Recall (X, d) metric space

metric topology on X : basis $B_{\epsilon, d}(x)$; all ϵ–balls, $\epsilon > 0$

$\exists$ metric $\overline{d}$ s.t. d and $\overline{d}$ have same metric topology.
e.g. $\overline{d} = \min(1, d)$

On $\mathbb{R}^{\Lambda}$ defined uniform metric

$\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup_{\lambda \in \Lambda}\left(\overline{d}(x_\lambda, y_\lambda)\right)$

Theorem 27

product topology $\subset$ uniform topology $\subset$ box topology

We proved 2nd "$\subset$" in last lecture
1st "$\subset$" We need to show that any basic open in the product topology is open in the uniform topology
Basic open : $\prod_{\lambda \in \Lambda} U_\lambda : U_\lambda = \mathbb{R},\ \lambda \notin \{\lambda_1, \lambda_2, \ldots \lambda_n\}\ U_{\lambda_i} \subset \mathbb{R}$ open for all i.

Pick $\mathbf{x} \in \prod_\Lambda U_\lambda,\ U_{\lambda_i}$ open $\Rightarrow \exists\, \epsilon_i > 0\ s.t.\ B_{\epsilon_i, \overline{d}}(x_{\lambda_i}) \subset U_{\lambda_i}$
let $\epsilon := \min\{\epsilon_i : 1 \leq i \leq n\}$ then $B_{\epsilon, \overline{\rho}}(\mathbf{x}) \subset \prod_\Lambda U_\lambda$

Reason : $\sup \overline{d}(x_\lambda, y_\lambda) < \epsilon \Rightarrow y_{\lambda_i} \in U_{\lambda_i}\ \forall\, i \Rightarrow \mathbf{y} \in \prod U_\lambda$. □

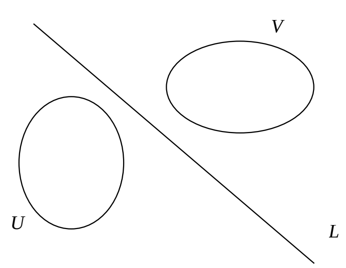
§ 23, 24 Connectedness

<u>Definition</u> A topological space X is ***connected*** if it cannot be written as disjoint union of two non–empty open subsets.

disjoint union of two non–empty open subsets : sometimes called separation of X.

Example

1) $X \subset \mathbb{R}^2$



$X = U \bigcup^* V$ (here $\bigcup^*$ denotes disjoint union)

U is open since $U = X \bigcap$(open half open to left of L) similary for V

$\Rightarrow X$ is disconnected.

2) $X = [0, 1] \bigcup (2, 3)$ in $\mathbb{R}$, let $U = [0, 1]$ and $V = (2, 3)$
U is open in X since $U = X \bigcap \left(-\infty, \frac{3}{2}\right)$.

3) $X = \mathbb{R} \setminus \mathbb{Q}$

Take $U = X \bigcap (-\infty, 0)$, $V = X \bigcap (0, \infty) \Rightarrow X$ is disconnected.

Definition $X \subset \mathbb{R}$ ***convex*** if $\forall\, x, y \in X \Rightarrow [x, y] \subset X$

Remark Convex sets are precisely all (possibly infinite) intervals.

(a, b) allow $a = -\infty$, $b = \infty$
$(a, b]$ allow $a = -\infty$
$[a, b)$ allow $b = \infty$
$[a, b]$ ($a = \inf X$, $b = \sup X$)

Example More generally suppose $X \subset \mathbb{R}$ not convex,

Claim X disconnected

X not connected $\Rightarrow \exists\, x < y$ in X and $\exists\, z \notin X$ $s.t.$ $x < z < y$

$U = X \bigcap (-\infty, z)$, $V = X \bigcap (z, \infty)$ open disjoint. $U \cap V = X$ since $z \notin X$, non–empty since $x \in U$, $y \in V$

Remark separation $X = U \bigcup^* V$ (so $V = U^c$) $\Leftrightarrow U \subset X$ open $U \neq \emptyset$, U^c open $U^c = \emptyset \Leftrightarrow \emptyset \subset U \subset X$ (open and closed)

So X conneceted $\Leftrightarrow$ the only open + closed subsets are $\emptyset$, X.

Example

1) $|X| > 1$, X discrete $\Rightarrow$ disconneceted b/c every subset is clopen (open and closed)

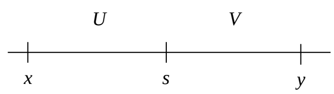
2) X trivial topology $\Rightarrow$ connected.

Theorem 28

If $X \subset \mathbb{R}$ convex, then X is connected, So $X \subset \mathbb{R}$ connected $\Leftrightarrow$ convex

Proof:

Suppose $X = U \bigcup^* V$ separation, pick $x \in U$, $y \in V$ can assume $x < y$ (otherwise swap U, V)
Let $s := \sup(U \cap [x, y])$ This means $U \bigcap [x, y] \subset (-\infty, s]$. and all $\epsilon > 0$, $U \cap [x, y] \not\subset (-\infty, s - \epsilon]$



For any $\epsilon > 0$, $(s - \epsilon, s + \epsilon) \cap X$ open nbd of s in X intersects U(as s is *least* upper bound) and V(as s is an upper bound)
$\Rightarrow s \in$ closure of U in $X = U$
$s \in$ closure of V in $X = V$ since U, V clopen.

$\Rightarrow s \in U \cap V = \emptyset$, contradiction. □