

ON THE IRREDUCIBILITY OF p -ADIC BANACH PRINCIPAL SERIES OF p -ADIC GL_3

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Dedicated to Pham Huu Tiep on the occasion of his 60th birthday

ABSTRACT. We establish an optimal (topological) irreducibility criterion for p -adic Banach principal series of $\mathrm{GL}_n(F)$, where F/\mathbb{Q}_p is finite and $n \leq 3$. This is new for $n = 3$ as well as for $n = 2$, $F \neq \mathbb{Q}_p$ and establishes a refined version of Schneider's conjecture [Sch06, Conjecture 2.5] for these groups.

1. INTRODUCTION

Suppose that F/\mathbb{Q}_p is a finite extension with normalized absolute value $|\cdot|_F$ and residue field of cardinality q . This paper concerns the continuous representations of $G = \mathrm{GL}_n(F)$ on p -adic Banach spaces over a coefficient field C that is a (sufficiently large) finite extension of \mathbb{Q}_p . Such Banach representations were introduced in the work of Schneider–Teitelbaum [ST02] and play a fundamental role in the p -adic Langlands program (see for example [Bre04], [BS07], [Col10], [Eme11], [Paš13], [CEG⁺16]). Little has been known about Banach representations outside the group $\mathrm{GL}_2(\mathbb{Q}_p)$ so far. The main goal of this paper is to determine an optimal (topological) irreducibility criterion for Banach principal series of $\mathrm{GL}_n(F)$ when $n \leq 3$. This goes further than Schneider's conjecture [Sch06, Conjecture 2.5] for these groups.

Let B denote the upper-triangular Borel subgroup, T the diagonal maximal torus. If $\chi = \chi_1 \otimes \cdots \otimes \chi_n : T \rightarrow C^\times$ is a continuous character, then we inflate χ to B and form the parabolic induction

$$(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}} := \{f : G \rightarrow C \text{ continuous} \mid f(gb) = \chi(b)^{-1}f(g) \text{ for any } g \in G, b \in B\},$$

which carries a natural Banach topology making it into an (admissible) Banach representation of G under left translation that we call a *Banach principal series*. If χ is smooth and we replace continuous functions by locally constant functions, then we obtain a dense smooth subrepresentation $(\mathrm{Ind}_B^G \chi)^{\mathrm{sm}}$ of $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$.

To state our main result, we say that a character $\lambda : F^\times \rightarrow C^\times$ is *non-positive algebraic* if it is of the form $\lambda(x) = \prod_{\kappa : F \rightarrow C} \kappa(t)^{a_\kappa}$ for some $(a_\kappa) \in \mathbb{Z}_{\leq 0}^{\mathrm{Hom}(F, C)}$.

Theorem 1.1. *Suppose that $n \leq 3$. Then the Banach principal series $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ is reducible if and only if there exists $1 \leq i < n$ such that $\chi_i \chi_{i+1}^{-1}$ is non-positive algebraic.*

This result was known for $n = 1$, as well as for $n = 2$ and $F = \mathbb{Q}_p$ [Sch06, Proposition 2.6]. It was also known when $d_{\chi_i, \kappa} - d_{\chi_j, \kappa} - (j - i) \notin \mathbb{Z}_{<0}$ for all $1 \leq i < j \leq n$ and all $\kappa : F \rightarrow C$ [Sch06, Proposition 2.6], [OS10], where $d_{\chi_i} : F \otimes_{\mathbb{Q}_p} C \cong \bigoplus_{\kappa} C \rightarrow C$ denotes the derivative of χ_i (noting that χ_i is \mathbb{Q}_p -locally analytic), by locally analytic representation theory and comparison with BGG category \mathcal{O} . It was known when $|\chi_i \chi_j^{-1}(\varpi_F)| < 1$ for all $1 \leq i < j \leq n$ [BH16], where $|\cdot|$ denotes a defining absolute value on C , by using continuous distribution algebras. Finally it was known when χ is unitary and $\bar{\chi}_i \neq \bar{\chi}_{i+1}$ for all $1 \leq i < n$, where $\bar{\chi}_i$ denotes the reduction of χ_i modulo the maximal ideal of \mathcal{O}_C , since the reduction of a G -stable unit ball in $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ is irreducible in this case [Oll06]. Note that the conditions in the last three results exclude the exceptional cases where $\chi_i \chi_{i+1}^{-1}$ is non-positive algebraic.

As a corollary to Theorem 1.1 we obtain a slightly improved version of [AH, Theorem 3.9] for $\mathrm{GL}_n(F)$ ($n \geq 1$), see Corollary 3.2. It is natural to wonder if Theorem 1.1 continues to hold for all n , but the evidence is rather limited at this point.

1991 *Mathematics Subject Classification.* 22E50.

Key words and phrases. p -adic representations, p -adic groups.

The first-named author was supported by JSPS KAKENHI Grant Number 18H01107.

The second-named author was partially supported by an NSERC grant.

We now indicate how we prove Theorem 1.1. The “if” direction is clear by transitivity of parabolic induction, because if $n = 2$ and $\chi_1\chi_2^{-1}$ is non-positive algebraic, then $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ contains (up to twist by $\chi_1 \circ \det$) a finite-dimensional algebraic subrepresentation.

To prove the “only if” direction we apply our results from [AH] – relying on locally analytic vectors [ST03] and the work of Orlik–Strauch [OS15], [OS14] – to reduce to the case where χ is smooth and moreover the smooth representation $(\mathrm{Ind}_B^G \chi)^{\mathrm{sm}}$ contains a non-generic irreducible subrepresentation. Our assumption then becomes that $\chi_i \neq \chi_{i+1}$ for all $1 \leq i < n$, and it remains to consider the case where $n = 3$ and $\chi_1\chi_3^{-1} = |\cdot|_F$ (by genericity considerations). We remark that generically $(\mathrm{Ind}_B^G \chi)^{\mathrm{sm}}$ is indecomposable of length 2 in this case. By symmetry, using the outer automorphism of GL_3 , we reduce to proving the following proposition.

Proposition 1.2. *Assume that χ is smooth and that $\chi_1 \neq \chi_2$, $\chi_2 \neq \chi_3$. If $\chi_1\chi_3^{-1} \neq |\cdot|_F^2$ and $|\chi_2^{-1}\chi_3(\varpi_F)| > |q|$, then $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ is irreducible.*

By [AH, Corollary 2.54] it suffices to show that any irreducible subrepresentation π of $(\mathrm{Ind}_B^G \chi)^{\mathrm{sm}}$ is dense in $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$. Let P be the parabolic corresponding to $3 = 2 + 1$, with unipotent radical N and Levi subgroup L containing T . Let Z_L denote the center of L . Let $N_0 := N \cap \mathrm{GL}_3(\mathcal{O}_F)$, $L^+ := \{z \in Z_L \mid zN_0z^{-1} \subset N_0\}$, and $Z_L^+ := Z_L \cap L^+$. Then π^{N_0} carries a Hecke action of L^+ by letting $\ell \in L^+$ act as $[N_0 : \ell N_0 \ell^{-1}]^{-1} \sum_{n \in N_0/\ell N_0 \ell^{-1}} n \ell v$ for $v \in \pi^{N_0}$, and $\pi^{N_0, Z_L^+ = \chi}$ becomes a smooth representation of L , with L^+ acting via the Hecke action and Z_L via χ . In fact, the natural map $\pi^{N_0, Z_L^+ = \chi} \rightarrow \pi_N^{Z_L = \chi}$ is an L -linear isomorphism [Eme06, Proposition 4.3.4], where π_N denotes the unnormalized Jacquet module. We use the geometric lemma to show that

$$0 \neq \pi^{N_0, Z_L^+ = \chi} \hookrightarrow ((\mathrm{Ind}_B^G \chi)^{\mathrm{sm}})^{N_0, Z_L^+ = \chi} \xrightarrow[\theta]{\sim} (\mathrm{Ind}_{B \cap L}^L \chi)^{\mathrm{sm}},$$

where the final isomorphism θ is induced by restriction of functions from G to L .

Our key step is to find an explicit inverse of the isomorphism θ , see Theorem 3.7. As $\chi_1 \neq \chi_2$, the L -subrepresentation $\pi^{N_0, Z_L^+ = \chi}$ has to either be the full principal series or a twist of the Steinberg representation, and we can apply θ^{-1} to obtain many explicit functions in π . We now put $\sigma := (\mathrm{Ind}_{B \cap L}^L \chi)^{\mathrm{cts}}$ and think of $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ as $(\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}}$. Then we are able to find sequences $h_n \in \pi$ and $h'_n \in (\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}}$ ($n \geq 1$) and a vector $v \in \sigma$ such that

- $\mathrm{supp}(h'_n) \subset \overline{N_0}P$, where $\overline{N_0}$ is the transpose of N_0 ;
- $h'_n(x) \in Cv$ for all $x \in \overline{N_0}$;
- the sequence h'_n is bounded away from 0;
- $\lim_{n \rightarrow \infty} (h_n - h'_n) = 0$.

Using Corollary 2.3, which may be of independent interest for establishing the irreducibility of continuous parabolic inductions, we deduce that π is dense in $(\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}}$.

1.1. Notation. Let C be a finite extension of \mathbb{Q}_p and \mathcal{O}_C the ring of integers in C . We fix a uniformizer ϖ_C of C and an absolute value $|\cdot|$ on C . In this paper, unless otherwise stated, the coefficient field of any representation is C . Let \underline{G} be a connected reductive group over \mathbb{Q}_p , \underline{Z}_G the center of \underline{G} , $G = \underline{G}(\mathbb{Q}_p)$ is the group of rational points. We use the same notation for other groups. For a parabolic subgroup \underline{P} of \underline{G} , Levi subgroup \underline{L} , unipotent radical \underline{N} and a representation σ of L , we consider the following two inductions. 1) If σ is a Banach representation, namely a continuous representation on a p -adic Banach space, then $(\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}}$ is the space of continuous maps $f: G \rightarrow \sigma$ such that $f(g\ell n) = \sigma(\ell)^{-1}f(g)$ for $g \in G$, $\ell \in L$ and $n \in N$, and G acts via $(\gamma f)(g) = f(\gamma^{-1}g)$ for $g, \gamma \in G$. Let K be a compact open subgroup of G such that $KP = G$ and $P \cap K = (L \cap K)(N \cap K)$. We can always choose a defining norm $|\cdot|$ on σ that is $L \cap K$ -invariant. Then we define the K -invariant norm $\|\cdot\|$ on $(\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}}$ by $\|f\| = \sup_{x \in K} |f(x)|$, and $(\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}}$ is a Banach representation with this norm. 2) If σ is smooth, then $(\mathrm{Ind}_P^G \sigma)^{\mathrm{sm}}$ is the space of locally constant functions $G \rightarrow \sigma$ such that $f(g\ell n) = \sigma(\ell)^{-1}f(g)$ for $g \in G$, $\ell \in L$ and $n \in N$, and G acts via $(\gamma f)(g) = f(\gamma^{-1}g)$ for $g, \gamma \in G$.

We say that a continuous representation of a topological group is *irreducible* if it is topologically irreducible.

1.2. Acknowledgement. Part of this work was done during a pleasant stay of the first-named author at University of Toronto.

2. PRELIMINARIES

2.1. Irreducibility criterion. We recall an irreducibility criterion from [AH]. We assume that the derived subgroup of \underline{G} is simply connected. Let $\underline{P} = \underline{LN}$ be a parabolic subgroup and σ an absolutely irreducible Banach representation of L . We assume that σ is finite-dimensional. Then σ is locally analytic (as the ground field is \mathbb{Q}_p) and by [AH, Lemma 2.51], after perhaps replacing C by a finite extension, there exist locally analytic representations σ_0, τ of L and a parabolic subgroup $\underline{Q} = \underline{L_Q N_Q}$ containing \underline{P} such that

- $\sigma \cong \sigma_0 \otimes \tau$.
- σ_0 is simple as $\text{Lie}(L) \otimes_{\mathbb{Q}_p} C$ -module and τ is smooth.
- Let $L(\sigma'_0)$ be the simple $\text{Lie}(G) \otimes_{\mathbb{Q}_p} C$ -module in the BGG category \mathcal{O} such that $L(\sigma'_0)^{\text{Lie}(N)} \cong \sigma'_0$, where σ'_0 is the dual of σ_0 . Then $L(\sigma'_0)$ is locally $\text{Lie}(Q) \otimes_{\mathbb{Q}_p} C$ -finite and \underline{Q} is maximal subject to this condition.
- $L(\sigma'_0)$ has the structure of a locally finite Q -locally analytic representation whose differential is the restriction of the action of $\text{Lie}(G)$ and whose restriction to P on $L(\sigma'_0)^{\text{Lie}(N)} \cong \sigma'_0$ is the given one. (This structure is unique if it exists, cf. [AH, Section 2.3].)

The decomposition $\sigma \cong \sigma_0 \otimes \tau$ is moreover unique up to smooth characters of L_Q [AH, Lemma 2.22(ii)].

Then we have the following.

Theorem 2.1 ([AH, Corollary 2.54, Theorem 2.56]). *Assume $p > 2$ (resp. $p > 3$) if the absolute root system of \underline{G} has irreducible components of type B, C or F_4 (resp. G_2). The following are equivalent.*

- (i) $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is irreducible;
- (ii) $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{cts}}$ is irreducible;
- (iii) any irreducible subrepresentation of $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}$ is dense in $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{cts}}$.

