Recall

Compactness in metric spaces:

Heine–Borel $X \subset \mathbb{R}^n$ compact $\Leftrightarrow$ closed and bounded in Euclidean metric.

Theorem 44

(X, d) metric space TFAE

(i) X is compact
(ii) X is liimit point compact
(iii) X is sequentially compact
(iv) X satisfies lebesgue lemma and X is totally bounded
(v) X is complete and totally bounded

Last time we showed : (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i)
Today : (iii) $\Leftrightarrow$(v)

Definition ***Limit point compact*** if infinite subsets have limit points
Definition ***Sequentially compact*** if any sequence has a convergent subsequence.
Definition ***Totally bounded*** if for any $\epsilon > 0$, X is a finite union of ϵ–balls.
Definition ***Lebesgue Lemma*** for any open cover $\{U_\lambda\}_{\lambda \in \Lambda}$, $\exists\, \delta > 0$ s.t. any δ–ball is contained in one of the U_λ' s.
Definition ***Complete*** if every Cauchy sequence converges.

Proof: (iii) $\Rightarrow$ (v)

Assume X is sequentially compact $\Rightarrow$ totally bounded as we proved in (iii) $\Rightarrow$ (iv). Suppose (x_n) is Cauchy sequence Sequentially compact $\Rightarrow \exists\, x_{n_i} \to x$ convergent subsequence.

Claim : $x_n \to x$ as $n \to \infty$. Given $\epsilon > 0$, since x_n is Cauchy, $d(x_n, x_m) < \epsilon/2$ for all $n, m \geq N$.
Since $x_{n_i} \to x$, $\exists\, i$ such that $n_i \geq N$ and $d(x_{n_i}, x) < \epsilon/2$. For $n \geq N$, $d(x_{n_i}, x_n) < \epsilon/2$.
$\Rightarrow d(x_n, x) \leq \epsilon/2 + \epsilon/2 = \epsilon$, for all $n \geq N$. Hence X is complete.

(v) $\Rightarrow$ (iii) : X is complete and totally bounded. (x_n) sequence in X, want a convergent subsequence. Use X totally bounded to construct a subsequence that is Cauchy. X complete $\Rightarrow$ subsequence convergent. If X is totally bounded, can cover X by finite number of 1–balls. $\Rightarrow$ one of them (call it B_1) contains $x_n$for infinitely many n. Pick n, smallest such that $x_{n_1} \in B_1$ and throw away all terms of the sequence that don't lie in B_1. Repeat : $\exists\ \frac{1}{2}$–ball, B_2 such that contains x_n for infinitely many n. Pick $n_2 > n_1$, smallest such that, $x_{n_2} \in B_2$ and throw away all $n > n_2$ such that $x_n \notin B_2$. By induction we get subsequence (x_{n_i}) such that $\forall\ j \geq i$, x_{n_j} lies in a $\frac{1}{j}$–ball B_j. So $d(x_{n_k}, x_{n_l}) < \frac{2}{j}$, $\forall\ k, l \geq j$.

So (x_{n_i}) is Cauchy $\xRightarrow{X \text{ complete}} (x_{n_i})$ converges. □

Application of Lebesgue Lemma

Theorem 45

Suppose X, Y metric spaces, Let $f : X \to Y$ continuous, if X is compact, then f is uniformly continuous.

Recall $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that $d(x_1, x_2) < \delta \Rightarrow d(f(x_1), f(x_2)) < \epsilon$.

Proof:
Given $\epsilon > 0$, $Y = \bigcup_{y \in Y} B_\epsilon(y)$ open cover. $\Rightarrow$ $X = \bigcup_y f^{-1}(B_\epsilon(y))$ open cover by continuity of f.
X compact $\Rightarrow \exists$ Lebesgue number $\delta > 0$, for this open cover. Claim : $d(x_1, x_2) < \delta \Rightarrow d(f(x_1), f(x_2)) < 2\epsilon$.
Reason: $x_2 \in B_\delta(x_1) \subset f^{-1}(B_\epsilon(y))$, some y. So $f(x_1), f(x_2) \in B_\epsilon(y) \Rightarrow d(f(x_1), f(x_2)) < 2\epsilon$. □

Another characterisation of compactness.

