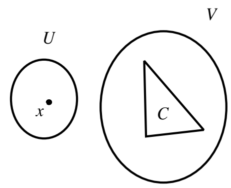
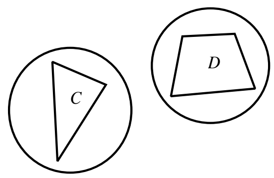
Recall X is T_3 if X is T_1 and for any closed subset $C \subset X$ and any $x \notin C$, $\exists$ disjoint open sets U containing x and, $V \supset C$.



X is T_4 if X is T_1 and



$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$

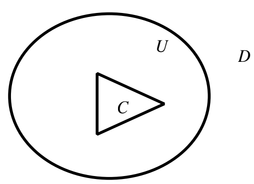
Lemma 50

X be T_1 space.

(i) X is $T_3 \Leftrightarrow \forall\, x \in U$ and $\forall\, U$ open nhd of x, $\exists\, V$ such that $\exists\, x \in V \subset \overline{V} \subset U$.
(ii) X is $T_4 \Leftrightarrow \forall\, C \subset U$ open, $\exists\, V$ open such that $C \subset V \subset \overline{V} \subset U$.

Proof: Do (ii), part (i) is analogous

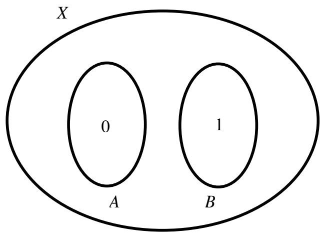
"$\Rightarrow$" X is T_4, suppose C closed $\subset U$ open. Let $D := U^c$, closed disjoint from C.
$T_4 \Rightarrow \exists\, U_1, U_2$ disjoint open such that $C \subset U_1$, $D \subset U_2$



Then, $C \subset U_1 \subset U_2^c \subset D^c = U$. Can take $V = U_1$ because $\overline{U_1} \subset U_2^c$(closed)
"$\Leftarrow$" Suppose C, D disjoint closed. Let $U := D^c$. Then C⊂U open, then ∃V open such that
$C \subset V \subset \overline{V} \subset U$. Then $C \subset V$ and $D \subset (\overline{V})^c$ disjoint open. □

§33 Urysohn's Lemma

Motivating Question : Let $A, B \subset X$ disjoint subsets



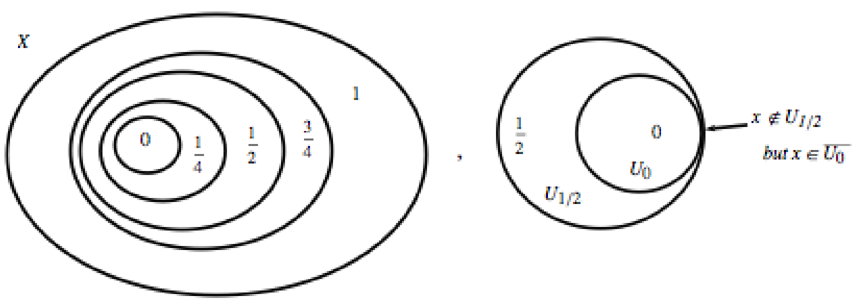
When does there exist a continuous function $f : X \to [0, 1]$ such that $f|_A \equiv 0$ and $f|_B \equiv 1$.
If f exists, $f^{-1}\{0\}$ closed subset containing A and $f^{-1}\{1\}$ closed subset containing B.
So we can restrict attention to A, B closed.

Moreover, $f^{-1}([0, 1/2))$ open subset containing A, $f^{-1}((1/2, 1])$ open subset containing B and disjoint.

Theorem 51 (Urysohn's Lemma)

Suppose X is T_4, then $\forall A, B$ disjoint closed subsets $\exists f : X \to [0, 1]$ cts such that $f|_A \equiv 0$ and $f|_B \equiv 0$

Idea: "successive approximation"



To avoid intersecting, we set $\overline{U_0} \subset U_{1/2}$

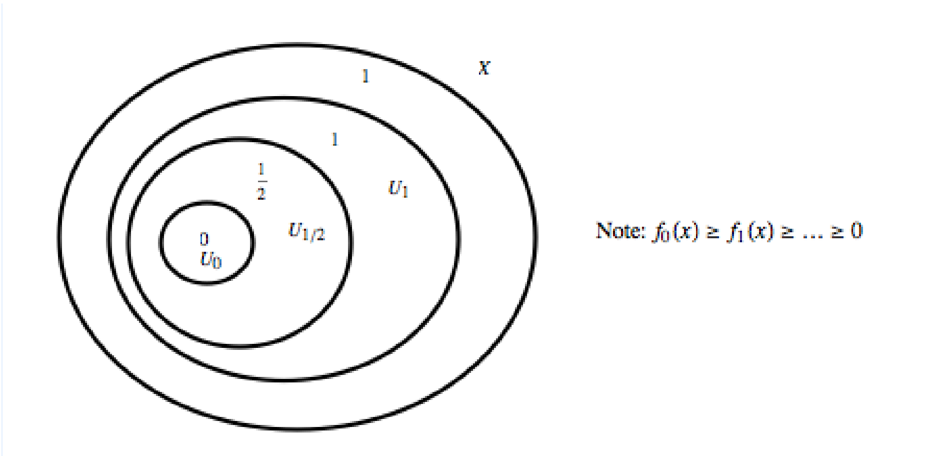
Proof: (nice pictures in Janich "Topology")