2.2. Density lemmas. To prove that a certain subrepresentation is dense, we will use the following. This is a generalization of [AH, Lemma 2.2].

Lemma 2.2. *Let $\underline{P} = \underline{LN}$ be a parabolic subgroup, $\overline{\underline{P}} = \underline{L\overline{N}}$ the opposite parabolic subgroup, and assume that $P \cap K = (L \cap K)(N \cap K)$. Let σ be an irreducible Banach representation of L with central character and an $L \cap K$ -stable unit ball σ^0 . Let $\pi \subset (\text{Ind}_P^G \sigma)^{\text{cts}}$ be a closed subrepresentation and $v \in \sigma$. We assume the following: there exists a compact open subgroup $\overline{N}_0 \subset \overline{N} \cap K$ such that for any $k \in \mathbb{Z}_{>0}$, there exists $f \in \pi$ satisfying the following conditions:*

- (i) we have $f(K) \subset \sigma^0$, $f(K) \not\subset \varpi_C \sigma^0$;
- (ii) for any $x \in K \setminus \overline{N}_0(P \cap K)$ we have $f(x) \in \varpi_C^k \sigma^0$;
- (iii) for any $n \in \overline{N}_0$ we have $f(n) \in Cv + \varpi_C^k \sigma^0$.

Then we have $\pi = (\text{Ind}_P^G \sigma)^{\text{cts}}$.

Proof. We choose an $L \cap K$ -invariant norm $|\cdot|$ on σ with unit ball σ^0 and $|\sigma| = |C|$. By the assumption we know that $v \neq 0$ (otherwise $f(K) \subset \varpi_C^k \sigma^0$), and for convenience we scale v such that $|v| = 1$. We get a K -invariant norm $\|\cdot\|$ on $(\text{Ind}_P^G \sigma)^{\text{cts}}$, as above, and we let $((\text{Ind}_P^G \sigma)^{\text{cts}})^0$ be its unit ball. Then condition (i) above just means that $\|f\| = 1$.

We introduce the following notation. For any open and closed subset $X \subset G/P$, set $V(X) := \{f \in (\text{Ind}_P^G \sigma)^{\text{cts}} \mid \text{supp}(f) \subset X\}$. This is a closed subspace of $(\text{Ind}_P^G \sigma)^{\text{cts}}$ and we have $(\text{Ind}_P^G \sigma)^{\text{cts}} = V(X) \oplus V((G/P) \setminus X)$ as Banach spaces. (Namely, $\|f_1 + f_2\| = \max(\|f_1\|, \|f_2\|)$ for $f_1 \in V(X)$ and $f_2 \in V((G/P) \setminus X)$). We set $\|f\|_X := \|f|_X\|$.

We say that $f \in \pi$ satisfies $\mathcal{P}(\overline{N}_0, f, k)$ if $f(K) \subset \sigma^0$ and conditions (ii), (iii) in the lemma hold.

First we prove that for any $k \in \mathbb{Z}_{>0}$ and for any compact open subgroup $\overline{N}'_0 \subset \overline{N}_0$ there exists $f \in \pi$ with $\|f\| = 1$ such that $\mathcal{P}(\overline{N}'_0, f, k)$ holds. There exists $z \in Z_L$ such that $z\overline{N}'_0 z^{-1} \subset \overline{N}'_0$. We set $X := \overline{N}'_0 P/P \subset G/P$ and $Y := (G/P) \setminus X$. The element z induces a topological isomorphism $(\text{Ind}_P^G \sigma)^{\text{cts}} \xrightarrow{\sim} (\text{Ind}_P^G \sigma)^{\text{cts}}$ which sends $V(X)$ to $V(zX)$. Hence it induces a topological isomorphism $V(X) \xrightarrow{\sim} V(zX)$ and likewise $V(Y) \xrightarrow{\sim} V(zY)$. Take $r_1 \in \mathbb{Z}_{\geq 0}$ (resp. $r_2 \in \mathbb{Z}_{\leq 0}$) such that $\|zf\|_{zX} \geq |\varpi_C^{r_1}| \|f\|_X$ (resp. $\|zf\|_{zY} \leq |\varpi_C^{r_2}| \|f\|_Y$) for any $f \in V(X)$ (resp. $f \in V(Y)$).

For a given $k \in \mathbb{Z}_{>0}$ (and $z \in Z_L$ as above), put $k' := k + r_1 - r_2 - \min(\text{val}(\omega_\sigma(z)), 0) \in \mathbb{Z}_{>0}$, where ω_σ denotes the central character of σ . We take $f \in \pi$, $\|f\| = 1$ such that $\mathcal{P}(\overline{N}_0, f, k')$ holds. Then we have $\|f\|_Y \leq |\varpi_C^{k'}| < 1$. Since $\|f\| = \max(\|f\|_X, \|f\|_Y)$ we have $\|f\|_X = 1$. Therefore $\|zf\|_{zX} \geq |\varpi_C|^{r_1}$ and $\|zf\|_{zY} \leq |\varpi_C|^{k'+r_2}$. Hence $\varpi_C^{-r_1} zf$ satisfies $\|\varpi_C^{-r_1} zf\|_X \geq 1$ and $\|\varpi_C^{-r_1} zf\|_{zY} \leq |\varpi_C|^{k - \min(\text{val}(\omega_\sigma(z)), 0)} \leq |\varpi_C|^k$. Taking $r \in \mathbb{Z}_{\geq 0}$ such that $f' = \varpi_C^{r-r_1} zf$ has norm $\|f'\| = 1$, we have

$\|f'\|_{zY} \leq |\varpi_C|^{k+r} \leq |\varpi_C|^k$. If $x \in K \setminus \overline{N}'_0(P \cap K)$, then the image of x in G/P is not in \overline{N}'_0P/P . Hence $z^{-1}xP/P = z^{-1}xzP/P$ is not in \overline{N}_0P/P since $\overline{N}_0 \subset z^{-1}\overline{N}'_0z$. Therefore $z^{-1}xP/P \notin X = (G/P) \setminus Y$, hence $xP/P \in zY$. Hence $|f'(x)| \leq \|f'\|_{zY} \leq |\varpi_C|^k$, i.e. $f'(x) \in \varpi_C^k \sigma^0$. For $n \in \overline{N}'_0$, if $n \in zY$, then $f'(n) \in \varpi_C^k \sigma^0$, as we have proved. If $n \in zX$, then $z^{-1}nz \in \overline{N}_0$. Hence $f'(n) = \varpi_C^{r-r_1} f(z^{-1}n) = \varpi_C^{r-r_1} \omega_\sigma(z) f(z^{-1}nz) \in \varpi_C^{r-r_1} \omega_\sigma(z) (Cv + \varpi_C^k \sigma^0) = Cv + \varpi_C^{k'+r-r_1} \omega_\sigma(z) \sigma^0$. We have $k' + r - r_1 = k + r - r_2 - \min(\text{val}(\omega_\sigma(z)), 0) \geq k - \text{val}(\omega_\sigma(z))$, as $r \geq 0$ and $r_2 \leq 0$. Hence $\varpi_C^{k'+r-r_1} \omega_\sigma(z) \in \varpi_C^k \mathcal{O}_F$ and $\mathcal{P}(\overline{N}'_0, f', k)$ holds.

Now we prove the lemma. Let $\pi(\overline{N}_0, k)$ be the \mathcal{O}_C -submodule of $((\text{Ind}_P^G \sigma)^{\text{cts}})^0$ consisting all $f \in \pi$ which satisfy $\mathcal{P}(\overline{N}_0, f, k)$. It is \overline{N}_0 -stable. Define a subspace $V(\overline{N}_0, v)$ of $V(\overline{N}_0P/P)$ as the space of functions $f \in V(\overline{N}_0P/P)$ such that $f(n) \in Cv$ for any $n \in \overline{N}_0$. This is a closed subspace of $V(\overline{N}_0P/P)$ and the restriction to \overline{N}_0 gives an isometric isomorphism $V(\overline{N}_0, v) \xrightarrow{\sim} C^0(\overline{N}_0, Cv)$, where $C^0(\overline{N}_0, Cv)$ is the space of continuous functions $f: \overline{N}_0 \rightarrow Cv$ with the supremum norm. Hence the submodule $V(\overline{N}_0, v)^0$ corresponds to $C^0(\overline{N}_0, Cv)^0 = C^0(\overline{N}_0, \mathcal{O}_C v)$. (The last equality follows from $|v| = 1$.) We prove that for any $1 \leq j \leq k$ and for any $h \in V(\overline{N}_0, v)^0$ there exists $f \in \pi(\overline{N}_0, k)$ such that $\|h - f\| \leq |\varpi_C^j|$ by induction on j .

Let $j = 1$. We first note that $(\text{Ind}_P^G \sigma)^{\text{cts}} \cong (\text{Ind}_{P \cap K}^K \sigma)^{\text{cts}}$ (as $PK = G$) and hence

$$(2.1) \quad ((\text{Ind}_P^G \sigma)^{\text{cts}})^0 / \varpi_C ((\text{Ind}_P^G \sigma)^{\text{cts}})^0 \cong (\text{Ind}_{P \cap K}^K \sigma^0 / \varpi_C \sigma^0)^{\text{sm}}$$

as K -representations. We introduce several subspaces of $(\text{Ind}_{P \cap K}^K \sigma^0 / \varpi_C \sigma^0)^{\text{sm}}$. For an open and closed subset $X \subset G/P \cong K/(P \cap K)$, we put $\overline{V}(X) := \{f \in (\text{Ind}_{P \cap K}^K \sigma^0 / \varpi_C \sigma^0)^{\text{sm}} \mid \text{supp}(f) \subset X\}$. The restriction to \overline{N}_0 induces an isomorphism $\overline{V}(\overline{N}_0(P \cap K)/(P \cap K)) \xrightarrow{\sim} C^\infty(\overline{N}_0, \sigma^0 / \varpi_C \sigma^0)$ as \overline{N}_0 -representations, where $C^\infty(\overline{N}_0, -)$ denotes the space of locally constant functions. Let $\overline{v} \in \sigma^0 / \varpi_C \sigma^0$ be the image of v . We denote the inverse image of the space of locally constant functions $C^\infty(\overline{N}_0, (\mathcal{O}_C / (\varpi_C)) \overline{v})$ in $\overline{V}(\overline{N}_0(P \cap K)/(P \cap K))$ by $\overline{V}(\overline{N}_0, \overline{v})$.

Let $\overline{\pi}(\overline{N}_0, k)$ be the image of $\pi(\overline{N}_0, k) \rightarrow (\text{Ind}_{P \cap K}^K \sigma^0 / \varpi_C \sigma^0)^{\text{sm}}$ (via the isomorphism (2.1)). Then $\overline{\pi}(\overline{N}_0, k)$ is non-zero, since there exists $f \in \pi(\overline{N}_0, k)$ with $\|f\| = 1$ by assumption. We also have $\overline{\pi}(\overline{N}_0, k) \subset \overline{V}(\overline{N}_0, \overline{v})$ by the definition of $\pi(\overline{N}_0, k)$. We prove that equality holds. The subspace $\overline{\pi}(\overline{N}_0, k)$ is non-zero and \overline{N}_0 -stable, so it contains non-zero \overline{N}_0 -fixed vectors. The space of \overline{N}_0 -fixed vectors in $\overline{V}(\overline{N}_0, \overline{v}) \cong C^\infty(\overline{N}_0, (\mathcal{O}_C / (\varpi_C)) \overline{v})$ is one-dimensional and spanned by $g_{\overline{N}_0}$ which is defined by $g_{\overline{N}_0}(n) = \overline{v}$ for any $n \in \overline{N}_0$. Hence $g_{\overline{N}_0} \in \overline{\pi}(\overline{N}_0, k)$. Recall that we have proved that for any compact open subgroup $\overline{N}'_0 \subset \overline{N}_0$, there exists $f \in \pi$ with $\|f\| = 1$ such that $\mathcal{P}(\overline{N}'_0, f, k)$ holds. Hence by the same argument, for any compact open subgroup $\overline{N}'_0 \subset \overline{N}_0$ we have $g_{\overline{N}'_0} \in \overline{\pi}(\overline{N}'_0, k) \subset \overline{\pi}(\overline{N}_0, k)$. Since these elements generate $\overline{V}(\overline{N}_0, \overline{v})$ as an \overline{N}_0 -representation, we deduce $\overline{\pi}(\overline{N}_0, k) = \overline{V}(\overline{N}_0, \overline{v})$. Now for any $h \in V(\overline{N}_0, v)^0$, the composition $\overline{N}_0 \xrightarrow{h} \mathcal{O}_C v \rightarrow (\mathcal{O}_C / (\varpi_C)) \overline{v}$ lies in $\overline{V}(\overline{N}_0, \overline{v})$. Hence there exists $f \in \pi(\overline{N}_0, k)$ such that $h - f \in \varpi_C ((\text{Ind}_P^G \sigma)^{\text{cts}})^0$. Therefore $\|h - f\| \leq |\varpi_C|$.

