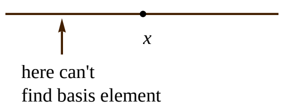
Recall

T_1 : points are closed($\Leftrightarrow$ finite subsets closed)
T_2 : distinct points have disjoint neighborhoods (Hausdorff)

Example

Non–T_1 Space

$\mathbb{R}$ with basis (a, ∞) where $a \in \mathbb{R}$



here can’t find basis element

Theroem 21

$f : X \to Y$, TFAE

(i) f is continuous
(ii) $f^{-1}(C)$ closed $\forall$ C closed
(iii) $f(\overline{A}) \subset \overline{f(A)}$
(iv) $\forall\, x \in X\ \forall$ nbd V of $f(x)\ \exists$ nbd U of x $s.t.$ $f(U) \subset V$.

(i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i)

Proof:
(i) $\Rightarrow$ (iv) want nbd U of x such that $f(U) \subset V \Leftrightarrow U \subset f^{-1}(V)$ open by (i). We can take $U = f^{-1}(V)$. It contains x, since $f(x) \in V$.
(iv) $\Rightarrow$ (iii) need $\forall\, x \in \overline{A}$, $f(x) \in f(\overline{A}) \Leftrightarrow$ any nbd V of $f(x)$ intersects $f(A)$. By (iv), $\exists$ nbd U of x, $s.t.$ $f(V) \subset V$. But $U \cap A \neq \emptyset$, becuase $x \in \overline{A} \Rightarrow$ if $y \in U \cap A$, then $f(y) \in f(U) \cap f(A) \subset V \cap f(A)$
(iii) $\Rightarrow$ (ii) want $f^{-1}(C)$ closed $\forall$ C closed. Apply (iii) with $A = f^{-1}(C)$. $f(\overline{f^{-1}(C)}) \subset \overline{f(f^{-1}(C))}$. We know $f(f^{-1}(C)) \subset C \Rightarrow \overline{f(f^{-1}(C))} \subset \overline{C} = C$. $f(\overline{f^{-1}(C)}) \subset C$. $\overline{f^{-1}(C)} \subset f^{-1}(C)$. $\Rightarrow f^{-1}(C) = \overline{f^{-1}(C)}$ as reverse inclusion is true. So $f^{-1}(C)$ is closed.
(ii) $\Rightarrow$ (i) need $f^{-1}(U)$ open $\forall$ U open. $f^{-1}(U) = f^{-1}(U^c) \Rightarrow f^{-1}(U)$ open.

Definition X, Y topological spaces: A map $f : X \to Y$ is a ***homeomorphism*** if it’s bijection, and f and f^{-1} are continuous. Equivalently, U is open in $Y \Leftrightarrow f^{-1}(U)$ is open in X.

In this case, X and Y have *same* topological properties.

Example

X in T_2(or T_1) $\Leftrightarrow$ Y is T_2(or T_1)
$C \subset X$ closed $\Leftrightarrow$ $f(C) \subset Y$ closed
$\mathcal{B}$ a basis for $X \Leftrightarrow f(\mathcal{B})$ a basis for Y.

Remark

A continuous bijection needn't be a homeomorphism.

Example

$(X, \tau) \to (X, \tau'), x \longmapsto x$ is continuously bijection $\Leftrightarrow \tau' \subset \tau$.
$(X, \tau) \to (X, \tau'), x \longmapsto x$ is homeomorphism $\Leftrightarrow \tau' = \tau$.

Example

$S^2 \setminus \{N\} \cong \mathbb{R}^2$ (Stereographic projection)

$f : \mathbb{R}^2 \to S^2 \setminus \{N\}$

$(x, y) \longmapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$, then $f^{-1}(a, b, c) \longmapsto \left(\frac{a}{1-c}, \frac{b}{1-c}\right)$ – homeomorphism

Remark

It can be hard to show 2 spaces are not homeomorphism.
One way is to show one is T_2, other isn't.

Definition

$f : X \to Y$ is an ***embedding*** if it is injective and the induced map $X \to f(X)$ is a homeomorphism.