Recall

$[a, b] \subset \mathbb{R}$ compact
closed $\subset$ compact $\Rightarrow$ compact
compact $\subset T_2 \Rightarrow$ closed
$f : X \to Y$ continuous, X compact $\Rightarrow$ $f(X)$ compact
$X_1, \ldots, X_n$ compact $\Rightarrow \prod_{i=1}^n X_i$ compact

Theorem 42

Suppose X is compact, Y is T_2. Then any continuous bijection $f : X \to Y$ is a homeomorphism

Proof:
We need f^{-1} is continuous. $\Leftrightarrow f(D)$ closed in Y $\forall D \subset X$ closed. Take $D \subset X$ closed, As X is compact, D is compact. $f(D)$ is compact (by Theorem 41) $\Rightarrow$ since Y is T_2, $f(D)$ is closed. □

§ 27, 28, 45 Compactness in metric spaces

Definition (X, d) is ***bounded*** if $\exists M > 0$ such that $d(x, y) \leq M$ for all $x, y \in X$.
Careful: This depends on the metric (not just the topology)
e.g. d and $\overline{d}$ have the same topology and $\overline{d} \leq 1$.

Theorem 43 (Heine–Borel)

A subspace $X \subset \mathbb{R}^n$ is compact $\Leftrightarrow X$ is closed and bounded in the Euclidean metric.

Example A closed ball $C_r(x) = \{y \in \mathbb{R}^n : |x - y| \leq r\}$ is compact.

Proof:
"$\Rightarrow$" X is compact, because $\mathbb{R}^n$ is $T_2 \Rightarrow X$ closed. $X \subset \mathbb{R}^n = \bigcup_{n \geq 1} B_n(0)$. Since X is compact, $X \subset B_n(0)$ for some n.
Take $M = 2n \Rightarrow X$ is bounded. "$\Leftarrow$" X is closed and bounded for d(Eucl). Since $d_\square(x, y) = \max_{1 \leq i \leq n}(|x_i - y_i|) \leq d(x, y)$
X bounded for $d_\square$. Fix $x_0 \in X$. Get : $d_\square(0, x) \leq d_\square(0, x_0) + d_\square(x_0, x)$. So if $M' = d_\square(0, x_0) + M$, then $X \subset [-M', M]^n$.
Theorem 35 + 40 $\Rightarrow [-M', M']^n$ is compact. X closed (in $\mathbb{R}^n \Rightarrow$ in $[-M', M']^n$). $\Rightarrow X$ compact. □

More generally

Theorem 44

(X, d) metric then the following are equivalent.

(i) X compact
(ii) X limit point compact
(iii) X sequentially compact.
(iv) X satisfies the Lebesgue Lemma + X totally bounded
(v) X is complete + X totally bounded..

Recall $x \in \overline{A} \Leftrightarrow$ all nbds U of x intersect A (i.e. $U \bigcap A \neq \emptyset$), $x \in A' \Leftrightarrow$ outside x. (i.e. $U \bigcap A \not\subset \{x\}$)
In metric space, $x \in A' \Leftrightarrow$ every neighborhood U of x contains infinitely many points of A.
Proof:
We'll show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) and (iii) $\Leftrightarrow$ (v)

Definition X is ***limit point compact*** if every infinite subset A has a limit point (i.e. $A' \neq \emptyset$)

(i) ⇒ (ii): X compact. Suppose X not limit point compact. $\forall\, x \in X$, $\exists\, U_x$ neighborhood of x such that $U_x \bigcap A \subset \{x\}$.
$X = \bigcup_{x \in X} U_x$ ⇒ since X is compact, $X = U_{x_1} \bigcup \ldots \bigcup U_{x_n}$ for some $x_i \in X$. ⇒ $A \subset \{x_1, x_2, \ldots, x_n\}$ contradiction. □

Definition X is ***sequentially compact*** if every sequence $x_n \in X\,(n \geq 1)$ has a convergent subsequence: i.e. $\exists$ $1 \leq n_1 < n_2 < \ldots$
such that $x_{n_i} \to x$ as $i \to \infty$, for some x. e.g. $x_n = (-1)^n \in \mathbb{R}$, take $n_i = 2\,i$ (or $2\,i + 1$)
(ii) ⇒ (iii): X limit point compact. $(x_n)_{n=1}^{\infty}$ sequence in X, $A := \{x_n : n \geq 1\}$.

