Oct 27 Term test

Office Hour : Wed 5:30–7:30
TA Office Hour : Tue 10:30–12:30

Last time

X ***connected*** if X can't be written as disjoint union $X = U \bigcup^* V$ of non–empty open subsets. ($\bigcup^*$ denotes disjoint union).
$\Leftrightarrow \nexists$ clopen subset $U \neq \emptyset, \ X$. ($X = U \bigcup^* V$ called separation.)

$X \subset \mathbb{R}$ connected $\Leftrightarrow$ X convex($x, y \in X : (x \leq y) \Rightarrow [x, y] \subset X$).

Theorem 29

If $f : X \to Y$ continuous, and X connected $\Rightarrow f(X)$ is connceted.

Proof:
Suppose not, then $f(x) = U \bigcup^* V$ separation (U, V non–empty open) then $X = f^{-1}(f(X)) = f^{-1}(U) \bigcup^* f^{-1}(V)$
$f^{-1}(U), \ f^{-1}(V)$ not epmty since $U, V \subset f(X)$, furthermore they are open because $f : X \to f(X)$ is continuous. (Theorem 7). □

<u>Remark</u> **Theorem 28 and 29** imply the *intermediate value theorem* for continuous functions $f : [a, b] \to \mathbb{R}$.

$[a, b]$ connceted (convex) $\Rightarrow$ its image $f([a, b])$ is connected $\Rightarrow$ convex.

Lemma 30

Suppose $X = U \bigcup^* V$ is a separation. If $A \subset X$ connceted subspace, then $A \subset U$ or $A \subset V$.

Proof:
Intersect with $A = (A \bigcap U) \bigcup^* (A \bigcap V)$, where $A \bigcap U$ and $A \bigcap V$ are open in A. If $A \bigcap U, \ A \bigcap V$ are both non–empty, this is a separation of A. contradiction. $\Rightarrow$ Either $A \bigcap U = \emptyset$ or $A \bigcap V = \emptyset$. i.e. $A \bigcap U = \emptyset \Rightarrow A \subset U^c = V, \ A \bigcap V = \emptyset \Rightarrow A \subset U$. □

Theorem 31

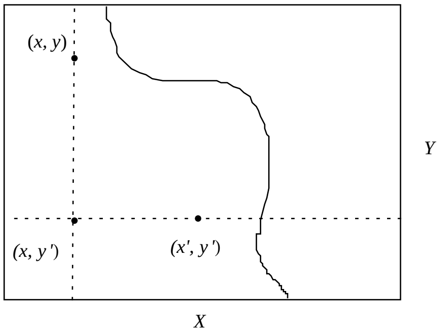
If $A \subset X$ connected subspace, and $A \subset B \subset \overline{A}$, then B is connected. In particular, A connected $\Rightarrow \overline{A}$ connected.

Proof:
Suppose $B = U \bigcup^* V$ (separation). By the lemma, we know $A \subset U$ or $A \subset V$. Can assume $A \subset U \Rightarrow \overline{A} \subset \overline{U} \Rightarrow B \subset B \bigcap \overline{U}$ = closure of U in B. (by Lemma 17) = U since U is clopen in B. Contradiction as $V = \emptyset$. □

Theorem 32

If $X_1, \ldots, X_n$ connected $\Rightarrow \prod_{i=1}^n X_i$ connected.

Proof:
$n = 2$ X, Y connected, want $X \times Y$ connected. Suppose $X \times Y = U \bigcup^* V$ separation



Fix $(x, y) \in U$. Say $(x', y') \in X \times Y$ arbitrary.

Recall (HW03 18.4) : $X \times \{b\} \cong X$ and $\{a\} \times Y \cong Y$ homeomorphic.$\Rightarrow X \times \{y\}$ connected.
Lemma 30 $\Rightarrow X \times \{y\} \subset U \Rightarrow (x', y) \in U$. Similary $\{x'\} \times Y \subset U \Rightarrow (x', y') \in U$.
Hence $X \times Y = U$. Contradiction □. $n > 2$ either imitate the same argument or use induction.
$\prod_{i=1}^n X_i \cong (X_1 \times \ldots \times X_{n-1}) \times X_n$. To check homeo i) use basis, ii) show $(X_1 \times \ldots \times X_{n-1}) \times X_n \to \prod_{i=1}^n X_i$ is continuous. By Theorem 22, we only have to check each coordinate function is continuous. True, because it is a composition of one $(i = n)$ or two $(i \neq n)$ projection maps. (projection maps are continuous). The other way is left as an exercise.