Let $j > 1$. From the case of $j = 1$, there exists $f \in \pi(\overline{N}_0, k)$ such that $\|h - f\| \leq |\varpi_C|$. Namely, $h - f = \varpi_C f'$ for some $f' \in ((\text{Ind}_P^G \sigma)^{\text{cts}})^0$. Take $f_1, f'_1 \in V(\overline{N}_0P/P)$ and $f_2, f'_2 \in V((G/P) \setminus (\overline{N}_0P/P))$ such that $f = f_1 + f_2$ and $f' = f'_1 + f'_2$. Since $h \in V(\overline{N}_0P/P)$, we have $h - f_1 = \varpi_C f'_1$ and $-f_2 = \varpi_C f'_2$. By the definition of $\pi(\overline{N}_0, k)$, we have $\|f_2\| \leq |\varpi_C^k|$ and therefore $\|f'_2\| \leq |\varpi_C^{k-1}|$. By the Hahn–Banach theorem there is a continuous linear map $a: \sigma \rightarrow C$ such that $a(v) = 1$ and $|a(w)| \leq |w|$ for any $w \in \sigma$. Define $f''_1, f'''_1 \in V(\overline{N}_0P/P)$ by $f''_1(n) = a(f'_1(n))v$ for $n \in \overline{N}_0$ and $f'''_1 = f'_1 - f''_1$. By the definition of $\pi(\overline{N}_0, k)$, we have $f(n) \in Cv + \varpi_C^k \sigma^0$ for any $n \in \overline{N}_0$ and we also have $h(n) \in Cv$ for any $n \in \overline{N}_0$. Hence $f''_1(n) = \varpi_C^{-1}(h(n) - f_1(n)) \in Cv + \varpi_C^{k-1} \sigma^0$. For $n \in \overline{N}_0$ take $c \in C$ such that $f''_1(n) - cv \in \varpi_C^{k-1} \sigma^0$. Then $|f''_1(n) - cv| = |a(f'_1(n) - cv)v| \leq |f'_1(n) - cv| \leq |\varpi_C^{k-1}|$. Hence $|f'''_1(n)| \leq \max(|f''_1(n) - cv|, |f''_1(n) - cv|) \leq |\varpi_C^{k-1}|$ for any $n \in \overline{N}_0$. Therefore $\|f'''_1\| \leq |\varpi_C^{k-1}|$, since $f'''_1 \in V(\overline{N}_0P/P)$. By definition, $f''_1 \in V(\overline{N}_0, v)$. We also have that for $\|f''_1\| = \|f'_1 - f'''_1\| \leq \max(\|f''_1\|, \|f'''_1\|) \leq 1$. Hence $f''_1 \in V(\overline{N}_0, v)^0$. Therefore by the inductive hypothesis there exists $f'' \in \pi(\overline{N}_0, k)$ such that $\|f''_1 - f''\| \leq |\varpi_C^{j-1}|$. Then $f + \varpi_C f'' \in \pi(\overline{N}_0, k)$ and

$$\begin{aligned} \|h - (f + \varpi_C f'')\| &= |\varpi_C| \|f' - f''\| = |\varpi_C| \|f'_1 + f'''_1 + f'_2 - f''\| \\ &= |\varpi_C| \|f'_2 + (f''_1 - f'') + f'''_1\| \leq |\varpi_C^j|, \end{aligned}$$

as required.

In particular, putting $j = k$, there exists $f = f_k \in \pi$ such that $\|h - f_k\| \leq |\varpi_C^k|$. Hence $h = \lim_{k \rightarrow \infty} f_k \in \pi$. In other words, for any continuous function $F: \overline{N}_0 \rightarrow C$, $h = F \otimes v \in \pi$, where $F \otimes v$ is

defined by $\overline{N}_0 \ni n \mapsto F(n)v$ and we regard this as an element in $V(\overline{N}_0P/P)$ as above. More generally, let F be a continuous function $\overline{N} \rightarrow C$ such that $\text{supp}(F)$ is compact and define $f = F \otimes v$. We extend this to $\overline{N}P$ by $f(np) = \sigma(p)^{-1}f(n)$ and further to G by $f|_{G \setminus \overline{N}P} = 0$. We take $z \in Z_L$ such that $z \text{supp}(F)z^{-1} \subset \overline{N}_0$. Then $\text{supp}(zf) \in \overline{N}_0P/P$ and for $n \in \overline{N}_0$ we have $(zf)(n) = \omega_\sigma(z)f(z^{-1}nz) \in Cv$. Hence $zf \in \pi$ and therefore $f \in \pi$.

Let $\ell \in L$ and suppose F is a continuous function $\overline{N} \rightarrow C$ such that $\text{supp}(F)$ is compact. We define $(\ell F)(n) = F(\ell^{-1}n\ell)$ for $n \in \overline{N}$. Then we have $F \otimes \ell v = \ell(\ell^{-1}F \otimes v) \in \pi$. As σ is irreducible and by continuity, we have $F \otimes v' \in \pi$ for any $v' \in \sigma$. (If $\text{supp}(F) \subset \overline{N} \cap K$, then $F \otimes v_i \rightarrow F \otimes v$ if $v_i \rightarrow v$, as $\|F \otimes v\| = \sup_{n \in \overline{N}} |F(n)| \cdot |v|$ in this case. In general, use the action of L to reduce to that case.) In particular, $C^0(\overline{N}_0, C) \otimes_C \sigma \subset \pi$.

Now let $f: \overline{N}_0 \rightarrow \sigma$ be a continuous function and define $f_k: \overline{N}_0 \xrightarrow{f} \sigma \rightarrow \sigma/\varpi_C^k \sigma^0$ for $k \geq 1$. This is a locally constant function, and therefore its image is a finite subset of $\sigma/\varpi_C^k \sigma^0$. Hence there exists $f'_k \in C^0(\overline{N}_0, C) \otimes_C \sigma$ such that $f(n) - f'_k(n) \in \varpi_C^k \sigma^0$ for any $n \in \overline{N}_0$. Consider f, f'_k as elements of $V(\overline{N}_0P/P)$ as above. Then $\|f - f'_k\| \leq |\varpi_C^k|$ and hence $f = \lim_{k \rightarrow \infty} f'_k \in \pi$. Now $C^0(\overline{N}_0, \sigma) \cong V(\overline{N}_0P/P)$ generates $(\text{Ind}_P^G \sigma)^{\text{cts}}$ as a G -representation. Hence $\pi = (\text{Ind}_P^G \sigma)^{\text{cts}}$. \square

We use this lemma in the following form, where again we choose an $L \cap K$ -invariant defining norm $|\cdot|$ on σ and denote by $\|\cdot\|$ the induced K -invariant norm on $(\text{Ind}_P^G \sigma)^{\text{cts}}$.

Corollary 2.3. *Let $\underline{P} = \underline{L}\underline{N}$ be a parabolic subgroup, $\overline{P} = \overline{L}\overline{N}$ be the opposite parabolic subgroup, and assume that $P \cap K = (L \cap K)(N \cap K)$. Let $\overline{N}_0 \subset \overline{N}$ be a compact open subgroup, σ an irreducible Banach representation of L having a central character, $\pi \subset (\text{Ind}_P^G \sigma)^{\text{cts}}$ a closed subrepresentation, and $v \in \sigma$. Assume that we have sequences $h_n \in \pi$ and $h'_n \in (\text{Ind}_P^G \sigma)^{\text{cts}}$ for $n \geq 1$ such that $\text{supp}(h'_n) \subset \overline{N}_0P$, $h'_n(x) \in Cv$ for any $x \in \overline{N}_0$, $\inf_n \|h'_n\| > 0$, and $\lim_{n \rightarrow \infty} (h_n - h'_n) = 0$. Then we have $\pi = (\text{Ind}_P^G \sigma)^{\text{cts}}$.*

Proof. By replacing $|\cdot|$ by an equivalent norm we may assume that $|\sigma| = |C|$. Note that $h'_n \neq 0$ for all n , as $\inf_n \|h'_n\| > 0$. Take $c_n \in C^\times$ such that $\|c_n h'_n\| = 1$. Then $|c_n|$ is bounded above because $\inf_n \|h'_n\| > 0$. Therefore $\lim_{n \rightarrow \infty} (c_n h_n - c_n h'_n) = 0$. By replacing h_n, h'_n with $c_n h_n, c_n h'_n$ respectively, we may assume that $\|h'_n\| = 1$ for all n . Let $k \in \mathbb{Z}_{>0}$ and take n such that $\|h_n - h'_n\| \leq |\varpi_C^k|$. Since $\|h'_n\| = 1$, we also have $\|h_n\| = 1$. If $x \in K$, then $h_n(x) - h'_n(x) \in \varpi_C^k \sigma^0$. Therefore, if $x \in \overline{N}_0$ we have $h_n(x) \in Cv + \varpi_C^k \sigma^0$ (as $h'_n(x) \in Cv$) and if $x \in K \setminus \overline{N}_0(P \cap K)$ we have $h_n(x) \in \varpi_C^k \sigma^0$ (as $h'_n(x) = 0$). \square

2.3. The group $GL_2(F)$. When $G = GL_2(F)$ for some finite extension F of \mathbb{Q}_p we have the following irreducibility criterion.

Let $\underline{G} := \text{Res}_{F/\mathbb{Q}_p} GL_2$. Let \underline{B} be the subgroup of upper-triangular matrices and \underline{T} the subgroup of diagonal matrices in \underline{G} . Let $\chi: T \rightarrow C^\times$ be a character given by $\chi(\text{diag}(t_1, t_2)) = \chi_1(t_1)\chi_2(t_2)$, where $\chi_i: F^\times \rightarrow C^\times$ is a continuous character for $i = 1, 2$. Let $\text{Hom}(F, C)$ be the set of field homomorphism $F \rightarrow C$.

Theorem 2.4. *The Banach representation $(\text{Ind}_B^G \chi)^{\text{cts}}$ is reducible if and only if there exists $(k_\kappa)_\kappa \in \mathbb{Z}_{\leq 0}^{\text{Hom}(F, C)}$ such that $\chi_1 \chi_2^{-1}(t) = \prod_{\kappa \in \text{Hom}(F, C)} \kappa(t)^{k_\kappa}$.*

When $F = \mathbb{Q}_p$, this is stated in [Sch06, Proposition 2.6] (though we do not know a reference for the complete proof).

Proof. Assume that $\chi_1 \chi_2^{-1}(t) = \prod_{\kappa \in \text{Hom}(F, C)} \kappa(t)^{k_\kappa}$ for some $(k_\kappa) \in \mathbb{Z}_{\leq 0}^{\text{Hom}(F, C)}$. We have $(\text{Ind}_B^G \chi)^{\text{cts}} \cong (\chi_2 \circ \det) \otimes (\text{Ind}_B^G \chi_1 \chi_2^{-1} \boxtimes \mathbf{1})^{\text{cts}}$, where $\mathbf{1}$ is the trivial character of F^\times . The space of rational functions in $(\text{Ind}_B^G \chi_1 \chi_2^{-1} \boxtimes \mathbf{1})^{\text{cts}}$ is an irreducible finite-dimensional (hence closed) G -subrepresentation. Hence $(\text{Ind}_B^G \chi)^{\text{cts}}$ is reducible. The converse follows from [AH, Theorem 3.9] (or [AH, Theorem 3.68]). \square

3. THE GROUP $GL_3(F)$

Let F be a finite extension of \mathbb{Q}_p , \mathcal{O}_F the ring of integers, ϖ_F a uniformizer of F , q the cardinality of the residue field of F and $\text{val}: F^\times \rightarrow \mathbb{Z}$ the normalized valuation of F . We normalize the norm $|\cdot|_F$ on F by $|\varpi_F|_F = q^{-1}$, namely $|x|_F = q^{-\text{val}(x)}$. Let $\underline{G} := \text{Res}_{F/\mathbb{Q}_p} GL_3$, \underline{B} the subgroup of upper-triangular matrices, \underline{T} the subgroup of diagonal matrices, and \underline{U} the subgroup of unipotent upper-triangular matrices. Let $\chi: T \rightarrow C^\times$ be a continuous character given by $\chi(\text{diag}(t_1, t_2, t_3)) = \chi_1(t_1)\chi_2(t_2)\chi_3(t_3)$, where $\chi_i: F^\times \rightarrow C^\times$ is a continuous character for $i = 1, 2, 3$. The main theorem of this paper is the following.

Theorem 3.1. *The Banach representation $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ is reducible if and only if there exists $(k_\kappa)_\kappa \in \mathbb{Z}_{\leq 0}^{\mathrm{Hom}(F,C)}$ such that $\chi_1 \chi_2^{-1}(t) = \prod_{\kappa \in \mathrm{Hom}(F,C)} \kappa(t)^{k_\kappa}$ or $\chi_2 \chi_3^{-1}(t) = \prod_{\kappa \in \mathrm{Hom}(F,C)} \kappa(t)^{k_\kappa}$.*

We can now slightly improve [AH, Theorem 3.9], adding the more restrictive condition (i).

