Recall Tietze Extension Theorem

X T_4 space, $A \subset X$ closed then $f : A \to [a, b]$ cts $\Rightarrow \exists\ \tilde{f} : X \to [a, b]$ cts, $\tilde{f}\big|_A = f$

Small remark $f_1, f_2 : X \to \mathbb{R}$ cts, then $f_1 + f_2$, $f_1 f_2$, … cts.

$X \to \mathbb{R}^2 \overset{+}{\to} \mathbb{R}$
$x \longmapsto (f_1(x), f_2(x)) \to f_1(x) + f_2(x)$.

Corollary 55

Tietze also true for functions to $\mathbb{R}$
Proof:
Note : $\mathbb{R} \cong (-1, 1)$
$x \longmapsto \frac{\arctan x}{\pi/2}$
Given $f : A \to (-1, 1)$ cts, by Theorem 54 $\exists\, g : X \to [-1, 1]$ such that $g|_A = f$
"Problematic set" : $g^{-1}(\{-1, 1\})$ closed and disjoint from A.
By Urysohn $\exists\, \lambda : X \to [0, 1]$ such that $\lambda|_A \equiv 1$, $\lambda_{g^{-1}(\{-1,1\})} \equiv 0$.
Define $\tilde{f} := \lambda\, g$. Then $\tilde{f}\big|_A = g\big|_A = f$. But $\tilde{f}(x) \notin \{-1, 1\}$ for all x, since $|\tilde{f}| \leq |g|$. □

Remark From $[a, b]$–version get $[a, b]^n$–version
From $\mathbb{R}$–version get $\mathbb{R}^n$–version
It is also true for $f : A \to \mathbb{R}^n$ such that $|f(x)| \leq r$ for all x.
Reason: $\{x \in \mathbb{R}^n : |x| \leq r\} \cong [-1, 1]^n$.

Compactness (lecture by Michael)

Compact spaces are small spaces.

3 questions for topologists when one finds a new property.

1) does any space have it?
2) perserved by continuous mappings?
3) do subspaces have these property?

Are these sets compact ?

$\{0\}^\omega$
$\{0, 1\}^\omega$

How to prove these sets are compact ?
1) Use Konig's Lemma
2) Cantor set $\cong \{0, 1\}^\omega$

How about these sets?
$[0, 1]^\omega$ product topology
$[0, 1]^\infty$ box topology

Use compactness definition which works for all spaces.

Theorem A space X is compact if and only if for every collection $\mathcal{F}$ of closed subsets of X with the finite intersection property, then $\bigcap \mathcal{F} \neq \emptyset$.

Recall In $\prod X_\lambda$, basic open sets are of the form $\prod^{-1}{}_{\lambda_1}(U_\lambda) \bigcap \ldots \bigcap \prod^{-1}{}_{\lambda_n}(U_\lambda)$.

Definition Let X be a nonempty set, $\mathcal{F}$ a collection of subsets of X. $\mathcal{F}$ is a filter(on X) if

i) $\mathcal{F}$, it has FIP$^+$ (i.e. for any collection $F \subseteq \mathcal{F}$, $\bigcap F \in \mathcal{F}$.)
ii) $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$
iii) $\mathcal{F}$ is closed upwards (if $A \in \mathcal{F}$, $B \supseteq A$, then $B \in \mathcal{F}$)

Example

Let $\mathcal{F} = \{(a, b) \subseteq \mathbb{R} : 100 \in (a, b)\}$ almost a filter (fails the third property)

$\hat{\mathcal{F}} = \{A \subseteq \mathbb{R} : \exists\, (a, b) \ni 100$ and $(a, b) \subseteq A\}$ This is a filter.

Biggest filter ? $\mathcal{F} \bigcup \{\{100\}\}$? No

Since $\{100\} \bigcup (0, 1) \notin \mathcal{F} \bigcap \{\{100\}\}$

So we define $\overline{\mathcal{F} \bigcup \{\{100\}\}} = \{A \subseteq \mathbb{R} : 100 \in A\}$ = maximal(ultra) filter. (the notion $\frown$ means closed upwards)

Definition A filter $\mathcal{F}$ is maximal(or an ultrafilter) if whenever $\mathcal{D} \supseteq \mathcal{F}$ is a filter, $\mathcal{D} = \mathcal{F}$.

Fact 1 For every filter $\mathcal{F}$, there is an ultrafilter $\mathcal{D} \supseteq \mathcal{F}$.
Fact 2 If $\mathcal{F}$ is an ultrafilter(on X) $A \subseteq X$, TFAE

i) $A \in \mathcal{F}$
ii) $\forall\, F \in \mathcal{F}$, $A \bigcap F \neq \emptyset$