Remark In the same way, can show: if $Y_\lambda \subset X$, Y_λ connected $\forall \lambda$, and all Y_λ have a point in common. Then $\bigcup_{\lambda \in \Lambda} Y_\lambda$ connected.

Definition X is path–conneceted if $\forall x, y \in X$, $\exists s : [0, 1] \to X$ such that $s(0) = x$ and $s(1) = y$.

Lemma 33

X path connected $\Rightarrow X$ connected

Proof:
Suppose $X = U \bigcup^* V$ separation. Pick $x \in U$, $y \in V$. We know $\exists s : [0, 1] \to X$ s.t. $s(0) = x$ and $s(1) = y$.
Then $s^{-1}(U) \bigcup^* s^{-1}(V) = [0, 1]$ separation. This is a contradiction because $[0, 1]$ is connected. □

Examples

Say $X \subset \mathbb{R}^n$ is ***convex*** if $\forall$ $\mathbf{x}, \mathbf{y} \in X$ then the whole segment connecting themis contained in X.
$t\mathbf{x} + (1 - t)\mathbf{y} \in X$, $\forall t \in [0, 1]$.

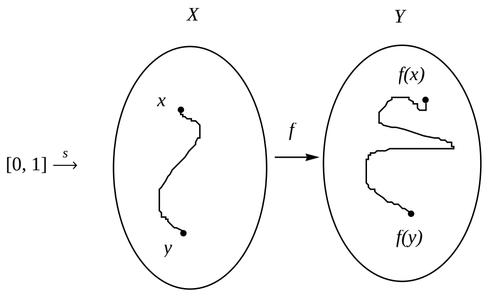
The unit ball $B = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq 1\}$, B is path connected $\Rightarrow B$ is connected.

$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}$ is path connected.

Theorem 34

i) $f : X \to Y$ continuous, X path connected $\Rightarrow f(X)$ path connected.
ii) $X_\lambda$path connected $\forall \lambda \Rightarrow \prod X_\lambda$ path connected.

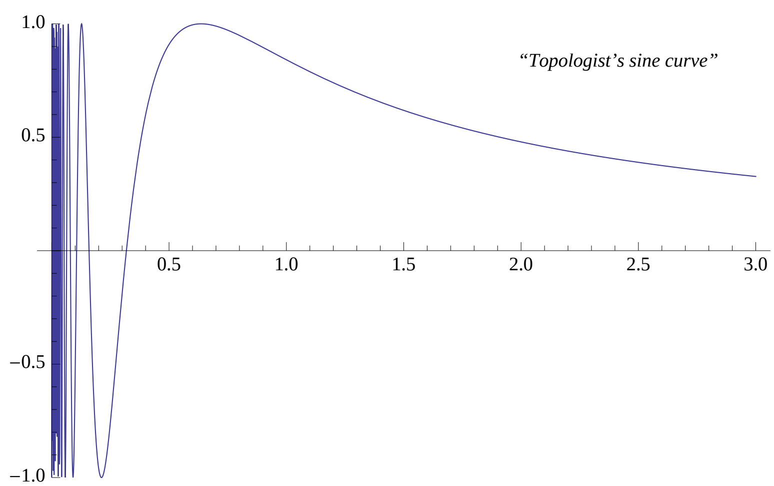
Proof:
i) pick $f(x), f(y) \in f(X)$



ii) $(x_\lambda)_{\lambda\in\Lambda},\ (y_\lambda)_{\lambda\in\Lambda} \in \prod X_\lambda$. We want $s : [0, 1] \to \prod X_\lambda$ s.t. $s(0) = (x_\lambda)_{\lambda\in\Lambda},\ s(1) = (y_\lambda)_{\lambda\in\Lambda}$.
So we need continuous coordinate functions $s_\lambda : [0, 1] \to X_\lambda$ such that $s_\lambda(0) = x_\lambda,\ s_\lambda(1) = y_\lambda$.
By assumption, $\exists\, s_\lambda$ for all λ.

Example

$X = \{(x, \sin(1/x)) : x > 0\} \cup \{0\}\times[-1, 1] \subset \mathbb{R}^2$



X is connected. X_+ is connected because $X_+ = f(\mathbb{R}_+)$, where $f : \mathbb{R}_+ \to \mathbb{R}^2$
$x \longmapsto (x, \sin(1/x))$

$\Rightarrow X$ connected because $X = \overline{X_+}$

However, X is *not* path connected. (exercise!)

<u>Remark</u> This example shows that A path connected ($\subset X$) does not imply that $\overline{A}$ path connected.