Corollary 3.2. *Let $G = \mathrm{GL}_n(F)$, B the upper-triangular Borel subgroup, and T the diagonal maximal torus with Lie algebra \mathfrak{t} . Let $\chi = \chi_1 \otimes \cdots \otimes \chi_n: T = (F^\times)^n \rightarrow C^\times$ be a continuous (hence locally \mathbb{Q}_p -analytic) character. We have $d\chi \in \mathrm{Hom}_{\mathbb{Q}_p}(\mathfrak{t}, C) \cong \bigoplus_{\kappa \in \mathrm{Hom}(F,C)} \mathrm{Hom}_C(\mathfrak{t} \otimes_{F,\kappa} C, C)$ and let $\lambda_\kappa = (\lambda_{\kappa,1}, \dots, \lambda_{\kappa,n})$ be the κ -component of $d\chi$, where $\lambda_{\kappa,k} \in \mathrm{Hom}_C(C, C) \cong C$. Choose $0 = n_0 < n_1 < \cdots < n_r = n$ such that $\lambda_{\kappa,i} - \lambda_{\kappa,i+1} \in \mathbb{Z}_{\leq 0}$ for all $\kappa: F \rightarrow C$ is equivalent to $i \notin \{n_1, \dots, n_r\}$. Assume that there exists no $n_k < i < j \leq n_{k+1}$ (for some $0 \leq k < r$) such that*

- (i) *if $n_{k+1} - n_k = 3$, then $j - i = 1$, and*
- (ii) *$\chi_i \chi_j^{-1}(t) = |t|_F^{j-i-1} \prod_{\kappa: F \rightarrow C} \kappa(t)^{\lambda_{\kappa,i} - \lambda_{\kappa,j}}$ for all $t \in F^\times$.*

Then $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ is absolutely irreducible.

Proof. Let $\underline{G} = \mathrm{Res}_{F/\mathbb{Q}_p} \mathrm{GL}_n$. For $\sigma := \chi$, take $\sigma_0, \tau, \underline{Q}$ as in subsection 2.1. Then condition (ii) is equivalent to $\tau_i \tau_j^{-1} \neq |\cdot|_F^{j-i-1}$ (cf. the proof of [AH, Theorem 3.9]), and \underline{Q} is the standard parabolic corresponding to the partition $\{1, \dots, n_1\}, \{n_1 + 1, \dots, n_2\}, \dots$ of $\{1, 2, \dots, n\}$. By Theorem 2.1 it suffices to show that $(\mathrm{Ind}_{B \cap L \underline{Q}}^{L \underline{Q}} \tau)^{\mathrm{cts}}$ is absolutely irreducible. By [AH, Proposition 2.59] we may reduce to the case where $r = 1$, i.e. $\underline{Q} = \underline{G}$. Then $(\mathrm{Ind}_B^G \tau)^{\mathrm{cts}}$ is irreducible by Theorem 3.1 if $n = 3$ and by [AH, Theorem 3.9] if $n \neq 3$. \square

We now prepare for the proof of Theorem 3.1. We first prove the ‘‘only if’’ part. Without loss of generality, using the same automorphism as in the proof of Lemma 3.6, suppose that $\chi_1 \chi_2^{-1}(t) = \prod_{\kappa \in \mathrm{Hom}(F,C)} \kappa(t)^{k_\kappa}$. Then by Theorem 2.4 there exists a subrepresentation $0 \subsetneq \pi \subsetneq (\mathrm{Ind}_{B \cap L}^L \chi)^{\mathrm{cts}}$, where $\underline{P} = \underline{LN}$ is the standard parabolic subgroup corresponding to $3 = 2 + 1$. Hence by transitivity of parabolic induction we get the closed subrepresentation $0 \subsetneq (\mathrm{Ind}_{\underline{P}}^{\underline{G}} \pi)^{\mathrm{cts}} \subsetneq (\mathrm{Ind}_{\underline{B}}^{\underline{G}} \chi)^{\mathrm{cts}}$.

For the ‘‘if’’ part, let $\sigma := \chi$ and take $\sigma_0, \tau, \underline{Q}$ as in subsection 2.1. More concretely, \underline{Q} is given as follows:

- $\underline{Q} = \underline{B}$ if $d\chi_i - d\chi_{i+1} \neq \sum_{\kappa \in \mathrm{Hom}(F,C)} k_\kappa \kappa$ for all $(k_\kappa) \in \mathbb{Z}_{\leq 0}^{\mathrm{Hom}(F,C)}$ and $i = 1, 2$.
- \underline{Q} is the standard parabolic subgroup corresponding to $3 = 2 + 1$ if there exists $(k_\kappa) \in \mathbb{Z}_{\leq 0}^{\mathrm{Hom}(F,C)}$ such that $d\chi_i - d\chi_{i+1} = \sum_{\kappa \in \mathrm{Hom}(F,C)} k_\kappa \kappa$ for $i = 1$ but not for $i = 2$.
- \underline{Q} is the standard parabolic subgroup corresponding to $3 = 1 + 2$ if there exists $(k_\kappa) \in \mathbb{Z}_{\leq 0}^{\mathrm{Hom}(F,C)}$ such that $d\chi_i - d\chi_{i+1} = \sum_{\kappa \in \mathrm{Hom}(F,C)} k_\kappa \kappa$ for $i = 2$ but not for $i = 1$.
- $\underline{Q} = \underline{G}$ if there exist $(k_{\kappa,i}) \in \mathbb{Z}_{\leq 0}^{\mathrm{Hom}(F,C)}$ such that $d\chi_i - d\chi_{i+1} = \sum_{\kappa \in \mathrm{Hom}(F,C)} k_{\kappa,i} \kappa$ for $i = 1, 2$.

By Theorem 2.1, $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ is irreducible if and only if $(\mathrm{Ind}_{B \cap L \underline{Q}}^{L \underline{Q}} \tau)^{\mathrm{cts}}$ is irreducible. Hence, if $\underline{Q} = \underline{B}$, $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ is irreducible and the theorem follows. If $\underline{Q} \neq \underline{B}$ and $\underline{Q} \neq \underline{G}$, then the theorem follows from Theorem 2.4. Therefore we may assume $\underline{Q} = \underline{G}$ and $\chi = \tau$. Namely we may assume χ is smooth. So our task is to prove the following.

Proposition 3.3. *Let χ be a smooth character of T . The Banach representation $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ is reducible if and only if $\chi_1 = \chi_2$ or $\chi_2 = \chi_3$.*

Lemma 3.4. *Proposition 3.3 is true if $\chi_1 \chi_3^{-1} \neq |\cdot|_F$.*

Proof. It is sufficient to prove that if $\chi_1 \neq \chi_2$, $\chi_2 \neq \chi_3$ and $\chi_1 \chi_3^{-1} \neq |\cdot|_F$, then $(\mathrm{Ind}_B^G \sigma)^{\mathrm{cts}}$ is irreducible. This follows from [AH, Theorem 3.9]. \square

In the rest of this paper we prove the following.

Proposition 3.5. *Assume that χ is smooth and that $\chi_1 \neq \chi_2$, $\chi_2 \neq \chi_3$. If $\chi_1 \chi_3^{-1} \neq |\cdot|_F^2$ and $|\chi_2^{-1} \chi_3(\varpi_F)| > |q|$, then $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ is irreducible.*

Lemma 3.6. *Proposition 3.5 implies Proposition 3.3, hence Theorem 3.1.*

Proof. By Lemma 3.4, it is sufficient to prove that if $\chi_1 \neq \chi_2$, $\chi_2 \neq \chi_3$ and $\chi_1 \chi_3^{-1} = |\cdot|_F$, then $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ is irreducible. As $\chi_1 \chi_3^{-1} \neq |\cdot|_F^2$, Proposition 3.5 implies that $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ is irreducible if $|\chi_2^{-1} \chi_3(\varpi_F)| > |q|$.

Define $\iota: \mathrm{GL}_3 \rightarrow \mathrm{GL}_3$ by $\iota(g) = \dot{w}_0 \cdot {}^t g^{-1} \cdot \dot{w}_0$, where \dot{w}_0 is a lift of the longest element of the Weyl group. Since $\iota(B) = B$, we have $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}} \circ \iota \cong (\mathrm{Ind}_B^G \chi \circ \iota)^{\mathrm{cts}}$. Therefore $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ is irreducible if

and only if $(\mathrm{Ind}_B^G \chi \circ \iota)^{\mathrm{cts}}$ is. Hence, from the first paragraph of this proof, $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ is irreducible if $|\chi_1^{-1} \chi_2(\varpi_F)| > |q|$. Since $|\chi_1^{-1} \chi_3(\varpi_F)| = |q|$, we have $|\chi_1^{-1} \chi_2(\varpi_F)| > |q|$ or $|\chi_2^{-1} \chi_3(\varpi_F)| > |q|$. Hence we get Theorem 3.1. \square

3.1. Jacquet modules. To prove Proposition 3.5, we use Jacquet modules in smooth representation theory. Let $\underline{P} = \underline{L}N$ be the standard parabolic subgroup corresponding to $3 = 2 + 1$, and recall that $\underline{Z}_{\underline{L}}$ denotes the center of \underline{L} . Fix a compact open subgroup N_0 of N and set $L^+ := \{\ell \in L \mid \ell N_0 \ell^{-1} \subset N_0\}$ and $Z_L^+ := Z_L \cap L^+$. If π is an admissible smooth representation of G , then $\ell \in L^+$ acts on the subspace of N_0 -fixed vectors π^{N_0} in π by the Hecke action

$$(3.1) \quad \tau_\ell(v) := \frac{1}{[N_0 : \ell N_0 \ell^{-1}]} \sum_{n \in N_0 / \ell N_0 \ell^{-1}} n \ell v = \int_{N_0} n \ell v \, dv$$

for $v \in \pi^{N_0}$, where the Haar measure is normalized such that the volume of N_0 is 1. Set $\pi^{N_0, Z_L^+ = \chi} := \{v \in \pi^{N_0} \mid \tau_z(v) = \chi(z)v \text{ for all } z \in Z_L^+\}$, which has a natural action of L , with L^+ acting via (3.1) and Z_L acting via χ . (This is well defined by [Eme06, Proposition 3.3.6].)

Let π_N be the space of N -coinvariants (i.e. the unnormalized Jacquet module) and define $\pi_N^{Z_L = \chi}$ analogously to above. Then by [Eme06, Propositions 3.4.9, 4.3.4], the natural projection $\pi^{N_0} \rightarrow \pi_N$ induces an L -linear isomorphism $\pi^{N_0, Z_L^+ = \chi} \cong \pi_N^{Z_L = \chi}$.

Let $\pi := (\mathrm{Ind}_B^G \chi)^{\mathrm{sm}}$. Then by the geometric lemma, π_N has a filtration $0 = F_0 \subset F_1 \subset F_2 \subset F_3 = (\mathrm{Ind}_B^G \chi)_N^{\mathrm{sm}}$ such that $F_3/F_2 \cong (\mathrm{Ind}_{B \cap L}^L \chi)^{\mathrm{sm}}$, $F_2/F_1 \cong (\mathrm{Ind}_{B \cap L}^L \chi')^{\mathrm{sm}}$ and $F_1/F_0 \cong (\mathrm{Ind}_{B \cap L}^L \chi'')^{\mathrm{sm}}$, where $\chi' := \chi_1 \boxtimes (\chi_3 \cdot | \cdot |_F) \boxtimes (\chi_2 \cdot | \cdot |_F^{-1})$ and $\chi'' := (\chi_2 \cdot | \cdot |_F) \boxtimes (\chi_3 \cdot | \cdot |_F) \boxtimes (\chi_1 \cdot | \cdot |_F^{-2})$. Hence if $\chi_2 \neq \chi_3 \cdot | \cdot |_F$ and $\chi_1 \neq \chi_3 \cdot | \cdot |_F^2$, then $\chi|_{Z_L} \neq \chi'|_{Z_L}$, and $\chi|_{Z_L} \neq \chi''|_{Z_L}$. Therefore $(\mathrm{Ind}_B^G \chi)^{\mathrm{sm}}_N^{Z_L = \chi} \cong (\mathrm{Ind}_{B \cap L}^L \chi)^{\mathrm{sm}}$. Hence we have an L -linear isomorphism

$$(3.2) \quad ((\mathrm{Ind}_B^G \chi)^{\mathrm{sm}})^{N_0, Z_L^+ = \chi} \cong (\mathrm{Ind}_{B \cap L}^L \chi)^{\mathrm{sm}}.$$

Note that the isomorphism is induced by the restriction $(\mathrm{Ind}_B^G \chi)^{\mathrm{sm}} \ni f \mapsto f|_L \in (\mathrm{Ind}_{B \cap L}^L \chi)^{\mathrm{sm}}$. (This follows either by the proof of the geometric lemma, or because the restriction is easily seen to induce a non-zero map $(\mathrm{Ind}_B^G \chi)^{\mathrm{sm}}_N^{Z_L = \chi} \rightarrow (\mathrm{Ind}_{B \cap L}^L \chi)^{\mathrm{sm}}$, hence an isomorphism.)

3.2. Explicit formulas. We now take $N_0 := N \cap K$ with $K := \mathrm{GL}_3(\mathcal{O}_F)$. We calculate the inverse of the map (3.2) explicitly.