Proof:(of fact 2)
"$\Rightarrow$" Obvious. "$\Leftarrow$" $\forall\, F \in \mathcal{F}$, $F \bigcap A \neq \emptyset$. Check that $\mathcal{F} \bigcup \{A\}$ has FIP$^+$.
Take $F_1, \ldots, F_n \in \mathcal{F}$, then $F_1 \bigcap \ldots \bigcap F_n \in \mathcal{F}$. So $F_1 \bigcap \ldots \bigcap F_n \bigcap A \neq \emptyset$.
Add to $\mathcal{F}$ all sets of the form $(F_1 \bigcap F_2 \bigcap \ldots \bigcap F_n \bigcap A)$ e.g. $(F_7 \bigcap F_9 \bigcap A)$
Let all sets of the above form $\mathcal{F}_2$
Then $\overline{\mathcal{F} \bigcup \mathcal{F}_2}$ is a filter, $\mathcal{F} \subseteq \overline{\mathcal{F} \bigcup \mathcal{F}_2}$. So by maximality, $\mathcal{F} = \overline{\mathcal{F} \bigcup \mathcal{F}_2}$

Proof(of fact 1) : use Zorn's lemma
Let $\mathcal{F}$ be a filter, and let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F}_n \subseteq \ldots \subseteq \mathcal{F}_\lambda \subseteq \ldots$
If we show $\bigcup_{\lambda \in I} \mathcal{F}_n$ is a filter, ZL says $\mathcal{F}$ can be extended to an ultra filter.

$\{\mathcal{F}_\lambda \geq \mathcal{F} : \lambda \in I\}$ is a ***chain*** if given $\lambda_1, \lambda_2 \in I$ then $\mathcal{F}_{\lambda_1} \subseteq \mathcal{F}_{\lambda_2}$ or $\mathcal{F}_{\lambda_1} \supseteq \mathcal{F}_{\lambda_2}$.
Call $\mathcal{F}_\infty = \bigcup_{\lambda \in I} \mathcal{F}_\lambda$. We know $X \in \mathcal{F}_\infty$, $\emptyset \notin \mathcal{F}_\infty$ and $\mathcal{F} \subseteq \mathcal{F}_\infty$.

Claim $\mathcal{F}_\infty$ has FIP$^+$.
Let $A_1, \ldots, A_n \in \mathcal{F}_\infty$. $\exists\, \mathcal{F}_{\lambda_1}, \mathcal{F}_{\lambda_2}, \ldots, \mathcal{F}_{\lambda_n}$ be in the chain such that $A_i \in \mathcal{F}_{\lambda_i}$.
So there is a λ_N such that $\mathcal{F}_{\lambda_i} \subseteq F_{\lambda_N}$, $\forall\, i$. Check closed upwards.

Tychonoff's Theorem

Let X_λ be compact then $\prod X_\lambda$ is compact.

Proof:
Let $\mathcal{F}$ be a collection of closed sets with the FIP. Let $\mathcal{F}_1 = \{F_1 \bigcap F_2 \bigcap \ldots \bigcap F_n : F_i \in \mathcal{F}\}$
Then $\overline{\mathcal{F} \bigcup \mathcal{F}_1}$ is a filter. From then we extend $\overline{\mathcal{F} \bigcup \mathcal{F}_1}$ to an ultrafilter $\mathcal{U}$. We want $\emptyset \neq \bigcap \mathcal{U} \subseteq \bigcap \mathcal{F}$
Let $\mathcal{U}_\lambda = \left\{\gamma \subseteq X_\lambda : \pi_\lambda^{-1}(\gamma) \in \mathcal{U}\right\}$. Let $\pi_\lambda^{-1}(A \bigcap B) = \pi_\lambda^{-1}(A) \bigcap \pi_\lambda^{-1}(B)$. Since π_λ are continuous, $\pi_\lambda^{-1}(C)$ is closed for C closed.

<u>Fact</u> $\mathcal{U}_\lambda$ is an ultrafilter on X_λ, we know $\bigcap \mathcal{U}_\lambda \neq \emptyset$. Pick $\mathcal{F}(\lambda) \in \bigcap \mathcal{U}_\lambda$
<u>Claim</u> $\mathcal{F} : I \to \bigcup X_\lambda$ given by $\mathcal{F}(\lambda)$
i) $\mathcal{F} \in \prod X_\lambda$
ii) $\mathcal{F} \in \bigcap \mathcal{U}$

Want if B a basic open set contains $\mathcal{F}$, then $B \in \mathcal{U}$. It is enough to show every subbasic open set $S \ni \mathcal{F}$ is in $\mathcal{U}$.
if (i), then $\mathcal{F} \in \bigcap_{B \in \mathcal{B}, \mathcal{F} \in B} \mathcal{B} \subseteq \bigcap \mathcal{U} = \overline{\bigcap \mathcal{U}}$

Let $S = \pi_\lambda^{-1}(U_\lambda) \ni \mathcal{F}$. ($U_\lambda$ open in X_λ). So $\mathcal{F}(\lambda) \in U_\lambda$. Also $\mathcal{F}(\lambda) \in \bigcap \mathcal{U}_\lambda$. $U_\lambda \bigcap \bigcap \mathcal{U}_\lambda \neq \emptyset$.
So $U_\lambda \bigcap F \neq \emptyset$, $\forall F \in \mathcal{U}$. So $U_\lambda \in \mathcal{U}_\lambda$. Now let $\mathcal{F} \in U$ an open sset in $\prod X_\lambda$. So there is a basic open B such that $\mathcal{F} \in B \subseteq U$.

TFAE

i) Zorn's lemma
ii) Axiom of choice
iii) Tychonoff's theorem