorem to $\mathbb{R}$, $\mathbb{R}_l$ – Note $\mathbb{R}$ represents the standard topology.

The basis of $\mathbb{R}$ is all open intervals $(a,\ b)$.



Any (a, b) is a union of $[c, d)$'s
So $\mathbb{R}_l$ is finer than $\mathbb{R}$ (standard topology)

$\mathbb{R}_l$ is strictly finer than $\mathbb{R}$, *i.e*. these are not the same topology, because a basic open $[a, b)$ in $\mathbb{R}_l$ is not open in $\mathbb{R}$.

Remark : Different basis may describe the same topology!

Example

$\mathbb{R}$ (standard topology) :

1) Open intervals (a, b)
2) All standard open subsets
3) Open intervals (a, b) with $a, b \in \mathbb{Q}$

$\mathbb{R}$ can have many different basis.

How can describe a basis for a given topology ?

Theorem 4

Let X be topological space, and $\mathcal{B}$ be collection of open subsets of X. If $\forall\, U \subset X$ is open $\forall\, x \in U \;\exists\, B \in \mathcal{B}$ *s.t.* $x \in B \subset U$. Then $\mathcal{B}$ is a basis of X. ($\Leftrightarrow \forall\, U \subset X$ is open are unions of elements of $\mathcal{B}$)

Proof : (left as an exercise)

Definition A ***subbasis*** for a topology on a set X is a collection $\mathcal{S}$ of subsets of X (whose union is X)

Theorem 5

If $\mathcal{S}$ is a subbasis, let $\tau = \{$all subsets $U \subset X \mid$ unions of finite intersections of elements of $\mathcal{S}\}$
$(\Leftrightarrow \forall\, x \in U \;\exists\, S_1, \ldots, S_n \in \mathcal{S}$ s.t $x \in \bigcap_{i=1}^n S_i \subset U)$

Then τ is the coarsest topology containing $\mathcal{S}$ (generated by $\mathcal{S}$).

Proof:

First we claim that $\mathcal{B} := \{S_1 \cap S_2 \cap \ldots \cap S_n :$ where $S_i \in \mathcal{S}\}$ is a basis and it generates τ. To see that we check the criterions i) union is everything, because empty intersection (i.e. $n = 0$) is X ($\Rightarrow \in \mathcal{B}$).

proof will be continued in next lecture.