Recall : Subspace Topology and Product topology

Definition Let X be a topological space with topology τ. If Y is a subset of X, the collection

$$\tau_Y = \{Y \cap U \mid U \in \tau\}$$

is a topology on Y, called the ***subspace topology***. With this topology, Y is called a ***subspace*** of X; its open sets consist of all intersections of open sets of X with Y.

Definition Let X and Y be topological spaces. The ***product topology*** on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Order of Topology

Let X be any set and $\leq$ linear order then

Definition The ***order topology*** of $(X, \leq)$ is generated by the basis

$(a, b) = \{x \in X : a < x < b\} \mid a, b \in X\}$
$[m, b) = \{x \in X : m \leq x < b\}$ if X has a smallest element m
$(a, M] = \{x \in X : a \leq x < M\}$ if X has a largest element M
$(X = [m, M]$ if $m = M \Leftrightarrow |X| = 1)$

Check this is a basis (left as an exercise)

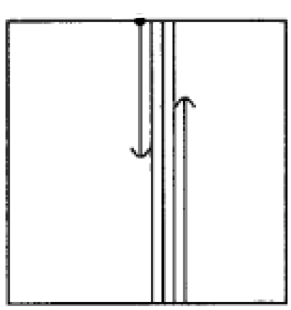
Example $X = \mathbb{R}$, usual $\leq$

Basis is just $(a, b) \to$ standard topology.

Example $X = \mathbb{Z}$, usual $\leq$

All singletons $\{n\} = (n - 1, n + 1)$ are in the basis.
Discrete topology

Example $X = [0, 1]^2$ with dictionary order (with usual $\leq$–order on $[0, 1]$)



Order

Note : The order topology $\neq$ product topology where $[0, 1]$ has order topology.

Exercise A subbasis of the order topology is given by the open rays, i.e. $\begin{aligned}(a,\ \infty) &:= \{x \in X : x > a\} \\ (-\infty,\ b) &:= \{x \in X : x < b\}\end{aligned}$

§17 Closed Sets and Limit Points

<u>Definition</u> X be a topological space, a subset $A \subset X$ is ***closed***, if $X \setminus A = A^c$ is open.

Example

(1) $[0, 1] \subset \mathbb{R}$ is closed.

$[0,\ 1]^c = (-\infty,\ 0) \cup (1,\ \infty)$

(2) Open and closed

e.g. if X has discrete topology, any subset is open and closed

(3) Neither open nor closed

e.g. $[0, 1) \subset \mathbb{R}$

Basic Properties

(i) $\emptyset$, X closed.
(ii) Intersection of closed sets are closed.
(iii) Finite unions of closed sets are closed.

<u>Definition</u> $A \subset X$ subset, ***closure*** of A in X : $\overline{A} := \bigcap\{\text{all closed sets } C \subset X \ s.t.\ A \subset C\}$
interior of A in X, $A^\circ = \bigcup\{\text{all open sets } U \subset X \ s.t.\ U \subset A\}$

Remark $\overline{A}$ is closed in X (since its $\bigcap$ of closed sets)
$\overline{A} \supset A$
$\therefore$ $\overline{A}$ is the smallest closed subset of X that contains A.

A° is open and $A^\circ \subset A$.
$\therefore A^\circ$ is the biggest open subset of X that is contained in A.

Example

$A = [0,\ 1) \subset \mathbb{R}$
$\overline{A} = [0,\ 1]$
$A^\circ = (0,\ 1)$

Lemma 13 $\overline{A^c} = (A^\circ)^c$

Proof:

$\{\text{closed sets containing } A^c\} \xleftrightarrow{\text{complement}} \{\text{open sets contained in } A\}$.
$U^c \supset A^c(\text{closed}) \Leftrightarrow U \subset A\ (\text{open})$
$\overline{A^c} = \text{smallest set on left} \xleftrightarrow{\text{complement}} A^\circ = \text{biggest set on right}$