Let $\eta_i := \chi_i^{-1} \chi_{i+1}$. For a character $\eta: F^\times \rightarrow C^\times$ let $c(\eta) \in \mathbb{Z}_{\geq 0}$ denote the conductor of η , i.e. the smallest integer $c \geq 0$ such that η is trivial on $(1 + (\varpi_F^c)) \cap \mathcal{O}_F^\times$. We also use the following representatives of simple reflections of the Weyl group:

$$(3.3) \quad \dot{s}_1 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \dot{s}_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We let $\dot{w}_0 := \dot{s}_1 \dot{s}_2 \dot{s}_1 = \dot{s}_2 \dot{s}_1 \dot{s}_2$. Then $\{e, \dot{s}_1, \dot{s}_2, \dot{s}_1 \dot{s}_2, \dot{s}_2 \dot{s}_1, \dot{w}_0\}$ is a full set of representatives for S_3 . Let I be the ‘‘upper’’ Iwahori subgroup, namely I is the set of $g \in \mathrm{GL}_3(\mathcal{O}_F)$ such that $g \pmod{\varpi_F}$ is an upper-triangular matrix. Then we have $G = \coprod_{w \in S_3} I \dot{w} B$, where \dot{w} are our representatives of $w \in S_3$, so to specify $f \in ((\mathrm{Ind}_B^G \chi)^{\mathrm{sm}})^{N \cap K, Z_L^+ = \chi}$ it suffices to describe the values of f on $(N \cap K) \backslash I \dot{w} B / B$ for each $w \in S_3$, and this is what we will do now. (In fact, it will be convenient to describe it on a slightly larger set.) We normalize the Haar measure on F such that the volume of \mathcal{O}_F is 1.

Theorem 3.7. *Assume that $\eta_2 \neq | \cdot |_F^{-1}$ and $\eta_1 \eta_2 \neq | \cdot |_F^{-2}$. Let $f \in ((\mathrm{Ind}_B^G \chi)^{\mathrm{sm}})^{N \cap K, Z_L^+ = \chi}$. Then*

$$\begin{aligned} f \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} &= \delta_{\mathrm{val}(b) \geq c(\eta_1 \eta_2)} \int_{\mathcal{O}_F} \eta_2(1 + ct) f \begin{pmatrix} 1 & 0 & 0 \\ a + bt & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} dt \quad (a \in \mathcal{O}_F, b, c \in (\varpi_F)), \\ f \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ c & b & 1 \end{pmatrix} \dot{s}_1 \right) &= \delta_{\mathrm{val}(b) \geq c(\eta_1 \eta_2)} \int_{\mathcal{O}_F} \eta_2(1 + ct) f \left(\begin{pmatrix} 1 & a + bt & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) dt \quad (a \in \mathcal{O}_F, b, c \in (\varpi_F)), \\ f \left(\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & 0 & 1 \end{pmatrix} \dot{s}_2 \right) &= \delta_{\mathrm{val}(c) \geq c(\eta_1 \eta_2)} \left\{ \int_{\mathcal{O}_F} \eta_2(t) \left[f \begin{pmatrix} 1 & 0 & 0 \\ a + ct & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - f \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] dt \right. \end{aligned}$$

$$\begin{aligned}
& + \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} f \left(\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \quad (a \in \mathcal{O}_F, c \in (\varpi_F)), \\
f \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix} \dot{s}_1 \dot{s}_2 \right) & = \delta_{\text{val}(c) \geq c(\eta_1 \eta_2)} \left\{ \int_{\mathcal{O}_F} \eta_2(t) \left[f \left(\begin{pmatrix} 1 & a-ct & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) - f \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) \right] dt \right. \\
& \quad \left. + \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} f \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) \right\} \quad (a \in \mathcal{O}_F, c \in (\varpi_F)), \\
f \left(\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \dot{s}_1 \right) & = \frac{\delta_{c(\eta_1 \eta_2)=0}(q-1)}{q(q^2 - \eta_1 \eta_2(\varpi_F))} \left\{ \int_{\mathcal{O}_F} \eta_2(t) \left[f \left(\begin{pmatrix} 1 & 0 & 0 \\ a+t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) - f \left(\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \right] dt \right. \\
& \quad \left. + \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} f \left(\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) + \int_{(\varpi_F)} \eta_2(1-at) f \left(\begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) dt \right\} \\
& \hspace{25em} (a \in \mathcal{O}_F), \\
f \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{w}_0 \right) & = \frac{\delta_{c(\eta_1 \eta_2)=0}(q-1)}{q(q^2 - \eta_1 \eta_2(\varpi_F))} \left\{ \int_{\mathcal{O}_F} \eta_2(t) \left[f \left(\begin{pmatrix} 1 & a-t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) - f \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) \right] dt \right. \\
& \quad \left. + \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} f \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) + \int_{(\varpi_F)} \eta_2(-1+at) f \left(\begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) dt \right\} \\
& \hspace{25em} (a \in \mathcal{O}_F).
\end{aligned}$$

Here we remark that the term

$$\delta_{c(\eta_2)=0} \frac{1}{q - \eta_2(\varpi_F)}$$

is well defined and independent of our choice of ϖ_F : if $c(\eta_2) \neq 0$, then this is zero and if $c(\eta_2) = 0$ then $\eta_2(\varpi_F) \neq |\varpi_F|_F^{-1} = q$ from the assumption $\eta_2 \neq |\cdot|_F^{-1}$. Similarly,

$$\frac{\delta_{c(\eta_1 \eta_2)=0}(q-1)}{q(q^2 - \eta_1 \eta_2(\varpi_F))}$$

is well defined and independent of our choice of ϖ_F because $\eta_1 \eta_2 \neq |\cdot|_F^{-2}$.

We prove Theorem 3.7 in this subsection. For $k = 2, 4, 6$, the k -th formula in the theorem follows from the $(k-1)$ -th formula by replacing f with $\dot{s}_1^{-1} f$. (When $k = 6$ it helps to observe that $\eta_2(-1) = \eta_1(-1)$ if the formula is nonzero, as this only happens when $\eta_1 \eta_2$ is unramified.)

Moreover, we may assume that $a = 0$. In general form can be obtained by replacing $f \in (\text{Ind}_B^G \chi)^{\text{sm}, N \cap K, Z_L^+ = x}$ with

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} f.$$

This reduction step is obvious for the first and third formulas. For the fifth formula, for $a \in \mathcal{O}_F \setminus \{0\}$, we use

$$\begin{aligned}
\int_{(\varpi_F)} f \left(\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) dt & = \int_{(\varpi_F)} f \left(\begin{pmatrix} 1 & t(1+at)^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \begin{pmatrix} 1+at & -a & 0 \\ 0 & (1+at)^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) dt \\
& = \int_{(\varpi_F)} \eta_1(1+at) f \left(\begin{pmatrix} 1 & t(1+at)^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) dt \\
& = \int_{1+(a\varpi_F)} \eta_1(t_1) f \left(\begin{pmatrix} 1 & a^{-1}(1-t_1^{-1}) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) |a|_F^{-1} dt_1 \\
& = \int_{1+(a\varpi_F)} \eta_1(t_2)^{-1} f \left(\begin{pmatrix} 1 & a^{-1}(1-t_2) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) |a|_F^{-1} dt_2
\end{aligned}$$

$$= \int_{(\varpi_F)} \eta_1(1 - at_3)^{-1} f \left(\begin{pmatrix} 1 & t_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) dt_3,$$

where $t_1 := 1 + at$, $t_2 := t_1^{-1}$ and $t_3 := a^{-1}(1 - t_2)$. Finally notice that if $\delta_{c(\eta_1\eta_2)=0} \neq 0$ then $\eta_1(1 - at_3)^{-1} = \eta_2(1 - at_3)$ for $t_3 \in (\varpi_F)$.

Fix $f \in (\text{Ind}_B^G \chi)^{\text{sm}, N \cap K, Z_L^+ = \chi}$.

Lemma 3.8. *Let $x \in F$ and $D \subset F$ a compact subset such that $x \notin D$. Then for sufficiently large y , we have*

$$f \left(\begin{pmatrix} 1 & 0 & y \\ x & 1 & ay \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \dot{s}_1 \right) = \eta_1 \eta_2(y) \eta_2(a - x) f \left(\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

for all $a \in D$.

Proof. By

$$\begin{pmatrix} 1 & 0 & y \\ x & 1 & ay \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \dot{s}_1 = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 1/y & -1/(y(x-a)) & 1 \end{pmatrix} \begin{pmatrix} y & -1 & 0 \\ 0 & a-x & -1 \\ 0 & 0 & 1/(y(a-x)) \end{pmatrix},$$

the left-hand side equals

$$\eta_1 \eta_2(y) \eta_2(a - x) f \left(\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 1/y & -1/(y(x-a)) & 1 \end{pmatrix} \right).$$

Hence the lemma follows from the smoothness of f . □

Lemma 3.9. *Let $x, y \in F$ such that $y + s \neq 0$, $x - t/(y + s) \neq 0$ for any $s, t \in \mathcal{O}_F$. Then*

$$f \left(\begin{pmatrix} 1 & 0 & y \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \dot{s}_1 \right) = \int_{\mathcal{O}_F} \int_{\mathcal{O}_F} \eta_1 \eta_2(y + s) \eta_2 \left(\frac{t}{y + s} - x \right) f \left(\begin{pmatrix} 1 & 0 & 0 \\ \frac{t}{y+s} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) ds dt.$$

Proof. Let $z := \text{diag}(\varpi_F^k, \varpi_F^k, 1)$. As $\tau_z f = \chi(z) f$, we have

$$\begin{aligned} f \left(\begin{pmatrix} 1 & 0 & y \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \dot{s}_1 \right) &= \chi_1 \chi_2(\varpi_F^{-k})(\tau_z f) \left(\begin{pmatrix} 1 & 0 & y \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \dot{s}_1 \right) \\ &= \chi_1 \chi_2(\varpi_F^{-k}) \int_{\mathcal{O}_F} \int_{\mathcal{O}_F} f \left(z^{-1} \begin{pmatrix} 1 & 0 & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & y \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \dot{s}_1 \right) ds dt \\ &= \chi_1 \chi_2(\varpi_F^{-k}) \chi_2 \chi_3(\varpi_F^k) \int_{\mathcal{O}_F} \int_{\mathcal{O}_F} f \left(\begin{pmatrix} 1 & 0 & \varpi_F^{-k}(y+s) \\ x & 1 & \varpi_F^{-k}t \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \dot{s}_1 \right) ds dt. \end{aligned}$$

By Lemma 3.8, if k is sufficiently large, this is equal to

$$\begin{aligned} &\eta_1 \eta_2(\varpi_F^k) \int_{\mathcal{O}_F} \int_{\mathcal{O}_F} \eta_1 \eta_2(\varpi_F^{-k}(y+s)) \eta_2 \left(\frac{t}{y+s} - x \right) f \left(\begin{pmatrix} 1 & 0 & 0 \\ \frac{t}{y+s} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) ds dt \\ &= \int_{\mathcal{O}_F} \int_{\mathcal{O}_F} \eta_1 \eta_2(y+s) \eta_2 \left(\frac{t}{y+s} - x \right) f \left(\begin{pmatrix} 1 & 0 & 0 \\ \frac{t}{y+s} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) ds dt. \end{aligned}$$

We get the lemma. □

We prove the first formula of Theorem 3.7.

Let $b, c \in (\varpi_F)$ and assume that $bc \neq 0$. By Lemma 3.9 we have

$$f \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & c & 1 \end{pmatrix} \right) = f \left(\begin{pmatrix} 1 & 0 & 1/b \\ -b/c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \dot{s}_1 \begin{pmatrix} b & c & 1 \\ 0 & c/b & 1/b \\ 0 & 0 & 1/c \end{pmatrix} \right)$$

$$\begin{aligned}
&= \eta_1(b)\eta_2(c) \int_{\mathcal{O}_F} \int_{\mathcal{O}_F} \eta_1\eta_2 \left(\frac{1}{b} + s \right) \eta_2 \left(\frac{t}{1/b+s} + \frac{b}{c} \right) f \begin{pmatrix} 1 & 0 & 0 \\ \frac{t}{1/b+s} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ds dt \\
&= \int_{\mathcal{O}_F} \int_{\mathcal{O}_F} \eta_1\eta_2 (1+bs) \eta_2 \left(\frac{c}{b} \right) \eta_2 \left(\frac{tb}{1+sb} + \frac{b}{c} \right) f \begin{pmatrix} 1 & 0 & 0 \\ \frac{tb}{1+sb} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ds dt.
\end{aligned}$$

Writing $\frac{tb}{1+sb} = bt'$ (i.e. $t' = \frac{t}{1+sb}$ and $dt' = dt$) we get

$$\begin{aligned}
&= \int_{\mathcal{O}_F} \int_{\mathcal{O}_F} \eta_1\eta_2 (1+bs) \eta_2 \left(\frac{c}{b} \right) \eta_2 \left(bt' + \frac{b}{c} \right) f \begin{pmatrix} 1 & 0 & 0 \\ bt' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ds dt' \\
&= \int_{\mathcal{O}_F} \int_{\mathcal{O}_F} \eta_1\eta_2 (1+bs) \eta_2(1+ct) f \begin{pmatrix} 1 & 0 & 0 \\ bt & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ds dt \\
&= \left(\int_{\mathcal{O}_F} \eta_1\eta_2(1+bs) ds \right) \left(\int_{\mathcal{O}_F} \eta_2(1+ct) f \begin{pmatrix} 1 & 0 & 0 \\ bt & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} dt \right).
\end{aligned}$$

As $\int_{\mathcal{O}_F} \eta_1\eta_2(1+bs) ds = 1$ if $\eta_1\eta_2$ is trivial on $1+b\mathcal{O}_F$ and zero otherwise, we get the first formula of Theorem 3.7 when $bc \neq 0$. By local constancy of f we can see that it also holds when $bc = 0$.

