§26, 27 Compact Spaces

Definition A collection $\mathcal{A}$ of subsets of a space X is said to ***cover*** X, or to be a ***covering*** of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an ***open covering*** of X if tis elements are open subsets of X.

Definition A space X is said to be ***compact*** if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Examples

1) $\mathbb{R}$ not compact, $\mathbb{R} = \bigcup_{n\geq 1}(-n,\ n)$

2) $(0,\ 1)$ is not compact $(0,\ 1) = \bigcup_{n\geq 1}\left(\frac{1}{n},\ 1\right)$, similary, $(0,\ 1]$, $[0,\ 1)$ not compact

3) If X is finite, then X is compact, Let $X = \bigcup_{\lambda\in\Lambda} U_\lambda$ (for each $x \in X$ pick $\lambda_x \in \Lambda$ such that $x \in U_{\lambda_x}$
then $\{U_{\lambda_x}\}_{x\in X}$ forms finite subcover.

4) $\{0\} \bigcup \left\{\frac{1}{n} : n \geq 1\right\} \subset \mathbb{R}$. This is compact because any open neighborhood of 0 contains all but finitely many $\frac{1}{n}$'s.

Theorem 35
Any closed interval $[a,\ b] \subset \mathbb{R}$ is compact

Suppose $\{U_\lambda\}_{\lambda\in\Lambda}$ open cover that doesn't have a finite subcover. Let $X := \{t \in [a,\ b]$: the interval $[a,\ t]$ is contained in the union of finitely many U_λ's$\}$. Let $x := \sup X \in [a,\ b]$. (Note $a < x < b$). There exists $\lambda_0 \in \Lambda$ such that $x \in U_{\lambda_0}$.
But $x - \frac{\epsilon}{2} \in X$, so $[a,\ x - (\epsilon/2)] \subset U_{\lambda_1} \bigcup \ldots \bigcup U_{\lambda_n}$(some $\lambda_i \subset \Lambda$). Then $\left[a,\ x + \frac{\epsilon}{2}\right] \subset U_{\lambda_1} \bigcup \ldots \bigcup U_{\lambda_n}$.
$\Rightarrow x + \frac{\epsilon}{2} \in X$. This is a contraiction as $x \in \sup X$. □

General facts

Lemma 36

$Y \subset X$ be subspace. Y is compact $\Leftrightarrow$ whenever $Y \subset \bigcup_{\lambda\in\Lambda} U_\lambda$ where $U_\lambda \subset X$ open then $\exists\ \lambda_1,\ \ldots,\ \lambda_n \in \Lambda$ s.t $Y \subset U_{\lambda_1} \bigcup \ldots \bigcup U_{\lambda_n}$.

Proof:
"$\Rightarrow$"
$Y \subset \bigcup_\lambda U_\lambda \Rightarrow Y = Y \cap \bigcup_\lambda U_\lambda = \bigcup_\lambda (Y \bigcap U_\lambda)$ where $Y \cap U_\lambda$ open in Y. Y compact $\Rightarrow Y = \bigcup_{i=1}^n (Y \bigcap U_{\lambda_i})$ some $\lambda_i \in \Lambda$ (fin subcover)
$\bigcup_{i=1}^n (Y \bigcap U_{\lambda_i}) \subset \bigcup_{i=1}^n U_{\lambda_i}$.
"$\Leftarrow$"
Say $Y = \bigcup_\lambda V_\lambda$ open cover of Y. Know $V_\lambda = Y \bigcap U_{\lambda_i}$ some $U_\lambda \subset X$ open. $\Rightarrow Y \subset \bigcup_\lambda U_\lambda$. By hypothesis, $Y \subset \bigcup_{i=1}^n U_{\lambda_i}$some $\lambda_i \in \Lambda$.
$\Rightarrow Y = \bigcup_i (Y \bigcap U_{\lambda_i}) = \bigcup_i Y_{\lambda_i}$. □

Theorem 37

If $Y \subset X$ closed subspace and X is compact, then Y is compact.

Proof:
Use Lemma 36. Say $Y \subset \bigcup_\lambda U_\lambda$ (where $U_\lambda \subset X$ open) Then $X = Y^c \cup \bigcup_\lambda U_\lambda$ open cover $\Rightarrow X = Y^c \cup \bigcup_{i=1}^n U_{\lambda_i}$ some $\lambda_i \in \Lambda$.
$\Rightarrow Y \subset \bigcup_{i=1}^n U_{\lambda_i}$ (as $Y \cap Y^c = \emptyset$)