1. Choose open sets $U_{i/2^n}$. Let $U_1 := B^c$(open), note A closed $\subset B^c$ open.
X is T_4 then by lemma 50, $\exists\, U_0$ open such that $A \subset U_0 \subset \overline{U_0} \subset U_1 = B^c$
Induct: $\exists\, U_{1/2}$ open such that $U_0 \subset \overline{U_0} \subset U_{1/2} \subset \overline{U_{1/2}} \subset U_1$
$\exists\, U_{1/4}, U_{3/4}$ open such that $U_0 \subset \overline{U_0} \subset U_{1/4} \subset \overline{U_{1/4}} \subset U_{1/2} \subset \overline{U_{1/2}} \subset U_{3/4} \subset \overline{U_{3/4}} \subset U_1$
$\exists\, U_{1/8}, U_{3/8}, U_{5/8}, U_{7/8}, \ldots$

2. Functions f_n and $f : X \to [0, 1]$

For $n \geq 0$, $U_0 \subset U_{1/2^n} \subset U_{2/2^n} \subset \ldots \subset U_{2^n-1/2^n} \subset U_1$

$$f_n(x) := \begin{cases} 0 & \text{if } x \in U_0 \\ \frac{i}{2^n} & \text{if } x \in U_{i/2^n} - U_{i-1/2^n} \\ 1 & \text{if } x \notin U_1 \end{cases}$$



and $|f_n(x) - f_{n+1}(x)| \leq \frac{1}{2^{n+1}}$, $\forall\, n$. Let $f(x) = \lim_{n \to \infty} f_n(x)$, (exists because bounded monotone sequence) and obtaion $f : X \to [0, 1]$, $f|_A \equiv 0$ (even zero on U_0) and $f|_B \equiv 1$ (because $B = U_1^c$).

3. Check that f is continuous :

Note: $|f(x) - f_n(x)| \leq \frac{1}{2^n}$, use Theorem 21(iv): $\forall\, x \in X$, $\forall\, \epsilon > 0$, need to show $\exists$ nbd U of x such that $f(U) \subset B_\epsilon(f(x))$.
Fix some n, $\exists\, i$ such that $x \in U_{i/2^n} - \overline{U_{i-2/2^n}}$ (this is open).
why does i exist?, take i minimal such that $x \in U_{i/2^n}$, then if $x \in \overline{U_{i-2/2^n}} \subset U_{i-1/2^n}$. contradiction.
For any $y \in U_{i/2^n} - \overline{U_{i-2/2^n}}$, $|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)| \leq \frac{1}{2^n} + \frac{1}{2^n} + \frac{1}{2^n} = \frac{3}{2^n}$.
Choose $n >> 0$ such that $\frac{3}{2^n} < \epsilon$. □

Theorem 49

$\mathbb{R}_l$ is T_4

Proof:
we know $\mathbb{R}_l$ is $T_2 \Rightarrow T_1$ (since finer than $\mathbb{R}$). For C, D disjoint closed.
For $x \in C$ since $x \in D^c$ open, $\exists\, c_x > x$ such that $[x, c_x) \subset D^c$
For $y \in D$ since $y \in C^c$ open, $\exists\, d_y > y$ such that $[y, d_y) \subset C^c$

$U := \bigcup_{x \in C} [x, c_x)$, $V := \bigcup_{y \in D} [y, d_y)$ where $C \subset U$, and $D \subset V$

Claim: $U \cap V = \emptyset$

If not, $\exists\ x \in C,\ y \in D$ such that $[x, c_x) \cap [y, d_y) \neq \emptyset$
WLOG, $x \leq y \Rightarrow y\,(\in D) \in [x, c_x)\,(\subset D^c)$. contradiction. □

Theorem 52

(a) X is $T_3 \Rightarrow$ any subspace is T_3
(b) X_λ is T_3 for all $\lambda \in \Lambda \Rightarrow \prod_{\lambda \in \Lambda} X_\lambda$ is T_3

Corollary $\mathbb{R}_l^2$ is T_3

Remark Since it's known that $\mathbb{R}_l^2$ is *not* T_4, the analogue of Theorem 52 (b) is not true. It is known that analogue of Theorem 52 (a) for T_4 also fails.

Proof:
$T_3 \Rightarrow T_2$ and we know subspace and products of T_2 spaces are T_2 ($\Rightarrow T_1$)
(a) Say $Y \subset X$ subspace. Suppose $D \subset Y$ closed, $y \in Y \setminus D$. Then $\exists\, C \subset X$ closed such that $D = Y \bigcap C$.
Note: $y \notin C$ (since $y \in Y$). X is $T_3 \Rightarrow y \in U$, $C \subset V$ disjoint open in X. Then, $U \bigcap Y$, $V \bigcap Y$ disjoint open in Y.
$C \bigcap Y = D$.