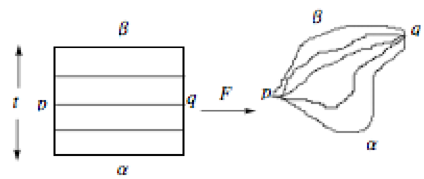
Path Homotopy

$\alpha, \beta : I \to X$ (cts) paths of some end points.

$\alpha \simeq_p \beta$ if $F : I \times I \to X$ such that



<u>constant path</u> $\epsilon_p(s) = p$ for all s

<u>reverse path</u> $\overline{\alpha}(s) = \alpha(1 - s)$

<u>concatenation</u> $(\alpha * \beta)(s) = \begin{cases} \alpha(2s) & s \in [0, 1/2] \\ \beta(2s - 1) & s \in [1/2, 1] \end{cases}$

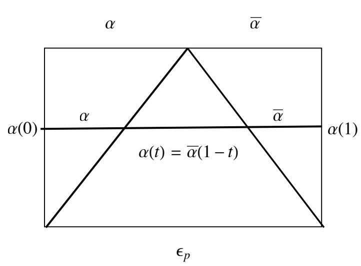
Theorem 57

α : path from p to q
β : path from q to r
γ : path from r to x

(i) $\alpha * \epsilon_q \simeq_p \alpha \simeq_p \epsilon_p * \alpha$
(ii) $\alpha * \overline{\alpha} \simeq_p \epsilon_p$, $\overline{\alpha} * \alpha \simeq_p \epsilon_q$
(iii) $(\alpha * \beta) * \gamma \simeq_p \alpha * (\beta * \gamma)$

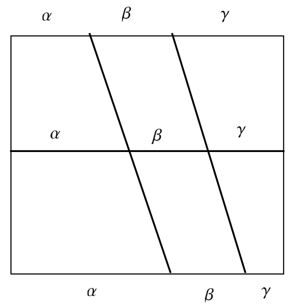
Proof:
(i) last time

(ii)



$$F(s, t) = \begin{cases} \alpha(2s) & s \leq t/2 \\ \alpha(t) & t/2 \leq s \leq 1 - t/2 \\ \overline{a}(2s - 1) & s \geq 1 - t/2 \end{cases}$$

(iii)

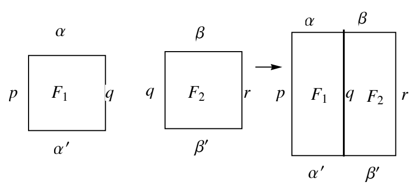


$$F(s,t)=\begin{cases}\alpha\left(\frac{4s}{t+1}\right) & t\geq 4s-1\\ \beta(4s-t-1) & 4s-2\leq t\leq 4s-1\\ \gamma\left(\frac{4s-2-t}{2-t}\right) & t\leq 4s-2\end{cases}$$

Proposition 58

If $\alpha\simeq_p\alpha'$, $\beta\simeq_p\beta'$ and $\alpha(1)=\beta(0)$, then $\alpha*\beta\simeq_p\alpha'*\beta'$ Also , $\overline{\alpha}\simeq_p\overline{\alpha'}$

Proof:



Fix $p\in X$. Let $\Omega(X,p)$:= "loops at p, i.e paths $\alpha: I\to X$ such that $\alpha(0)=\alpha(1)=p$ ". Then $\epsilon_p\in\Omega(X,p)$
$\alpha,\beta\in\Omega(X,p)\Rightarrow\overline{\alpha},\ \alpha*\beta\in\Omega(X,p)$.

