Recall X compact if every open cover of X has a finite subcover.

i.e. $X = \bigcup_{\lambda\in\Lambda} U_\lambda \Rightarrow X = U_{\lambda_1} \cup \ldots \cup U_{\lambda_n}$ for some $\lambda_i \in \Lambda$

Lemma 36

Let $Y \subset X$ subspace. Then, Y compact $\Leftrightarrow$ ($Y \subset \bigcup_\lambda U_\lambda$ with each U_λ open in $X \Rightarrow Y \subset U_{\lambda_1} \cup \ldots \cup U_{\lambda_n}$ some $\lambda_i \in \Lambda$)

Example $[a, b] \subset \mathbb{R}$) compact

Theorem 37

X compact, $Y \subset X$ closed $\Rightarrow$ Y compact

Theorem 38

If X is T_2 and $Y \subset X$ compact, then Y is closed

Proof:
We want Y^c to be open. Pick any $x \in Y^c$: want a neighborhood of x that is disjoint from Y.
For any $y \in Y$, $\exists$ disjoint neighborhood U_y of y and V_y of x (as X is T_2)

So $Y \subset \bigcup_{y\in Y} U_y \xRightarrow{Y \text{ compact}} Y \subset U_{y_1} \cup \ldots \cup U_{y_n}$ for some $y_i \in Y_i$ Let $V := V_{y_1} \cap \ldots \cap V_{y_n}$ open and neighborhood of x.
Claim : $V \cap Y = \emptyset$. Take $y \in V \cap Y \Rightarrow y \in U_{y_i}$ for some i. $y \in V \subset V_{y_i}$. contradiction □

Corollary 39

If X is compact T_2 then for any closed $Y \subset X$ and any $x \notin Y$, $\exists$ open $U, V \subset X$ such that $Y \subset U$ and $x \in V$
Proof:
$Y \subset X$ closed, X compact $\Rightarrow$ Y compact. Then the preceding proof shows that we can take $U := \bigcup_{i=1}^n U_{y_i}$, $V = \bigcap_{i=1}^n V_{y_i}$. □

Theorem 40

If $X_1, \ldots, X_n$ compact, then $\prod_{i=1}^n X_i$ is compact. (remark : for infinite products, later!)

Proof:
Enough to show this for $n = 2$. For $n > 2$, use $(X_1 \times \ldots \times X_{n-1}) \times X_n \cong \prod_{i=1}^n X_i$ and induction.
Let $X = X_1$, $Y = X_2$. Suppose $X \times Y = \bigcup_{\lambda\in\Lambda} U_\lambda$ open cover. Fix $x \in X$, $x \times Y \cong Y$, so it's compact.
So by Lemma 36, $x \times Y \subset W_{\lambda_i} \cup \ldots \cup W_{\lambda_n}$ (some $\lambda_i \in \Lambda$). Declare $W = W_{\lambda_i} \cup \ldots \cup W_{\lambda_n}$.
Claim: $\exists$ neighborhood U of x such that $U \times Y \subset W$. Assume this is true. So for all $x \in X$, $\exists$ neighborhood U_x of x such that

$U_x \times Y$ contained in a finite union of W_λ's. $X = \bigcup_{x\in X} U_x \xRightarrow{X \text{ compact}} X = U_{x_i} \cup \ldots \cup U_{x_m}$ for some $x_i \in X$.
$\Rightarrow X \times Y = (U_{x_1} \times Y) \cup \ldots \cup (U_{x_m} \times Y)$. Each is contained in finite union of W_λ's $\Rightarrow X \times Y$ is a finite union of W_λ's.
Now we prove the claim. $x \times Y \subset W$ open $\Rightarrow \forall y \in Y$, $(x, y) \in W \Rightarrow \exists U_y \subset X$, $V_y \subset Y$ open such that $(x, y) \in U_y \times V_y \subset W$.

$Y = \bigcup_{y\in Y} V_y \xRightarrow{Y \text{ compact}} Y = V_{y_1} \cup \ldots \cup V_{y_s}$ for some $y_i \in Y$. Take $U := \bigcap_{i=1}^n U_{y_i}$. □

Theorem 41

Suppose $f : X \to Y$ continuous and X compact then $f(X)$ is compact.

Proof:

Use Lemma 36, suppose $f(X) \subset \bigcup_{\lambda\in\Lambda} V_\lambda \Leftrightarrow X = f^{-1}(\bigcup_\lambda V_\lambda) = \bigcup_\lambda f^{-1}(V_\lambda) \Leftrightarrow X = \bigcup_\lambda f^{-1}(V_\lambda)$ open cover.

$\xRightarrow{X \text{ is compact}} X = \bigcup_{i=1}^n f^{-1}(V_\lambda) = f^{-1}(\bigcup_{i=1}^n V_{\lambda_i}) \Leftrightarrow f(X) \subset \bigcup_{i=1}^n V_{\lambda_i}$. □