<u>Recall</u> X topological space, $p \in X$

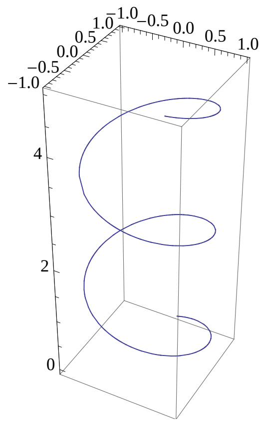
$$\pi_1(X,\ p) := \text{loops in } X \text{ at } p \ / \simeq_p$$
$$= \{\text{path homotopy equivalence classes } [\alpha] \text{ of loops } \alpha \text{ at } p\}$$
$$[\alpha] = [\beta] \Leftrightarrow \alpha \simeq_p \beta$$

$\pi_1(X,\ p)$ has only one element, if $X \subset \mathbb{R}^n$ convex.
$\pi_1(X,\ p)$ is a group: $[\alpha] * [\beta] := [\alpha * \beta]$.

$$\overline{[\alpha]} := [\overline{\alpha}]$$
$$\left[\epsilon_p\right] \text{ (identity)}$$

Goal: $\pi_1\left(S^1,\ 1\right) \cong (\mathbb{Z},\ +)$

$$\mathbb{R} \xrightarrow{\pi} S^1$$
$$x \longmapsto e^{2\pi i x}$$



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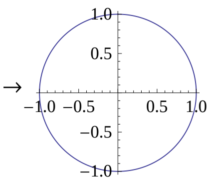


Proposition 61 ("path lifting")

Suppose $\alpha : I \to S^1$is a path. Fix $x \in \pi^{-1}(\alpha(0))$. Then there is a unique path $\tilde{\alpha} : I \to \mathbb{R}$ such that $\tilde{\alpha}(0) = x$ and $\pi \circ \tilde{\alpha} = \alpha$.
Proof uses U_+ and U_- (can be found in the textbook)

Proposition 62 ("Homotopy lifting")

Suppose $\alpha,\ \beta : I \to S^1$ have same endpoints. Fix $x \in \pi^{-1}(\alpha(0))$. Then $\alpha \simeq_p \beta \Rightarrow \tilde{\alpha} \simeq_p \tilde{\beta}$, where $\tilde{\alpha},\ \tilde{\beta}$ lifted path with starting point x.

Theorem 63

$\left(\pi_1\left(S^1,\ 1\right),\ *\right) \cong (\mathbb{Z},\ +)$ or simply $\pi_1\left(S^1,\ 1\right) \cong \mathbb{Z}$.

Proof:

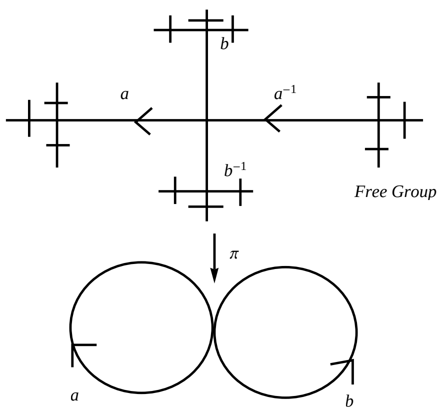
Define $\theta : \pi_1\left(S^1,\ 1\right) \to \mathbb{Z}$, where $\tilde{\alpha}$ = lifted path from Proposition 61 with $\tilde{\alpha}(0) = 0$.

$$[\alpha] \longmapsto \tilde{\alpha}(1)$$

This does not depend on the choice of α : $[\alpha] = [\beta] \Rightarrow [\tilde{\alpha}] = [\tilde{\beta}]$ by Proposition 62. $\Rightarrow \tilde{\alpha}(1) = \tilde{\beta}(1)$.

Claim 1: θ is surjective.
Given $n \in \mathbb{Z}$, take any path $\tilde{\alpha} : I \to \mathbb{R}$ such that $\tilde{\alpha}(0) = 0$, $\tilde{\alpha}(1) = n$ $\quad$ (*e.g.* $\tilde{\alpha}(s) = ns$)
Let $\alpha := \pi \circ \tilde{\alpha}$ $\qquad$ (*e.g.* $\alpha\,(s) = e^{2\pi i n s}$).
Then $\tilde{\alpha}$ is a path lifting α and starting at 0, so it's the lift of Proposition 61 gives $\Rightarrow \theta\,([\alpha]) = n$.

