Theorem 5

If $\mathcal{S}$ is a subasis, the τ = {unions of finite intersection of elements of τ}. This is a topology that is the coarsest topology containing $\mathcal{S}$.

Proof :

$\mathcal{B}$ = {finite intersection $\bigcap_{i=1}^{n} S_i$, $S_i \in \mathcal{S}$}, we need $\mathcal{B}$ to be a basis ($\Rightarrow \tau$ is topology generated by it).

Checking basis criterions

(i) Trivial
(ii) $B_1, B_2 \in \mathcal{B}$, Let $B_1 = S_2 \bigcap \ldots \bigcap S_r$, $B_2 = S_{r+1} \bigcap \ldots \bigcap S_{r+t}$
$B_1 \bigcap B_2 = \bigcap_{i=1}^{r+t} S_i \in \mathcal{B}$

Now we wish to show that this topology is the coarsest topology. Suppose $\tau' \supset \mathcal{S}$ is any topology. It is required to show that $\tau \subset \tau'$. This is true because τ' closed under finite intersections and any unions.

Subspace and Product Topology §15, 16

Definition   Suppose (X, τ_X) is a topological space and $Y \subset X$ is a *subset*. Then the ***subspace topology*** of Y in X is
$T_Y = \{Y \bigcap U | \, U \in \tau_X\}$.

Check this is a topology!

Theorem 6

The subspace topology is the coarsest topology on Y s.t. the inclusion map i : $Y \to X$ is continuous.

Proof : The map i continuous $\Leftrightarrow$ $i^{-1}(U) = (Y \cap U)$ open in Y, $\forall$ U open in X. The inclusion map is continuous when Y has topology τ'. $\Leftrightarrow \tau_Y = \{Y \cap U \,|\, U \subset X\} \subset \tau'$. Hence τ_Y is coarsest. □

Theorem 7 (Restriction of (co)domain)

Suppose $f : X \to Y$ is continuous map of topological spaces.

i) If $Z \subset X$ is subset, then $f|_Z \, Z \to Y$ is continuous. (if Z has subspace topology).
ii) If $W \subset Y$ is a subset containing $f(X)$, then $g : X \to W$ is continuous. (if W has subspace topology).

Proof :

i) $f|_Z$ is a composite map : $Z \overset{i}{\to} X \overset{f}{\to} Y$. Both i, f are continuous and since composites of continuous maps are continuous. (By Thm 1)
ii) We need to show that if $V \subset W$ is open, then $g^{-1}(V) \subset X$ is open. Note that V is of the form $W \cap U$, where $U \subset Y$ is open. So we have $g^{-1}(V) = g^{-1}(W \cap U) = f^{-1}(W \cap U) = f^{-1}(U)$ because $f(X) \subset W$. Hence $f^{-1}(U)$ is open in X by the continuity of f. □

Theorem 8

Let X be a topological space and Z, Y be subspaces such that $Z \subset Y \subset X$. The natural topologies on Z coincide.

1) Subspace topology in X
2) Subspace topology in Y, where Y has subspace topology in X.

Proof : (left as an exercise)

Theorem 9

Let X be a topological space and Y be a subset of X. If $\mathcal{B}_X$ is a basis for the topology of X then $\mathcal{B}_Y = \{Y \cap B,\ B \in \mathcal{B}_X\}$ is a basis for the subspace topology on Y.

Proof : Use Thm 4.

<u>Definition</u> Suppose X, Y are topological spaces. Then the projection is $p_1 : X \times Y \to X,\ p_2 : X \times Y \to Y$. i.e. $p_1(x, y) = x$ and $p_2(x, y) = y$.

Theorem 10

There is a coarsest topology on $X \times Y$ such that projection maps p_1 and p_2 are continuous.

Proof :

$p_1,\ p_2$ are continuous $\Leftrightarrow p_1{}^{-1}(U),\ p_2^{-1}(V)$ are open in $X \times Y$, for all open U and V in X and Y, respectively. Let $\mathcal{S} := \left\{p_1{}^{-1}(U),\ p_2{}^{-1}(V) \middle| U \subset X,\ V \subset Y \text{ open}\right\}$ The topology generated by this subbasis is the coarsest containing $\mathcal{S}$, i.e. $p_1,\ p_2$ are both continuous. ▫

This topology is called the ***product topology*** on $X \times Y$.

In fact, we can get basis out of the subbasis by taking all finite $\cap$:

$p_1^{-1}(U_1) \cap \ldots \cap p_1^{-1}(U_r) \cap p_2^{-1}(V_1) \cap \ldots \cap p_2^{-1}(V_s)$ where $U_i \subset X,\ V_j \subset Y$ is open $\forall\ i,\ j$
So the basis $= \left\{p_1^{-1}(U) \cap p_2^{-1}(V) \middle| U \subset X,\ V \subset Y \text{ open}\right\} = \left\{U \times V \mid U \subset X,\ V \subset Y \text{ open}\right\}$

We call $p_1^{-1}(U),\ p_2^{-1}(V)$ "open cylinders" and $U \times V$ "open box".

Examples

$X = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Two topologies :
(1) product topology ($\mathbb{R}$ has standard topology)
(2) standard topology on $\mathbb{R}^2$.

These two are the same! (Use Thm 3)

- (1) has basis $U \times V,\ U, V \subset \mathbb{R}$ open
- (2) open balls(or disks), $B_\delta(x, y)$

Theorem 11

If $\mathcal{B}_X$ is a basis for X and $\mathcal{B}_Y$ is a basis for Y, then $\mathcal{B} := \{B_1 \times B_2 | B_1 \in \mathcal{B}_X,\ B_2 \in \mathcal{B}_Y\}$ is a basis for the product topology.

Proof : Use Thm 4.

Theorem 12

If $A \subset X$, $B \subset Y$ are subsets of topological spaces X, Y then on $A \times B$ the two natural topologies coincide.
i) Product topology of the subspace topology on A, B
ii) subspace topology of the product topology on $X \times Y$.

Basis of topology (i)

i.e. $(A \cap U) \times (B \cap Y)$ where $U \subset X$ and $V \subset Y$ are open. i.e. (Open subsets of A) $\times$ (Open subsets of B)

Basis of topology (ii)

$\overset{\text{Thm 9}}{\Rightarrow}$ basis for subspace $A \times B$: $(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V)$. Same basis $\Rightarrow$ Same topology.

Order Topology §14

$(X, \leq)$ is a set together with a linear(or total) order

Example

$(\mathbb{R}, \leq)$, $(\mathbb{Z}, \leq)$ – standard order

If $(X, \leq)$, $(Y, \leq')$ then have dictionary order on $X \times Y$:
say $(x, y) \leq (x', y') \Leftrightarrow (x < x')$ or $(x = x'$ and $y < y')$