Let X be topological space.

<u>Definition</u> A collection $\mathcal{C}$ of subsets of X has the *finite intersection property* if any finite subcollection has a non–empty intersection : $\bigcap_{i=1}^n C_i \neq \emptyset$ for all $C_i \in \mathcal{C}$, $\forall n \in \mathbb{N}$.

Proposition 46

X is compact $\Leftrightarrow$ any collection of <u>closed</u> subsets having the finite intersection property has non–empty intersection $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

Proof:
Define $\mathcal{U} := \{C^c : C \in \mathcal{C}\}$, a collection of open subsets $(\bigcap_{C \in \mathcal{C}} C)^c = \bigcup_{U \in \mathcal{U}} U$.
So $\mathcal{C}$ has non–empty intersection $\Leftrightarrow$ $\mathcal{U}$ not a cover
criterion of prop 46 $\Uparrow$ $\Uparrow$ X compact

$\mathcal{C}$ has finite intersection prop $\Leftrightarrow$ $\mathcal{U}$ has no finite subcover. □

An example : the Cantor set.

Start with $C_0 = [0, 1] \in \mathbb{R}$

(http://en.wikipedia.org/wiki/Cantor_set)

$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right], \ldots$
$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right)$.

Let $C := \bigcap_{n=0}^\infty C_n$ be cantor set. It is metrisable (since $C \subset \mathbb{R}$) and compact (closed and bounded in Euclidean metric)
What are its elements?

Use base 3 (e.g. $3 = (10)_3$, $4 = (11)_3$, $5\,(12)_3$, $6 = (20)_3$, $9 = (100)_3$, …
also, $\frac{1}{3} = (0.1)_3$, … since, $(0.1)_3 = 0 + \frac{1}{3}$

The numbers between 0 and $\frac{1}{3}$: first decimal digit is 0,
The numbers between $\frac{1}{3}$ and $\frac{2}{3}$: first decimal digit is 1,
The numbers between $\frac{2}{3}$ and 1 : first decimal digit is 2,

⋮

Similarly , between 0 and $\frac{1}{9} = (0.01)_3$: first and second digit is 0.
between $\frac{2}{9} = (0.02)_3$ and $\frac{1}{3} = (0.1)_3$: first digit is 0, second digit is 2.

⋮

$C = \{x \in [0, 1]$, that can be written in base 3 without using 1 as digit$\}$

Endpoints:
In base 3: $0, d_1, \ldots, d_n 22 \ldots$ (with $d_n \neq 2) = 0, d_1, \ldots, d_n (d_{n+1})$

For example, $(0.020202 \ldots)_3 = x$, multiplying by 9
$-\quad (2.0202 \ldots)_3 = 9x$ (subtracting from each other)

$$2 = 8x \Rightarrow x = \frac{1}{4}$$

In fact, C is totally disconnected, $X \subset C$, $|X| > 1$ is not conneceted.
The point is, by picking $x \neq y$ in X. At some stage, a point between x and y is removed.

C'(limit point of C) $= C$: Just change n^{th} digit $0 \leftrightarrow 2$ (for $n >> 0$).

Length: at n^{th} step, length $= \left(\frac{2}{3}\right)^n \to 0$ as $n \to \infty$.

<u>Definition</u> A set A is ***countable*** if $\exists$ bijection $\mathbb{N} \to A$

C is uncountable.

Proof:
Assume C is countable, So we have $C = \{x_0, x_1, \ldots\}$

e.g.
$x_0 = 0.20220202 \ldots$
$x_1 = 0.20020222 \ldots$
$x_2 = 0.2000022 \ldots$
$x_3 = 0.20220202 \ldots$
$x_4 = 0.20002002 \ldots$

pick $x = 0.20022\ldots$ cannot be one of the x_i's. (just as the proof real numbers are not countable). contradiction. □