Definition The ***fundamental group*** $\pi_1(X,p)$ is the set of path homotopy equivalence classes in $\Omega(X,p)$
i.e. $\pi_1(X,p)=\Omega(X,p)/\simeq_p$

Example $X\subset\mathbb{R}^n$ convex subset, then $\pi_1(X,p)$ consists of one element only (see Ex. last time)

Write $[\alpha]$ for the $\simeq_p$ −equivalence class of a path α. By prop 58, $[\alpha]=[\alpha']\Rightarrow[\overline{\alpha}]=[\overline{\alpha'}]$.
So can define $\overline{[\alpha]}:=[\overline{\alpha}]$. as the right–hand side only depends on $[\alpha]$.
Similarly, can define $[\alpha]*[\beta]:=[\alpha*\beta]$ since the right–hand side only depends on $[\alpha]$ and $[\beta]$. (by prop 58)

From Theorem 57 we get

Corollary 59

Suppose $[\alpha],[\beta],[\gamma]\in\pi_1(X,p)$, then
(i) $[\alpha]*[\epsilon_p]=[\alpha]=[\epsilon_p]*[\alpha]$
(ii) $[\alpha]*\overline{[\alpha]}=[\epsilon_p]=\overline{[\alpha]}*[\alpha]$
(iii) $([\alpha]*[\beta])*[\gamma]=[\alpha]*([\beta]*[\gamma])$.

This means that $\pi_1(X, p)$ together with the operation $*$ ($* : \pi_1 \times \pi_1 \to \pi_1$)
(i): $[\epsilon_p]$ ***identity*** element.
(ii) $\overline{[\alpha]}$ ***inverse*** of $[\alpha]$ (identity and inverse elements are infact unique).
(iii) $*$ is ***associative***

Examples

$(\mathbb{Z}, +)$ group: id $= 0$, inverse of $n = -n$, associative.
Similarly, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, …

D_6 = symmetries of equilateral triangle. (2 rotations, 3 reflections, identity)

Find group of the circle §53, 54

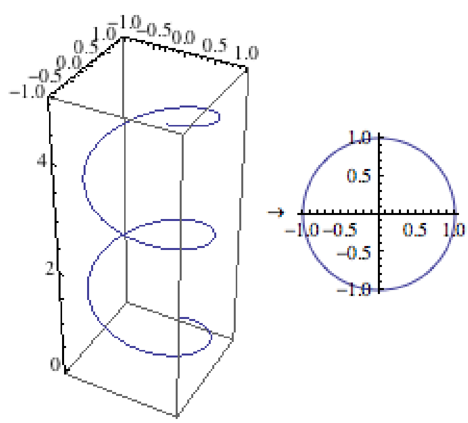
$S^1 := \{z \in \mathbb{C} : |z| = 1\}$

Goal: $\pi_1(S^1, 1) \cong (\mathbb{Z}, +)$ – bijection that preserves the group operation.

Intuition: we can assign its winding number $\in \mathbb{Z}$.

$\pi : \mathbb{R} \to S^1$ (cts)
$x \longmapsto e^{2\pi i x}$

$e^{2\pi i x} = 1$, $e^{2\pi i x} = \cos(2\pi x) + i\sin(2\pi x) \Leftrightarrow x \in \mathbb{Z}$.
$e^{2\pi i x} = e^{2\pi i y} \Leftrightarrow x - y \in \mathbb{Z}$



The Idea is can lift a loop at $1 \in S^1$ to a path starting at $0 \in \mathbb{R}$. Endpoint $\in \mathbb{Z}$

Consider $U_+ := S^1 \setminus \{-1\}$, $U_- := S^1 \setminus \{1\}$, open subsets.

Proposition 60

$\pi^{-1}(U_+) = \bigcup_{n\in\mathbb{Z}} \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$ (disjoint union) and $\pi|_{(n-1/2, n+1/2)} : (n - 1/2, n + 1/2) \to U_+$ is a homeomorphism.
Similarly, for U_-. So for any point $x \in S^1$, there is a neighborhood (either U_+ or U_-) that has a simple preimage.

Proposition 61 (path lifting)

Suppose $\alpha : I \to S^1$ is a path. Fix $x \in \pi^{-1}(\alpha(0))$. Then there is a unique path $\tilde{\alpha} : I \to \mathbb{R}$ such that $\tilde{\alpha}(0) = x$ and $\pi \circ \tilde{\alpha} = \alpha$.