We prove the third formula in Theorem 3.7. Note that our formula does not depend on ϖ_F . We take ϖ_F such that $\eta_2(\varpi_F) \neq q$, which is possible as $\eta_2 \neq |\cdot|_F^{-1}$. Let $c \in (\varpi_F)$. For $z = \text{diag}(\varpi_F^k, \varpi_F^k, 1)$ with $k \geq 0$, we have

$$\begin{aligned}
f \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{pmatrix} \dot{s}_2 \right) &= \chi(z)^{-1} (\tau_z f) \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{pmatrix} \dot{s}_2 \right) \\
&= \chi_2^{-1} \chi_3(\varpi_F^k) \int_{\mathcal{O}_F} \int_{\mathcal{O}_F} f \left(\begin{pmatrix} 1+cv & 0 & \varpi_F^{-k}v \\ cw & 1 & \varpi_F^{-k}w \\ \varpi_F^k c & 0 & 1 \end{pmatrix} \dot{s}_2 \right) dv dw.
\end{aligned}$$

We have

$$\begin{pmatrix} 1+cv & 0 & \varpi_F^{-k}v \\ cw & 1 & \varpi_F^{-k}w \\ \varpi_F^k c & 0 & 1 \end{pmatrix} \dot{s}_2 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{cw}{1+cv} & 1 & \varpi_F^{-k}w \\ \frac{\varpi_F^k c}{1+cv} & 0 & 1 \end{pmatrix} \dot{s}_2 \begin{pmatrix} 1+cv & \varpi_F^{-k}v & 0 \\ 0 & 1/(1+cv) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$(3.4) \quad f \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{pmatrix} \dot{s}_2 \right) = \eta_2(\varpi_F^k) \int_{\mathcal{O}_F} \int_{\mathcal{O}_F} \eta_1(1+cv) f \left(\begin{pmatrix} 1 & 0 & 0 \\ \frac{cw}{1+cv} & 1 & \varpi_F^{-k}w \\ \frac{\varpi_F^k c}{1+cv} & 0 & 1 \end{pmatrix} \dot{s}_2 \right) dv dw.$$

Note that, by construction, the integrand in (3.4) only depends on v and w modulo ϖ_F^k .

The $w \equiv 0$ part of (3.4) equals, for k sufficiently large (by smoothness),

$$\begin{aligned}
\frac{\eta_2(\varpi_F^k)}{q^k} \int_{\mathcal{O}_F} \eta_1(1+cv) f \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\varpi_F^k c}{1+cv} & 0 & 1 \end{pmatrix} \dot{s}_2 \right) dv &= \frac{\eta_2(\varpi_F^k)}{q^k} \int_{\mathcal{O}_F} \eta_1(1+cv) f \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \right) dv \\
&= \delta_{\text{val}(c) \geq c(\eta_1)} \frac{\eta_2(\varpi_F^k)}{q^k} f \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \right).
\end{aligned}$$

We calculate the $w \not\equiv 0$ part. For such w we have

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{cw}{1+cv} & 1 & \varpi_F^{-k}w \\ \frac{\varpi_F^k c}{1+cv} & 0 & 1 \end{pmatrix} \dot{s}_2 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{cw}{1+cv} & 1 & 0 \\ \frac{\varpi_F^k c}{1+cv} & \varpi_F^k/w & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varpi_F^{-k}w & -1 \\ 0 & 0 & \varpi_F^k/w \end{pmatrix}.$$

Note that $cw/(1+cv) \in \mathcal{O}_F$, $\varpi_F^k c/(1+cv)$, $\varpi_F^k/w \in (\varpi_F)$. Hence by the first formula of Theorem 3.7 for k sufficiently large, we get

$$f \left(\begin{pmatrix} 1 & 0 & 0 \\ \frac{cw}{1+cv} & 1 & \varpi_F^{-k}w \\ \frac{\varpi_F^k c}{1+cv} & 0 & 1 \end{pmatrix} \dot{s}_2 \right) = \eta_2(\varpi_F^{-k}w) \int_{\mathcal{O}_F} \eta_2 \left(1 + \frac{\varpi_F^k}{w}t \right) f \left(\begin{pmatrix} 1 & 0 & 0 \\ \frac{c(w+\varpi_F^k t)}{1+cv} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) dt.$$

Hence the $w \neq 0$ part of (3.4) equals

$$\begin{aligned} & \frac{1}{q^k} \sum_{w \in (\mathcal{O}_F / (\varpi_F^k)) \setminus \{0\}} \int_{\mathcal{O}_F} \eta_1(1+cv) \int_{\mathcal{O}_F} \eta_2(w + \varpi_F^k t) f \left(\begin{pmatrix} 1 & 0 & 0 \\ \frac{c(w+\varpi_F^k t)}{1+cv} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) dt dv \\ &= \frac{1}{q^k} \sum_{w \in (\mathcal{O}_F / (\varpi_F^k)) \setminus \{0\}} \int_{\mathcal{O}_F} \eta_1(1+cv) \int_{w+(\varpi_F^k)} \eta_2(1+cv) \eta_2(t') f \left(\begin{pmatrix} 1 & 0 & 0 \\ ct' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \frac{dt'}{|\varpi_F^k|_F} dv, \end{aligned}$$

where $t' := \frac{w+\varpi_F^k t}{1+cv}$, hence

$$\begin{aligned} &= \left(\int_{\mathcal{O}_F} \eta_1 \eta_2(1+cv) dv \right) \left(\int_{\mathcal{O}_F \setminus (\varpi_F^k)} \eta_2(t') f \left(\begin{pmatrix} 1 & 0 & 0 \\ ct' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) dt' \right) \\ &= \delta_{\text{val}(c) \geq c(\eta_1 \eta_2)} \left(\int_{\mathcal{O}_F \setminus (\varpi_F^k)} \eta_2(t') f \left(\begin{pmatrix} 1 & 0 & 0 \\ ct' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) dt' \right) \\ &= \delta_{\text{val}(c) \geq c(\eta_1 \eta_2)} \left(\int_{\mathcal{O}_F} \eta_2(t) \left[f \left(\begin{pmatrix} 1 & 0 & 0 \\ ct & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) - f(1) \right] dt + \left(\int_{\mathcal{O}_F \setminus (\varpi_F^k)} \eta_2(t) dt \right) f(1) \right). \end{aligned}$$

In particular, when $c = 0$ equation (3.4) gives

$$f(\dot{s}_2) \left(1 - \frac{\eta_2(\varpi_F^k)}{q^k} \right) = \left(\int_{\mathcal{O}_F \setminus (\varpi_F^k)} \eta_2(t) dt \right) f(1).$$

Lemma 3.10. *We have*

$$\int_{\mathcal{O}_F \setminus (\varpi_F^k)} \eta_2(t) dt = \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q - \eta_2(\varpi_F)} \left(1 - \frac{\eta_2(\varpi_F^k)}{q^k} \right).$$

Proof. Let n be the valuation of t . Then

$$\int_{\mathcal{O}_F \setminus (\varpi_F^k)} \eta_2(t) dt = \sum_{n=0}^{k-1} \int_{\varpi_F^n \mathcal{O}_F^\times} \eta_2(t) dt = \sum_{n=0}^{k-1} \left(\frac{\eta_2(\varpi_F)}{q} \right)^n \int_{\mathcal{O}_F^\times} \eta_2(s) ds,$$

where $t := \varpi_F^n s$ and $dt = |\varpi_F^n|_F ds$, so

$$= \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q} \cdot \frac{1 - (q^{-1} \eta_2(\varpi_F))^k}{1 - q^{-1} \eta_2(\varpi_F)},$$

where we used our assumption that $\eta_2(\varpi_F) \neq q$. \square

By our assumption that $\eta_2(\varpi_F) \neq q$ we can choose our sufficiently large k such that $\eta_2(\varpi_F^k) \neq q^k$ (equality cannot hold for two consecutive values of k). Hence we get

$$(3.5) \quad f(\dot{s}_2) = \delta_{c(\eta_2)=0} \frac{q-1}{q - \eta_2(\varpi_F)} f(1).$$

In general, substituting (3.5) into the $w \equiv 0$ part we get from our analysis of (3.4) and Lemma 3.10 that

$$\begin{aligned} f \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{pmatrix} \dot{s}_2 \right) &= \delta_{\text{val}(c) \geq c(\eta_1)} \delta_{c(\eta_2)=0} \frac{\eta_2(\varpi_F)^k}{q^k} \frac{q-1}{q - \eta_2(\varpi_F)} f(1) \\ &\quad + \delta_{\text{val}(c) \geq c(\eta_1 \eta_2)} \left(\int_{\mathcal{O}_F} \eta_2(t) \left[f \left(\begin{pmatrix} 1 & 0 & 0 \\ ct & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) - f(1) \right] dt \right) \end{aligned}$$

$$\begin{aligned}
& + \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} \left(1 - \frac{\eta_2(\varpi_F)^k}{q^k} \right) f(1) \\
& = \delta_{\text{val}(c) \geq c(\eta_1 \eta_2)} \left(\int_{\mathcal{O}_F} \eta_2(t) \left[f \begin{pmatrix} 1 & 0 & 0 \\ ct & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - f(1) \right] dt \right. \\
& \quad \left. + \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} f(1) \right).
\end{aligned}$$

Here we used $\delta_{\text{val}(c) \geq c(\eta_1)} \delta_{c(\eta_2)=0} = \delta_{\text{val}(c) \geq c(\eta_1 \eta_2)} \delta_{c(\eta_2)=0}$ in the last line. We get the third formula of Theorem 3.7.

Finally we prove the fifth formula of Theorem 3.7. Note that our formula does not depend on ϖ_F . We take our uniformizer ϖ_F such that $\eta_1 \eta_2(\varpi_F) \neq q^2$, which is possible as $\eta_1 \eta_2 \neq |\cdot|_F^{-2}$. For $z_1 = \text{diag}(\varpi_F, \varpi_F, 1)$, we have as in the proof of Lemma 3.9,

$$\begin{aligned}
f(\dot{s}_2 \dot{s}_1) & = \chi(z_1)^{-1} (\tau_{z_1} f)(\dot{s}_2 \dot{s}_1) \\
& = \eta_1 \eta_2(\varpi_F) \int_{\mathcal{O}_F} \int_{\mathcal{O}_F} f \left(\begin{pmatrix} 1 & 0 & \varpi_F^{-1} v \\ 0 & 1 & \varpi_F^{-1} w \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \dot{s}_1 \right) dv dw.
\end{aligned}$$

We calculate the right-hand side, recalling that the integrand only depends on v, w modulo ϖ_F .

(i) $v, w \equiv 0 \pmod{\varpi_F}$. We have

$$\frac{\eta_1 \eta_2(\varpi_F)}{q^2} f(\dot{s}_2 \dot{s}_1).$$

(ii) $v \not\equiv 0, w \equiv 0 \pmod{\varpi_F}$. We have

$$\begin{pmatrix} 1 & 0 & \varpi_F^{-1} v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \dot{s}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varpi_F v^{-1} & 0 & 1 \end{pmatrix} \dot{s}_2 \begin{pmatrix} \varpi_F^{-1} v & -1 & 0 \\ 0 & \varpi_F v^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, by using the third formula of Theorem 3.7, the right-hand side equals

$$\begin{aligned}
& \frac{\eta_1 \eta_2(\varpi_F) \eta_1(\varpi_F^{-1})}{q} \int_{\mathcal{O}_F^\times} \eta_1(v) f \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varpi_F v^{-1} & 0 & 1 \end{pmatrix} \dot{s}_2 \right) dv \\
& = \delta_{1 \geq c(\eta_1 \eta_2)} \frac{\eta_2(\varpi_F)}{q} \int_{\mathcal{O}_F^\times} \eta_1(v) \left(\int_{\mathcal{O}_F} \eta_2(t) \left[f \begin{pmatrix} 1 & 0 & 0 \\ \varpi_F v^{-1} t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - f(1) \right] dt \right. \\
& \quad \left. + \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} f(1) \right) dv.
\end{aligned}$$

Hence, by letting $t' := \varpi_F v^{-1} t$ (so $dt' = |\varpi_F|_F dt$) and noting that $\eta_2(v) = 1$ if the final term contributes, we get

$$\begin{aligned}
& \frac{\delta_{1 \geq c(\eta_1 \eta_2)}}{q} \left(\int_{\mathcal{O}_F^\times} \eta_1 \eta_2(v) dv \right) \left(\int_{(\varpi_F)} \eta_2(t') \left[f \begin{pmatrix} 1 & 0 & 0 \\ t' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - f(1) \right] \frac{dt'}{|\varpi_F|_F} \right. \\
& \quad \left. + \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} \eta_2(\varpi_F) f(1) \right), \\
& = \delta_{c(\eta_1 \eta_2)=0} \frac{q-1}{q} \left(\int_{(\varpi_F)} \eta_2(t') \left[f \begin{pmatrix} 1 & 0 & 0 \\ t' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - f(1) \right] dt' \right. \\
& \quad \left. + \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} \frac{\eta_2(\varpi_F)}{q} f(1) \right).
\end{aligned}$$

