Recall   Urysohn's Lemma: X is T_4 then $\forall\, A, B$ disjoint closed subset, $\exists\ f : X \to [0, 1]$ cts such that $f\,|_A = 0$ and $f\,|_B = 1$.

Remark   This works for $[a, b]$ instead of $[0, 1]$ since $[0, 1] \cong [a, b]$.

Theorem 52 (b)

X_λ is $T_3\ \forall\, \lambda \Rightarrow \prod_{\lambda\in\Lambda} X_\lambda$ is T_3.
Proof:
Use criterion of Lemma 50 $\left(\exists\, V \text{ open such that } x \in V \subset \overline{V} \subset U\right)$, Fix $\mathbf{x} = (x_\lambda) \in \prod X_\lambda,\ \mathbf{x} \in U$
So $\exists$ basic open nbd, $\prod_{\lambda\in\Lambda} V_\lambda$ of x contained in U. (V_λ open in X_λ and $V_\lambda = X_\lambda$ for all but finitely many λ.)
Since X_λ is $T_3\ \exists\, W_\lambda$ such that $x_\lambda \in W_\lambda \subset \overline{W_\lambda} \subset V_\lambda$. Whenever $V_\lambda = X_\lambda$ take $W_\lambda := X_\lambda$
Then $W := \prod W_\lambda$ is open in $\prod X_\lambda$.

Claim : $\overline{W} \subset \prod V_\lambda$
For any $\mu \in \Lambda,\ p_\mu : \prod X_\lambda \to X_\mu$ is cts. $\Rightarrow p_\mu(\overline{W}) \subset \overline{p_\mu(W)}$ (by Theorem 21) $= \overline{W_\mu} \subset V_\mu$. □

§35 Tietze Extension Theorem

Proposition 53

Y metric space, suppose $f_n : X \to Y$ cts $(n \geq 1)$, and $f : X \to Y$. If $f_n \to f$ uniformly, then f is continuous.

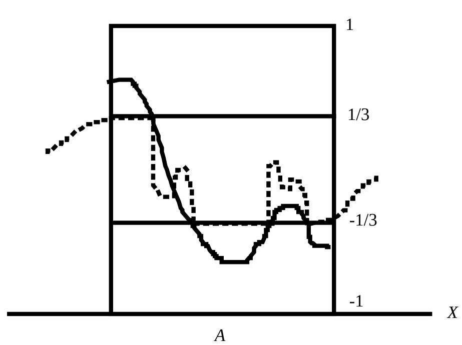
uniformly means : $\forall\, \epsilon > 0,\ \exists\, n_0 \geq 1$ such that $d(f_n(x),\ f(x)) < \epsilon,\ \forall\, x \in X,\ \forall\, n \geq n_0$.

Example

$f_n : X \to \mathbb{R}$ cts, $M_n \in \mathbb{R}_{\geq 0}$ such that $|f_n(x)| \leq M_n,\ \forall\, x$ and $\sum M_n$ converges $\Rightarrow \sum f_n$ converges uniformly and its continuous.
(Weierstrass M test)

Theorem 54 (Tietze)

X be T_4 space, $A \subset X$ closed, $f : A \to [a, b]$ cts, then $\exists\ \tilde{f} : X \to [a, b]$ cts such that $\tilde{f}\,|_A = f$.
Proof:
WLOG, $[a, b] = [-1, 1]$ (they are homeo)
Idea: use successive approximation.



If $g : A \to [-1, 1]$ satisfies.

$$g(x) = \begin{cases} 1/3 & \text{if } f(x) \geq 1/3 \\ \in [-1/3, 1/3] & \text{if } f(x) \in [-1/3, 1/3] \\ -1/3 & \text{if } f(x) \leq -1/3 \end{cases}$$

Then $|f(x) - g(x)| \leq \frac{2}{3}\ \forall x \in A$

$C := f^{-1}\left(\left[\frac{1}{3}, 1\right]\right)$ closed in $A \to$ closed in X (as A is)

$D := f^{-1}\left(\left[-1, -\frac{1}{3}\right]\right)$ closed in $A \to$ closed in X (as A is)

$\Rightarrow C, D$ is disjoint closed

Uryusohn's Lemma $\Rightarrow \exists\, f_1 : X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ cts such that $f_1|_C \equiv \frac{1}{3},\ f_1|_D \equiv -\frac{1}{3}$

Moreoever, $|f(x) - f_1(x)| \leq \frac{2}{3},\ \forall x \in A.$

Hence, $f - f_1|_A : A \to \left[-\frac{2}{3}, \frac{2}{3}\right]$ cts

The same argument gives: $\exists\, f_2 : X \to \left[-\frac{2}{3^2}, \frac{2}{3^2}\right]$ cts such that $|f(x) - f_1(x) - f_2(x)| \leq \left(\frac{2}{3}\right)^2,\ \forall x \in A$

$\exists\, f_n : X \to \left[-\frac{1}{3}\left(\frac{2}{3}\right)^{n-1}, \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}\right]$ cts such that $|f(x) - f_1(x) - \ldots - f_n(x)| \leq \left(\frac{2}{3}\right)^n$ (✲)

Let $\tilde{f} := \sum_{n=1}^{\infty} f_n$. Note: $|f_n| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$

Weierstrass M test $\Rightarrow \sum_{n=1}^{\infty} f_n$ converges uniformly, so $\tilde{f}$ cts.

$\left|\tilde{f}(x)\right| \leq \left|\sum_{n=1}^{\infty} f_n(x)\right| \leq \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = 1.$

Get $\tilde{f} : X \to [-1, 1]$ cts. Let $n \to \infty$ in (✲) $\Rightarrow \tilde{f}|_A = f.$ □