§17 Closed sets and Limit points

Recall $A \subset X$ closed $\Leftrightarrow A^c$ open

closure of $A = \overline{A}$ = smallest closed set in X containing A
interior of $A = A^\circ$ = largest open set in X contained in A

<u>Definition</u> An open neighborhood of $x \in X$ is an open set U *s.t.* $x \in U$

Theorem 14

$A \subset X$ subset, then $x \in \overline{A} \Leftrightarrow$ all nbds of x intersect A

Proof
x $\notin \overline{A} = \bigcap \{C \text{ closed} : C \supset A\} \Leftrightarrow \exists\, C$ closed, $C \supset A$, $x \notin C$. Let $U := C^c \Leftrightarrow \exists\, U$ open, $A \bigcap U = \emptyset$, $x \in U$
$\Leftrightarrow \exists$ nbd U of x that doesn't intersect A. □

Example Compute $\left\{1 - \frac{1}{n},\ n \geq 1\right\}$
Check 1 $\in$ closure : any nbd of 1 contains open interval $I = (1 - \delta,\ 1 + \delta)$ for some $\delta > 0$.
Since $\text{limit}_{n\to\infty}\, 1 - \frac{1}{n} = 1 \Rightarrow 1 - \frac{1}{n} \in I$ for $n >> 0 \Rightarrow 1 \in$ closure
$\left\{1 - \frac{1}{n},\ n \geq 1\right\} \bigcup \{1\}$ is already closed $\Rightarrow$ it has to be the closure.

Exercise Check $\overline{\left\{1 - \frac{1}{n},\ n \geq 1\right\}}$ is in $\mathbb{R}_l$

<u>Definition</u> $x \in X$ is a ***limit point*** of A if every nhd of x intersect A outside x. Notation : A' = set of limit points of A.

Theorem 15

$\overline{A} = A \cup A'$

Example $\left\{1 - \frac{1}{n},\ n \geq 1\right\} \subset \mathbb{R}$, limit point = $\{1\}$

Example $A = [0,\ 1)$

$A' = [0,\ 1] = \overline{A}$

Example $A = \mathbb{Q} \subset \mathbb{R} \Rightarrow \overline{A} = \mathbb{R}$ Since any interval contains some rational numbers.

$A' = \mathbb{R}$, Since any interval contains infinitely many rational numbers.

Proof: (left as an exercise)

Theorem 16

$\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof: hw 2 □

More on subspaces

Lemma 17

$Y \subset X$ subspace

(i) The closed subsets are $\{Y \cap C : C \text{ closed in } X\}$
(ii) If $Z \subset Y$ closed subset and $Y \subset X$ closed $\Rightarrow$ $Z \subset X$ closed
If $Z \subset Y$ open subset and $Y \subset X$ open $\Rightarrow$ $Z \subset X$ open
(iii) If $A \subset Y$ subset, then the closure of A in Y is $Y \cap \overline{A}$

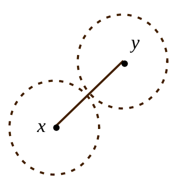
Proof:

(i) exercise
(ii) open case: $Z \subset Y$ open $\Rightarrow$ $\exists$ U open in X $s.t.$ $Z = Y \cap U$ $\Rightarrow$ Z is open in X because Y, U are open.
(iii) To check $Y \cap \overline{A}$ is the closure, verify it is the smallest closed set in Y containing A. $Y \cap \overline{A}$ is closed in Y(by (i)) and contains A since $Y \supset A$, $\overline{A} \supset A$. We need to show whenever a subset $D \subset Y$ closed, $D \supset A$, then $D \supset Y \cap \overline{A}$. By part (i) $D = Y \cap C$.
$Y \cap C \supset A \Rightarrow A \subset C \Rightarrow \overline{A} \subset C \Rightarrow Y \cap \overline{A} \subset A \cap C = D$. □

Some separation axiom

Definition A topological space X is ***Hausdorff*** or T_2 if $\forall\, x \neq y \in X$, $\exists$ nbds U of x and V of y s.t. $U \cap V = \emptyset$.

