§51, 52 Homotopy and Fundamental Groups

Suggested textbook : Armstrong "Basic Topology"

Let $I := [0, 1]$ ($\subset \mathbb{R}$)

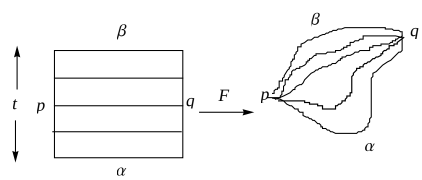
<u>Definition</u> $f, g : X \to Y$ cts, then f, g are ***homotopic*** (write $f \simeq g$) if $\exists\, F : X \times I \to Y$ such that such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$, $\forall\, x \in X$. F is called ***homotopy*** (between f, g)

<u>Example</u>

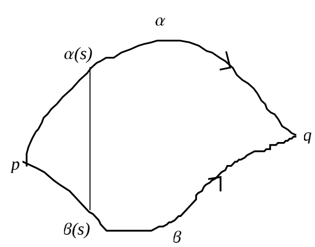
$f : \mathbb{R}^2 \to \mathbb{R}^2$ rotation by angle α (around 0), then $f \simeq$ id(identity map)
Homotopy at time t : rotation by tα.

Focus on $X = I$, i.e. consider paths $I \to Y$.

<u>Definition</u> $\alpha, \beta : I \to X$ (cts) are path homotopic (write $\alpha \simeq_p \beta$) if α, β have same endpoints p, q.
$\exists\, F : I \times I$ (time) $\to X$ such that $F(s, 0) = \alpha(s)$, $F(s, 1) = \beta(s)$, $F(0, t) = p$, $F(1, t) = q$.



<u>Examples</u>
1) $X = \mathbb{R}^2$, Fix $p, q \in X$. Any two paths from p to q are homotopic.

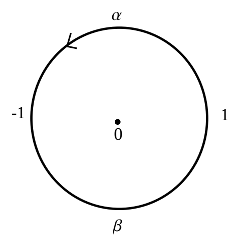


Fix s, idea : connect by straight lines.

Explicitly: $F(s, t) = (1 - t)\,\alpha(s) + t\beta(s)$, clearly continuous.

Similarly this works fro all convex subsets in $\mathbb{R}^n$

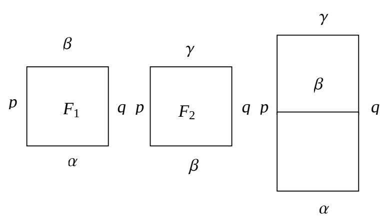
2) $X = \mathbb{R}^2 \setminus \{0\} \cong \mathbb{C} \setminus \{0\}$. We'll see that $\alpha(s) = e^{\pi i s}$, $\beta(s) = e^{-\pi i s}$ <u>not</u> path homotopic.



Proposition 56

(Path) homotopy is an equivalence relation.
Proof: do only for $\simeq_p$
$\alpha \simeq_p \alpha$ obvious : $F(s,\ t) = \alpha(s)$, $\alpha \simeq_p \beta \Rightarrow \beta \simeq_p \alpha$: revser time $t \leftrightarrow 1 - t$

$$\alpha \simeq_p \beta,\ \ \beta \simeq_p \gamma \Rightarrow \alpha \simeq_p \gamma$$



$$F(s,\ t) = \begin{cases} F_1(s,\ 2\,t) & 0 \le t \le 1/2 \\ F_2(s,\ 2\,t - 1) & 1/2 \le t \le 1 \end{cases}$$

By pasting lemma (book 18.3) shows F is continuous. □

Constant path: given $p \in X$, let $\epsilon_p : I \to X$ for all s
$$s \longmapsto p$$

Reverse path: given path α, define $\overline{\alpha} : I \to X$
$$s \longmapsto \alpha(1 - s)$$

Concatenation: If $\alpha,\ \beta : I \to X$ such that $\alpha(1) = \beta(0)$, then can define

$$\alpha * \beta : I \to X,\ \ s \longmapsto \begin{cases} \alpha(2\,s) & 0 \le s \le 1/2 \\ \beta(2\,s - 1) & 1/2 \le s \le 1 \end{cases} \text{ (continuous by pasting lemma)}$$

Proof: $(\alpha * \beta) * \gamma \neq \alpha * (\beta * \gamma)$

Theorem 57

(i) α path from p to q, $\alpha \neq \epsilon_q \simeq_p \alpha$ and $\epsilon_p * \alpha \simeq_p \alpha$

Proof: Only do $\epsilon_p * \alpha \simeq_p \alpha$.

Let $F(s,\ t) = \begin{cases} p & t \geq 2\,s \\ \alpha\left(\frac{2\,s-t}{2-t}\right) & t \leq 2\,s \end{cases}$ □