Last Time

Theorem 44(v) states “complete+totally bounded” $\Leftrightarrow$ compact. This generalizes Heine–Borel.

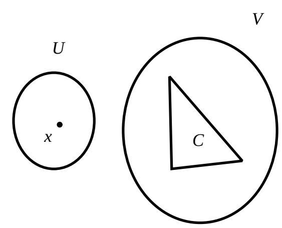
$$X \subset \mathbb{R}^n, \quad \begin{array}{ccc} X \text{ bounded in Euclidean metric} & \Leftrightarrow & \text{totally bounded} \\ X \text{ closed} & \Leftrightarrow & X \text{ compete, since } \mathbb{R}^n \text{ complete} \end{array}$$

Any compact metric space is continuous image of Cantor set! (proof will be linked on course website)

Separation Axioms

T_1 is equivalent of "points are closed", T_2 = Hausdorff. $T_2 \Rightarrow T_1$.
T_2 does not imply T_1, example : finite complement topology.

<u>Definition</u> X is ***regular*** or $\boldsymbol{T_3}$ if X is $T_1$and for any closed subset $C \subset X$ and any $x \notin C$, $\exists$ disjoint open sets U containing x and, $V \supset C$.



<u>Remark</u> $T_3 \Rightarrow T_2$ because T_3 space is T_1, so we can take $C = \{y\}$.

<u>Definition</u> X is ***normal*** or $\boldsymbol{T_4}$ if X is T_1 and for any disjoint closed subsets C, D $\exists$ disjoint open subsets $U \supset C, V \supset D$.

Remark $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$

T_2 does not imply T_3. Take $X = \mathbb{R}_K$, with basis (a, b), $(a, b) \setminus K$, where $K = \left\{\frac{1}{n} : n \geq 1\right\}$.
Finer top. than $\mathbb{R} \Rightarrow \mathbb{R}_K$ is $T_2 (\Rightarrow T_1)$

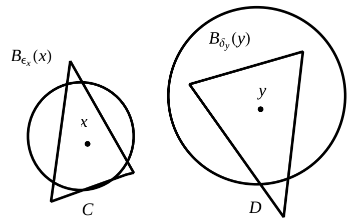
Take $C = K$ closed, and choose $x = 0$. Suppose $\exists\, U, V$ disjoint open, such that $0 \in U$, $V \supset K$.
So $0 \in (-\epsilon, \epsilon) \setminus K$(basis) $\subset U$ for some $\epsilon > 0$. Pick n, such that $\frac{1}{n} < \epsilon$. Since $\frac{1}{n} \in V \Rightarrow \left(\frac{1}{n} - \delta, \frac{1}{n} + \delta\right) \subset V$ some $\delta > 0$.
Then points close enough to $\frac{1}{n}$ will be in $U \bigcap V$. contradiction. □

Also, T_3 does not imply T_4. For example: $\mathbb{R}_l^2$. We will see next time that is T_3. It is not T_4, see book pg .198.

Theorem 47

If X is a metrizable topological space, then X is T_4.

Proof:
We know X is $T_2 \Rightarrow T_1$. Say topology comes from the metric d and $C, D \subset X$ disjoint closed.



For $x \in C$, $\exists\, \epsilon_x > 0$ such that $B_{\epsilon_x}(x) \cap D = \emptyset$ (since D^c is open)
For $y \in D$, $\exists\, \delta_y > 0$ such that $B_{\delta_y}(y) \cap C = \emptyset$.
Let $U := \bigcup_{x \in C} B_{\epsilon_x/2}(x)$ open, contains C. $V := \bigcup_{y \in D} B_{\delta_y/2}(y)$ open, contains D.
U, V disjoint: if not, $\exists\, x \in C,\ y \in D$ such that $B_{\epsilon_x/2}(x) \cap B_{\delta_y/2}(y) \neq \emptyset$. So $d(x, y) < \frac{\epsilon_x + \delta_y}{2} \leq \max(\epsilon_x, \delta_y)$.
Say $\epsilon_x \geq \delta_y$: then $d(x, y) < \epsilon_x$, $y \in B_{\epsilon_x}(x) \cap D = \emptyset$. Contradiction. □

Theorem 48

If X is compact T_2, then X is T_4.

In Corollary 39, we saw that X is T_3.

$C, D \subset X$ disjoint + closed. $\Rightarrow$ C, D compact as X is compact.
$\forall\, x \in C$, $\exists\, U_x$ containing x, $V_x \supset D$ open + disjoint since X is T_3. (note: $x \notin D$). As in corollary 39 (as C is compact).
$C \subset U_{x_1} \cup \ldots \cup U_{x_n}$ (some $x_i \in C$), $D \subset V_{x_1} \cap \ldots \cap V_{x_n}$. Clearly disjoint. □