(iii) $v \equiv 0, w \not\equiv 0 \pmod{\varpi_F}$. We have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \varpi_F^{-1}w \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \dot{s}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \varpi_F w^{-1} & 1 \end{pmatrix} \dot{s}_1 \begin{pmatrix} \varpi_F^{-1}w & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & \varpi_F w^{-1} \end{pmatrix}.$$

Hence the right-hand side equals, by using the second formula of Theorem 3.7,

$$\begin{aligned} \frac{\eta_1 \eta_2(\varpi_F) \eta_1 \eta_2(\varpi_F^{-1})}{q} \int_{\mathcal{O}_F^\times} \eta_1 \eta_2(w) f \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \varpi_F w^{-1} & 1 \end{pmatrix} \dot{s}_1 \right) dw \\ = \frac{\delta_{1 \geq c(\eta_1 \eta_2)}}{q} \int_{\mathcal{O}_F^\times} \eta_1 \eta_2(w) \int_{\mathcal{O}_F} f \left(\begin{pmatrix} 1 & \varpi_F w^{-1}t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) dt dw. \end{aligned}$$

By letting $t' := \varpi_F w^{-1}t$ (so $dt' = |\varpi_F|_F dt$), we get

$$\delta_{c(\eta_1 \eta_2)=0} \frac{q-1}{q} \int_{(\varpi_F)} f \left(\begin{pmatrix} 1 & t' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) dt'.$$

(iv) $v \not\equiv 0, w \not\equiv 0 \pmod{\varpi_F}$. We have

$$\begin{pmatrix} 1 & 0 & \varpi_F^{-1}v \\ 0 & 1 & \varpi_F^{-1}w \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \dot{s}_1 = \begin{pmatrix} 1 & 0 & 0 \\ v^{-1}w & 1 & 0 \\ v^{-1}\varpi_F & \varpi_F w^{-1} & 1 \end{pmatrix} \begin{pmatrix} \varpi_F^{-1}v & -1 & 0 \\ 0 & wv^{-1} & -1 \\ 0 & 0 & \varpi_F w^{-1} \end{pmatrix}.$$

Hence we get

$$\frac{\eta_1 \eta_2(\varpi_F)}{q} \int_{\mathcal{O}_F^\times} \sum_{w \in (\mathcal{O}_F / (\varpi_F)) \setminus \{0\}} \eta_1 \eta_2(\varpi_F^{-1}) \eta_1(v) \eta_2(w) f \left(\begin{pmatrix} 1 & 0 & 0 \\ v^{-1}w & 1 & 0 \\ v^{-1}\varpi_F & \varpi_F w^{-1} & 1 \end{pmatrix} dv \right).$$

Therefore, by applying the first formula of Theorem 3.7, we get

$$\frac{\delta_{1 \geq c(\eta_1 \eta_2)}}{q} \int_{\mathcal{O}_F^\times} \sum_{w \in (\mathcal{O}_F / (\varpi_F)) \setminus \{0\}} \eta_1(v) \eta_2(w) \int_{\mathcal{O}_F} \eta_2(1 + \varpi_F w^{-1}t) f \left(\begin{pmatrix} 1 & 0 & 0 \\ v^{-1}(w + \varpi_F t) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} dt dv \right).$$

By letting $t' := v^{-1}(w + \varpi_F t)$ (so $dt' = |\varpi_F|_F dt$ and $vt' = w + \varpi_F t$), we get

$$\begin{aligned} \frac{\delta_{1 \geq c(\eta_1 \eta_2)}}{q} \int_{\mathcal{O}_F^\times} \sum_{w \in (\mathcal{O}_F / (\varpi_F)) \setminus \{0\}} \eta_1(v) \int_{\mathcal{O}_F} \eta_2(w + \varpi_F t) f \left(\begin{pmatrix} 1 & 0 & 0 \\ v^{-1}(w + \varpi_F t) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} dt dv \right) \\ = \frac{\delta_{1 \geq c(\eta_1 \eta_2)}}{q} \int_{\mathcal{O}_F^\times} \eta_1 \eta_2(v) \sum_{w \in (\mathcal{O}_F / (\varpi_F)) \setminus \{0\}} \int_{v^{-1}w + (\varpi_F)} \eta_2(t') f \left(\begin{pmatrix} 1 & 0 & 0 \\ t' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{dt'}{|\varpi_F|_F} dv \right) \\ = \delta_{c(\eta_1 \eta_2)=0} \frac{q-1}{q} \int_{\mathcal{O}_F^\times} \eta_2(t') f \left(\begin{pmatrix} 1 & 0 & 0 \\ t' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} dt' \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \left(1 - \frac{\eta_1 \eta_2(\varpi_F)}{q^2} \right) f(\dot{s}_2 \dot{s}_1) &= \frac{\delta_{c(\eta_1 \eta_2)=0}(q-1)}{q} \left(\int_{(\varpi_F)} \eta_2(t') \left[f \left(\begin{pmatrix} 1 & 0 & 0 \\ t' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) - f(1) \right] dt' \right. \\ &\quad \left. + \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q - \eta_2(\varpi_F)} \frac{\eta_2(\varpi_F)}{q} f(1) \right) \\ &\quad + \frac{\delta_{c(\eta_1 \eta_2)=0}(q-1)}{q} \int_{(\varpi_F)} f \left(\begin{pmatrix} 1 & t' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) dt' \\ &\quad + \frac{\delta_{c(\eta_1 \eta_2)=0}(q-1)}{q} \int_{\mathcal{O}_F^\times} \eta_2(t') f \left(\begin{pmatrix} 1 & 0 & 0 \\ t' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} dt' \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{\delta_{c(\eta_1\eta_2)=0}(q-1)}{q} \left(\int_{\mathcal{O}_F} \eta_2(t') \left[f \begin{pmatrix} 1 & 0 & 0 \\ t' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - f(1) \right] dt' \right. \\
&\quad \left. + \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} f(1) \right) \\
&\quad + \frac{\delta_{c(\eta_1\eta_2)=0}(q-1)}{q} \int_{(\varpi_F)} f \left(\begin{pmatrix} 1 & t' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) dt',
\end{aligned}$$

where we used Lemma 3.10 (with $k = 1$) to combine the first and third terms. We obtain the fifth formula and hence conclude the proof of Theorem 3.7.

3.3. Density argument. We continue to assume that $\eta_2 \neq |\cdot|_F^{-1}$ and $\eta_1\eta_2 \neq |\cdot|_F^{-2}$, and we now also assume $\eta_2 \neq 1$. For $n \geq 1$, $\gamma \in C^\times$ and a smooth function $g: F \rightarrow C$ that vanishes outside \mathcal{O}_F , we define $f_n \in (\text{Ind}_B^G \chi)^{\text{sm}, N \cap K, Z_L^+ = \chi}$ as follows:

$$\begin{aligned}
f_n \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= 0 & (a \in (\varpi_F)), \\
f_n \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) &= \gamma^n g \left(\frac{a}{\varpi_F^n} \right) & (a \in \mathcal{O}_F).
\end{aligned}$$

Here we use that the restriction map (3.2) is an isomorphism. Let $\pi_0(g)$ be the smallest closed subrepresentation of $(\text{Ind}_B^G \chi)^{\text{cts}}$ that contains all f_n ($n \geq 1$).

Lemma 3.11. *There exists g such that $\pi_0(g) = (\text{Ind}_B^G \chi)^{\text{cts}}$ and $\int_{\mathcal{O}_F} g(t) dt = 0$.*

To prove the lemma, we calculate f_n on each Iwahori orbit $I\dot{w}B$ ($w \in S_3$) with the help of Theorem 3.7. (Recall that the Iwahori subgroup I and the representatives \dot{w} were defined in subsection 3.2.) ($w = e$) We have for $a, b, c \in (\varpi_F)$:

$$f_n \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} = \delta_{\text{val}(b) \geq c(\eta_1\eta_2)} \int_{\mathcal{O}_F} \eta_2(1+ct) f_n \begin{pmatrix} 1 & 0 & 0 \\ a+bt & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} dt = 0.$$

($w = s_1$) We have for $a \in \mathcal{O}_F$ and $b, c \in (\varpi_F)$:

$$\begin{aligned}
f_n \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ b & c & 1 \end{pmatrix} \dot{s}_1 \right) &= \delta_{\text{val}(c) \geq c(\eta_1\eta_2)} \int_{\mathcal{O}_F} \eta_2(1+bt) f_n \left(\begin{pmatrix} 1 & a+ct & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) dt \\
&= \delta_{\text{val}(c) \geq c(\eta_1\eta_2)} \cdot \gamma^n \int_{\mathcal{O}_F} \eta_2(1+bt) g \left(\frac{a+ct}{\varpi_F^n} \right) dt.
\end{aligned}$$

For simplicity, we put

$$k(a, b, c) := \int_{\mathcal{O}_F} \eta_2(1+at) g(b+ct) dt$$

for $a \in (\varpi_F)$ and $b, c \in F$. Then

$$f_n \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ b & c & 1 \end{pmatrix} \dot{s}_1 \right) = \delta_{\text{val}(c) \geq c(\eta_1\eta_2)} \gamma^n k \left(b, \frac{a}{\varpi_F^n}, \frac{c}{\varpi_F^n} \right).$$

($w = s_2$) We have for $a, c \in (\varpi_F)$:

$$\begin{aligned}
&f_n \left(\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & 0 & 1 \end{pmatrix} \dot{s}_2 \right) \\
&= \delta_{\text{val}(c) \geq c(\eta_1\eta_2)} \left\{ \int_{\mathcal{O}_F} \eta_2(t) \left[f_n \begin{pmatrix} 1 & 0 & 0 \\ a+ct & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - f_n \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] dt + \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} f_n \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} = 0.
\end{aligned}$$

($w = s_1 s_2$) We have for $a \in \mathcal{O}_F$, $c \in (\varpi_F)$:

$$f_n \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix} \dot{s}_1 \dot{s}_2 \right) = \delta_{\text{val}(c) \geq c(\eta_1 \eta_2)} \cdot \gamma^n \left\{ \int_{\mathcal{O}_F} \eta_2(t) \left(g \left(\frac{a-ct}{\varpi_F^n} \right) - g \left(\frac{a}{\varpi_F^n} \right) \right) dt \right. \\ \left. + \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} g \left(\frac{a}{\varpi_F^n} \right) \right\}.$$

We calculate

$$\int_{\mathcal{O}_F} \eta_2(t) \left(g \left(\frac{a-ct}{\varpi_F^n} \right) - g \left(\frac{a}{\varpi_F^n} \right) \right) dt$$

for $a, c \in \mathcal{O}_F$ (not only for $c \in (\varpi_F)$).

- (i) Assume $\text{val}(a) < \text{val}(c)$ and $\text{val}(a) < n$. then $\text{val}(a-ct) = \text{val}(a) < n$ for any $t \in \mathcal{O}_F$. Hence the value is zero.
- (ii) Assume that $\text{val}(a) \geq n$. Then $\text{val}(a-ct) \geq n$ if and only if $t \in (\varpi_F^n/c)$. Hence if $\text{val}(c) < n$ then the value is equal to the sum of

$$\int_{(\varpi_F^n/c)} \eta_2(t) \left(g \left(\frac{a-ct}{\varpi_F^n} \right) - g \left(\frac{a}{\varpi_F^n} \right) \right) dt \\ = \eta_2 \left(\frac{\varpi_F^n}{c} \right) \left| \frac{\varpi_F^n}{c} \right|_F \int_{\mathcal{O}_F} \eta_2(t') \left(g \left(\frac{a}{\varpi_F^n} - t' \right) - g \left(\frac{a}{\varpi_F^n} \right) \right) dt'$$

and

$$\left(\int_{\mathcal{O}_F \setminus (\varpi_F^n/c)} \eta_2(t) dt \right) \left(-g \left(\frac{a}{\varpi_F^n} \right) \right) \\ = -\delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} \left(1 - \frac{\eta_2(\varpi_F)^{n-\text{val}(c)}}{q^{n-\text{val}(c)}} \right) g \left(\frac{a}{\varpi_F^n} \right) \\ = -\delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} \left(1 - \eta_2 \left(\frac{\varpi_F^n}{c} \right) \left| \frac{\varpi_F^n}{c} \right|_F \right) g \left(\frac{a}{\varpi_F^n} \right)$$

by Lemma 3.10, noting that this expression is zero unless η_2 is unramified.