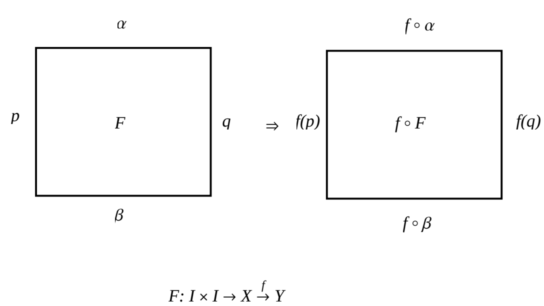
Claim 2: θ is injective.
If $\theta([\alpha]) = \theta([\beta])$, then $\tilde{\alpha}(1) = \tilde{\beta}(1)$. So $\tilde{\alpha}$, $\tilde{\beta}$ are paths in $\mathbb{R}$ with same endpoints $\Rightarrow \tilde{\alpha} \simeq_p \tilde{\beta}$.
$\Rightarrow (\pi \circ \tilde{\alpha} = \alpha) \simeq_p (\pi \circ \tilde{\beta} = \beta)$, which means $[\alpha] = [\beta]$.
Claim 3: $\theta([\alpha] * [\beta]) = \theta([\alpha]) + \theta([\beta])$ ("group homomorphism")
(left as an exercise). □



Lemma 64

If $\alpha, \beta : I \to X$ are paths and $f : X \to Y$ continuous then $\alpha \simeq_p \beta \Rightarrow f \circ \alpha \simeq_p f \circ \beta$.

Proof:
If F is a path homotopy between α and β then $f \circ F$ is a path homotopy between $f \circ \alpha$ and $f \circ \beta$.



Disc $D^2 = \{x \in \mathbb{R}^2 : |x| \leq 1\}$
$S^2 = \{x \in \mathbb{R}^3, |x| = 1\}$

Theorem 65 (Brouwer fixed point theorem)

Any continuous function $f : D^2 \to D^2$ has a fixed point ie. $\exists\, x \in D^2$ such that $f(x) = x$.

Proof:
Suppose that f has a no fixed point. Then for $x \in D^2$, we can define $r(x)$ in S^1 as follows:

It is the intersection point of the line from $f(x)$ to x with S^1(the point that is closer to x)
This works because we assumed $f(x) \neq x$.
$r: D^2 \to S^1$; (not bad to check that r is continous.) Note that $r(x) = x$ if $x \in S^1$.

$$S^1 \xrightarrow{i} D^2 \xrightarrow{r} S^1, \text{ hence } r \circ i = \text{Id}$$

Suppose $\alpha: I \to S^1$ in a loop at 1, then $i \circ \alpha = I \to D^2$. Then $i \circ \alpha \simeq \epsilon_1$ constant loop since $D^2 \subset \mathbb{R}^2$ is a convex subset.
$\Rightarrow \overbrace{r \circ i \circ \alpha}^{\alpha} \simeq_p \overbrace{r \circ \epsilon_1}^{\epsilon_1}$ in S^1. So $\pi_1(S^1, 1) = \{[\epsilon_1]\}$: contradicting Theorem 63. □

Theorem 66 (Borsuk-Ulam Theorem)

$\forall f: S^2 \to \mathbb{R}^2$ continous, $\exists x \in S^2$ such that $f(x) = f(-x)$.
Proof:
If not, define $g(x) := \frac{f(x) - f(-x)}{|f(x) - f(-x)|} \in S^1$, note $f(x) - f(-x) \neq 0$ by assumption.
$g: S^2 \to S^1$ continous. Note $g(-x) = -g(x)$.

Consider the loop $\alpha: I \to S^2$
$s \longmapsto (\cos(2\pi s), \sin(2\pi s), 0)$

Let $\beta = g \circ \alpha: I \to S^1$ (loop)
Note: $\beta\left(s + \frac{1}{2}\right) = -\beta(s)$ for all $s \in \left[0, \frac{1}{2}\right]$

$\beta\left(s + \frac{1}{2}\right) = g\left(\alpha\left(s + \frac{1}{2}\right)\right) = g(-\alpha(s)) = -g(\alpha(s))$ (by oddness) $= -\beta(s)$.

By Proposition 61, can lift β to a path $\tilde{\beta}: I \to \mathbb{R}$ $\left(\text{pick any starting point } \in \pi^{-1}(\beta(0))\right)$

For any $s \in \left[0, \frac{1}{2}\right]$, $\tilde{\beta}\left(s + \frac{1}{2}\right) - \tilde{\beta}(s) = \frac{1}{2} + q_s$ where $q_s \in \mathbb{Z}$
$\Rightarrow q_s = \tilde{\beta}\left(s + \frac{1}{2}\right) - \tilde{\beta}(s) - \frac{1}{2}$. This is a continuous function $\left[0, \frac{1}{2}\right] \to \mathbb{Z}$ = discrete topology.
So this is constant, $q_s = q \in \mathbb{Z}$ (independent of s)
Then $\tilde{\beta}(1) = \tilde{\beta}\left(\frac{1}{2}\right) + \frac{1}{2} + q = \tilde{\beta}(0) + \frac{1}{2} + q + \frac{1}{2} + q = \tilde{\beta}(0) + 2q + 1$.
So β winds around S^1, $2q + 1$ times. So $\beta \not\simeq_p \epsilon$, but $\alpha \simeq_p \epsilon$ inside S^2
$\Rightarrow \overbrace{g \circ \alpha}^{\beta} \simeq_p \overbrace{g \circ \epsilon}^{\epsilon}$, contradiction by Lemma 64. □