Example $\mathbb{R}^n$ is T_2



open balls, radius $= \frac{1}{2}\,|x - y|$ or $\leq$.

Example Trivial topology (non–Hausdorff)

Definition A sequence $\{x_n\}_{n=1}^{\infty}$ $(x_n \in X)$ converges to $x \in X$ if $\forall$ U nbd of x, $\exists\, n_0$ $s.t.$ $x_n \in U$ $\forall\, n \geq n_0$
(write $x_n \to x$, as $n \to \infty$)

Theorem 18

If X is T_2, then any sequence converge to at most one point.

Example : If X has trivial topology $x_n \to x$ for any x!

Proof:

Suppose $\exists$ $\{x_n\}_{n=1}^{\infty}$ s.t. $x_n \to x$, $x_n \to y$ where $x \neq y$. $T_2 \Rightarrow \exists$ U, V disjoint nbds of x, y

$$\left.\begin{array}{l} x_n \to x \Rightarrow x_n \in U,\ \forall\ n \geq n_0 \\ x_n \to y \Rightarrow x_n \in V,\ \forall\ n \geq n_1 \end{array}\right\} \Rightarrow x_n \in U \cap V = \emptyset,\ \forall\ n \geq \max\{n_0, n_1\}. \quad \text{Contradiction. } \square$$

Theorem 19

(i) If X is T_2, then so is any subspace
(ii) If X, Y are T_2, so is $X \times Y$.

Proof:

(i) $Y \subset X$ subspace, say $y_1 \neq y_2$ in Y. X is T_2 $\Rightarrow \exists\ U_1, U_2$ disjoint nbds of y_1, y_2 in X.
$\Rightarrow \exists\ U_1 \cap Y_1, U_2 \cap Y_2$ disjoint nbds of y_1, y_2 in Y.
(ii) Pick $(x_1, y_1) \neq (x_2, y_2)$ in $X \times Y$. If $x_1 \neq x_2$: X is T_2 $\Rightarrow\ \exists\ U_1, U_2$ disjoint nbds of x_1, x_2 in X.
$\Rightarrow U_1 \times Y$, $U_2 \times Y$ disjoint nhds of $(x_1, y_1), (x_2, y_2)$ in $X \times Y$. If $y_1 \neq y_2$ is analogous! (use Y Hausdorff)

Definition A topological space X is T_1 if $\forall\ x \neq y$ in X if $\exists$ nbd of y $s.t.$ $x \notin V$.

Clearly $T_2 \Rightarrow T_1$

Theorem 20

X is $T_1 \Leftrightarrow \forall\ x$, $\{x\}$ is closed in X)

Proof: " $\Rightarrow$" Take $x \in X$. Want $\{x\}^c$ open. $T_1 \Rightarrow \forall\ y \in \{x\}^c\ \exists$ open nbd V_y of y $s.t.$ $V_y \subset \{x\}^c$.
See $\{x\}^c = \bigcup_{y \neq x} V_y \Rightarrow \{x\}^c$ open. "$\Leftarrow$": Given $x \neq y$. Then can take $V = \{x\}^c$ (open by assumption) □

Example X any set with the finite complement topology. (recall U open $\Leftrightarrow U = \emptyset$ or U^c finite). X is T_1 since $\{x\}^c$ is open.
If X is finite, it has the discrete topology, so it is T_2. If X is infinite, U^c finite, so U is infinite. So X is T_1 but not T_2.

More on countinuous functions

Theorem 21

$f : X \to Y$ between topological spaces.

TFAE (the followings are equivalent)

(i) f is continuous.
(ii) $f^{-1}(C)$ closed in X , $\forall\ C \subset Y$ closed.
(iii) $f(\overline{A}) \subset \overline{f(A)}$ $\forall$ subset A
(iv) $\forall\ x \in X$, $\forall\ V$ nbd of $f(x)$ $\exists$ nbd U of x $s.t.$ $f(U) \subset V$ (This is similar to ϵ–δ definition in a metric space.)