Case 1 : A infinite

X limit point compact ⇒ $\exists\, x \in A'$. Construct subsequence by induction

Pick n_1 such that $x_{n_1} \in B_1(x)$ and $n_2 > n_1$ such that $x_{n_2} \in B_{1/2}(x)$. (can do that b/c any neighborhood contains infinitelyy many point of A). By construction, $x_{n_i} \to x$ as $i \to \infty$.

Case 2 : A is finite

$A = \{a_1, a_2, \ldots, a_n\}$. $\exists\, j$ such that $\{n : x_n = a_j\}$ is infinite. Pick $n_1 < n_2 < \ldots$ such that $x_{n_i} = a_j$. $\forall\, i$
Then $x_{n_i} \to a_j$. □

Definition X satisfies Lebesgue's Lemma if $\forall$ open covers $\{U_\lambda\}$ of X, $\exists\, \delta > 0$ such that any δ–ball is contained in one of the U_λ.

Example $\mathbb{R}$ doesn't satisfy LL

Definition X is ***totally bounded*** if $\forall\, \epsilon > 0$, X is a finite union of ϵ–balls.

Remark Totally bounded ⇒ bounded.
Pick any $\epsilon > 0$. Then $X = \bigcup_{i=1}^{n} B_\epsilon(x_i)$. Then $\forall\, x, y \in X$. $\exists\, i, j$ such that $x \in B_\epsilon(x_i)$, $y \in B_\epsilon(x_j)$
$d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) \leq 2\,\epsilon + \max_{1 \leq i, j \leq n} d(x_i, x_j)$.

bounded *does not* imply totally bounded.
e.g. Take d = discrete metric. X infinite, $\epsilon < 1$.

But bounded ⇔ totally bounded for $X \subset \mathbb{R}^n$ with Euclidean metric.

(iii) ⇒(iv)

If X is not totally bounded ⇒ $\exists\, \epsilon > 0$ such that $X \neq$ finite union of ϵ–balls.

Constrct sequence $x_n \in X$ by induction. Pick x_1 arbitrarily. Pick $x_2 \in X \setminus B_\epsilon(x_1)$, … , Pick $x_n \in X \setminus \bigcup_{i=1}^{n-1} B_\epsilon(x_i)$, …
X sequentially compact ⇒ $\exists$ convergent subsequence $x_{n_i} \to x$ as $i \to \infty$.
⇒ $d(x_{n_i}, x) < \epsilon/2$ for $i >> 0$. ⇒ $d(x_{n_{i+1}}, x_{n_i}) < \epsilon/2 + \epsilon/2 = \epsilon$. Contradicts that $x_{n_i} \notin B_\epsilon(x_{n_i})$.

If X does not satisfy Lebesgue, $\exists$ open cover $\{U_\lambda\}$, $\forall\, \delta > 0$. $\exists\, \delta$–ball in X that is not contained in any U_λ.
For $\delta = 1/n$: $\exists\, x_n$ such that $B_{1/n}(x_n)$ is not contained in any U_λ. X sequentially compact ⇒ $\exists$ convergent subsequence $x_{n_i} \to x$.
$x \in U_\lambda$ some λ. U_λ open ⇒ $\exists\, \epsilon > 0$ such that $B_\epsilon(x) \subset U_\lambda$.

Pick i>>0 such that $d(x_{n_i}, x) < \frac{\epsilon}{2}$ and $\frac{1}{n_i} < \frac{\epsilon}{2}$. $\Rightarrow B_{1/n_i}(x_{n_i}) \subset B_2(x) \subset U_\lambda$. contradiction. □

(iv) ⇒ (i): X Lebesgue + totally bounded. Suppose $\{U_\lambda\}$ open cover, Lebesque ⇒ $\exists\, \delta > 0$ (Lebesque #), such that any δ–ball in X
is contained in one of the U_λ.
totally bounded ⇒ $X = \bigcup_{i=1}^{n} B_\delta(x_i)$ for some x_i. $\exists\, \lambda_i$ such that $B_\delta(x_i) \subset U_{\lambda_i}$. So $X = \bigcup_{i=1}^{n} U_{\lambda_i}$ (finite subcover).

Recall (X, d) is complete if every Cauchy sequence converges.
Recall (X, d) complete, Y complete ⇔ Y closed.
Remark Let Y be subset of X, and (X, d) metric. $x \in \overline{Y} \Leftrightarrow \exists$ sequence $y_n \in Y$ such that $y_n \to x$.