- (iii) Otherwise $\text{val}(c) \leq \text{val}(a) < n$. Then the value is equal to

$$\int_{a/c + (\varpi_F^n/c)} \eta_2(t) g \left(\frac{a-ct}{\varpi_F^n} \right) dt = \left| \frac{\varpi_F^n}{c} \right|_F \eta_2 \left(\frac{a}{c} \right) \int_{\mathcal{O}_F} \eta_2 \left(1 - \frac{\varpi_F^n}{a} t' \right) g(t') dt' \\ = \left| \frac{\varpi_F^n}{c} \right|_F \eta_2 \left(\frac{a}{c} \right) k \left(-\frac{\varpi_F^n}{a}, 0, 1 \right).$$

To simplify the notation, we put

$$h(a, b) := \int_{\mathcal{O}_F} \eta_2(t) (g(a+bt) - g(a)) dt + \delta_{c(\eta_2)=0} \frac{q-1}{q-\eta_2(\varpi_F)} g(a)$$

for $a, b \in \mathcal{O}_F$. Then we have

$$f_n \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix} \dot{s}_1 \dot{s}_2 \right) \\ = \delta_{\text{val}(c) \geq c(\eta_1 \eta_2)} \cdot \gamma^n \begin{cases} h \left(\frac{a}{\varpi_F^n}, -\frac{c}{\varpi_F^n} \right) & (\text{val}(a) \geq n, \text{val}(c) \geq n), \\ \eta_2 \left(\frac{\varpi_F^n}{c} \right) \left| \frac{\varpi_F^n}{c} \right|_F h \left(\frac{a}{\varpi_F^n}, -1 \right) & (\text{val}(a) \geq n, \text{val}(c) < n), \\ \left| \frac{\varpi_F^n}{c} \right|_F \eta_2 \left(\frac{a}{c} \right) k \left(-\frac{\varpi_F^n}{a}, 0, 1 \right) & (\text{val}(a) < n, \text{val}(c) \leq \text{val}(a)), \\ 0 & (\text{val}(a) < n, \text{val}(c) > \text{val}(a)). \end{cases}$$

($w = s_2 s_1$) We have for $a \in (\varpi_F)$:

$$f_n \left(\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \dot{s}_1 \right) = \frac{\delta_{c(\eta_1 \eta_2)=0} (q-1)}{q(q^2 - \eta_1 \eta_2(\varpi_F))} \left\{ \int_{\mathcal{O}_F} \eta_2(t) \left[f_n \begin{pmatrix} 1 & 0 & 0 \\ a+t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - f_n \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] dt \right\}$$

$$+ \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} f_n \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \int_{(\varpi_F)} \eta_2(1-at) f_n \left(\begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) dt \Big\}.$$

The first two terms are zero, so

$$\begin{aligned} f_n \left(\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_2 \dot{s}_1 \right) &= \delta_{c(\eta_1\eta_2)=0} \frac{q-1}{q^2-\eta_1\eta_2(\varpi_F)} \frac{\gamma^n}{q^{n+1}} \int_{\mathcal{O}_F} \eta_2(1-a\varpi_F^n t) g(t) dt \\ &= \delta_{c(\eta_1\eta_2)=0} \frac{q-1}{q^2-\eta_1\eta_2(\varpi_F)} \frac{\gamma^n}{q^{n+1}} k(-a\varpi_F^n, 0, 1). \end{aligned}$$

($w = w_0$) We have for $a \in \mathcal{O}_F$:

$$\begin{aligned} f_n \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{w}_0 \right) &= \frac{\delta_{c(\eta_1\eta_2)=0}(q-1)}{q(q^2-\eta_1\eta_2(\varpi_F))} \gamma^n \left\{ \int_{\mathcal{O}_F} \eta_2(t) \left(g\left(\frac{a-t}{\varpi_F^n}\right) - g\left(\frac{a}{\varpi_F^n}\right) \right) dt \right. \\ &\quad \left. + \delta_{c(\eta_2)=0} \cdot \frac{q-1}{q-\eta_2(\varpi_F)} g\left(\frac{a}{\varpi_F^n}\right) \right\}. \end{aligned}$$

We calculated the term in parentheses already. We have

$$\begin{aligned} &f_n \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{w}_0 \right) \\ &= \delta_{c(\eta_1\eta_2)=0} \frac{q-1}{q(q^2-\eta_1\eta_2(\varpi_F))} \gamma^n \begin{cases} \eta_2(\varpi_F^n) |\varpi_F^n|_F h\left(\frac{a}{\varpi_F^n}, -1\right) & (\text{val}(a) \geq n), \\ \eta_2(a) |\varpi_F^n|_F k\left(-\frac{\varpi_F^n}{a}, 0, 1\right) & (\text{val}(a) < n). \end{cases} \end{aligned}$$

We start to prove Lemma 3.11.

Lemma 3.12.

- (i) *The functions h and k are smooth. In particular h is bounded on $\mathcal{O}_F \times \mathcal{O}_F$.*
- (ii) *There exists a smooth function $g: F \rightarrow C$ that vanishes outside \mathcal{O}_F such that $\int_{\mathcal{O}_F} g(t) dt = 0$ and $h(0, -1) \neq 0$.*
- (iii) *Assume that $\int_{\mathcal{O}_F} g(t) dt = 0$. Then the function $k(a, b, c)$ is compactly supported, hence bounded on $(\varpi_F) \times F \times F$. Moreover, $k(a, 0, 1) = 0$ if $\text{val}(a) \geq c(\eta_2)$.*

Proof. For (i), take $\ell \in \mathbb{Z}_{>0}$ such that $g(x+t) = g(x)$ for any $t \in (\varpi_F^\ell)$. Then we have $h(a+a_1, b+b_1) = h(a, b)$ for any $a_1, b_1 \in (\varpi_F^\ell)$. Hence h is smooth and a similar argument applies for k .

We prove (ii). Set $c := \max(c(\eta_2), 1)$ and define $g: \mathcal{O}_F \rightarrow C$ by

$$g(x) = \begin{cases} -1 & (x \in \mathcal{O}_F \setminus (-1 + (\varpi_F^c))), \\ q^c - 1 & (x \in -1 + (\varpi_F^c)). \end{cases}$$

Then $g(x+y) = g(x)$ for any $y \in (\varpi_F^c)$ and $\sum_{x \in \mathcal{O}_F / (\varpi_F^c)} g(x) = 0$. Hence $\int_{\mathcal{O}_F} g(x) dx = 0$. We have

$$\begin{aligned} h(0, -1) &= \int_{\mathcal{O}_F} \eta_2(t) (g(-t) - g(0)) dt + \delta_{c(\eta_2)=0} \frac{q-1}{q-\eta_2(\varpi_F)} g(0) \\ &= \sum_{a \in \mathcal{O}_F / (\varpi_F^c)} \int_{(\varpi_F^c)} \eta_2(t-a) (g(a-t) - g(0)) dt + \delta_{c(\eta_2)=0} \frac{q-1}{q-\eta_2(\varpi_F)} g(0) \\ &= \sum_{a \in \mathcal{O}_F / (\varpi_F^c)} (g(a) - g(0)) \int_{(\varpi_F^c)} \eta_2(t-a) dt + \delta_{c(\eta_2)=0} \frac{q-1}{q-\eta_2(\varpi_F)} g(0) \\ &= q^c \int_{(\varpi_F^c)} \eta_2(t+1) dt - \delta_{c(\eta_2)=0} \frac{q-1}{q-\eta_2(\varpi_F)}. \end{aligned}$$

We have $\eta_2(1+t) = 1$ for any $t \in \varpi_F^c \mathcal{O}_F$. Hence

$$h(0, -1) = 1 - \delta_{c(\eta_2)=0} \frac{q-1}{q-\eta_2(\varpi_F)} = \begin{cases} \frac{1-\eta_2(\varpi_F)}{q-\eta_2(\varpi_F)} & (c(\eta_2) = 0), \\ 1 & (c(\eta_2) > 0). \end{cases}$$

This is not zero, as $\eta_2 \neq 1$.

Consider $k(a, b, c) = \int_{\mathcal{O}_F} \eta_2(1+at)g(b+ct)dt$ and recall that $\text{supp}(g) \subset \mathcal{O}_F$. The p -adic balls $b+c\mathcal{O}_F$ and \mathcal{O}_F are either disjoint or nested. If $(b+c\mathcal{O}_F) \cap \mathcal{O}_F = \emptyset$, then $k(a, b, c) = 0$, and $b+c\mathcal{O}_F \subset \mathcal{O}_F$ is equivalent to $b, c \in \mathcal{O}_F$ (compact). Hence it remains to consider the case where $\mathcal{O}_F \subset b+c\mathcal{O}_F$, or equivalently $\text{val}(c) \leq \min(0, \text{val}(b))$. Letting $t' := b+ct$ we obtain

$$\begin{aligned} k(a, b, c) &= |c|_F^{-1} \int_{\mathcal{O}_F} \eta_2 \left(1 + a \cdot \frac{t' - b}{c} \right) g(t') dt' \\ &= |c|_F^{-1} \eta_2 \left(1 - \frac{ab}{c} \right) \int_{\mathcal{O}_F} \eta_2 \left(1 + \frac{a}{c-ab} t' \right) g(t') dt'. \end{aligned}$$

Hence if $\text{val}(a) - \text{val}(c-ab) \geq c(\eta_2)$, then $k(a, b, c) = 0$. Note that $\text{val}(c-ab) = \text{val}(c)$, as $\text{val}(ab) > \text{val}(b) \geq \text{val}(c)$. Therefore in this region, k is supported on the compact subset $\text{val}(b) \geq \text{val}(c) > \text{val}(a) - c(\eta_2) > -c(\eta_2)$. \square

We assume that g satisfies the condition of Lemma 3.12(ii) and prove that $\pi_0(g) = (\text{Ind}_B^G \chi)^{\text{cts}}$. Let us assume that $n \geq c(\eta_2)$ from now on. Since $\int_{\mathcal{O}_F} g(x)dx = 0$, the formula for f_n simplifies and we have the following, where $a \in \mathcal{O}_F$ and $b, c \in (\varpi_F)$:

$$\begin{aligned} f_n \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ b & c & 1 \end{pmatrix} \dot{s}_1 \right) &= \delta_{\text{val}(c) \geq c(\eta_1 \eta_2)} \cdot \gamma^n k \left(b, \frac{a}{\varpi_F^n}, \frac{c}{\varpi_F^n} \right), \\ f_n \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix} \dot{s}_1 \dot{s}_2 \right) &= \delta_{\text{val}(c) \geq c(\eta_1 \eta_2)} \cdot \gamma^n \begin{cases} h \left(\frac{a}{\varpi_F^n}, -\frac{c}{\varpi_F^n} \right) & (\text{val}(a) \geq n, \text{val}(c) \geq n), \\ \eta_2 \left(\frac{a}{\varpi_F^n} \right) \left| \frac{\varpi_F^n}{c} \right|_F h \left(\frac{a}{\varpi_F^n}, -1 \right) & (\text{val}(a) \geq n, \text{val}(c) < n), \\ \left| \frac{\varpi_F^n}{c} \right|_F \eta_2 \left(\frac{a}{c} \right) k \left(-\frac{a}{\varpi_F^n}, 0, 1 \right) & (\text{val}(a) < n, \text{val}(c) \leq \text{val}(a)), \\ 0 & (\text{val}(a) < n, \text{val}(c) > \text{val}(a)), \end{cases} \\ f_n \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{w}_0 \right) &= \delta_{c(\eta_1 \eta_2) = 0} \frac{q-1}{q(q^2 - \eta_1 \eta_2 (\varpi_F))} \gamma^n \begin{cases} \eta_2 (\varpi_F^n) |\varpi_F^n|_F h \left(\frac{a}{\varpi_F^n}, -1 \right) & (\text{val}(a) \geq n), \\ \eta_2 (a) |\varpi_F^n|_F k \left(-\frac{a}{\varpi_F^n}, 0, 1 \right) & (\text{val}(a) < n). \end{cases} \end{aligned}$$

On the other orbits, f_n vanishes. (In case of $w = s_2 s_1$, this is because $n \geq c(\eta_2)$ and Lemma 3.12(iii).)

From now on, we moreover assume that $|\eta_2(\varpi_F)| > |q|$ and put $\gamma := q/\eta_2(\varpi_F)$, so $|\gamma| < 1$. We define $v \in (\text{Ind}_{B \cap L}^L \chi)^{\text{cts}}$ by

$$v \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0, \quad v \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{\text{val}(a) \geq c(\eta_1 \eta_2)} \frac{q^{\text{val}(a)}}{\eta_2(a)}, \quad v \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) = \delta_{c(\eta_1 \eta_2) = 0} \frac{q-1}{q(q^2 - \eta_1 \eta_2 (\varpi_F))},$$

where $a \in (\varpi_F) \setminus \{0\}$, respectively $a \in \mathcal{O}_F$. Note that this defines a continuous function since $\lim_{a \rightarrow 0} |q^{\text{val}(a)}/\eta_2(a)| = \lim_{a \rightarrow 0} |\gamma|^{\text{val}(a)} = 0$. We also define $h'_n \in C^0(\bar{N}, Cv)$ by

$$h'_n \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & c & 1 \end{pmatrix} = \begin{cases} h \left(\frac{b}{\varpi_F^n}, -1 \right) v & (\text{val}(b) \geq n, c \in \mathcal{O}_F), \\ \eta_2 \left(\frac{b}{\varpi_F^n} \right) k \left(-\frac{\varpi_F^n}{b}, 0, 1 \right) v & (0 \leq \text{val}(b) < n, c \in \mathcal{O}_F), \\ 0 & (\text{otherwise}). \end{cases}$$

This function can be regarded as an element of $(\text{Ind}_P^G (\text{Ind}_{B \cap L}^L \chi)^{\text{cts}})^{\text{cts}}$ as usual: $h'_n(\bar{n}p) = p^{-1}h'_n(\bar{n}) \in (\text{Ind}_{B \cap L}^L \chi)^{\text{cts}}$ for $p \in P$, $\bar{n} \in \bar{N}$ and $h'_n|_{G \setminus \bar{N}P} = 0$. Since $(\text{Ind}_P^G (\text{Ind}_{B \cap L}^L \chi)^{\text{cts}})^{\text{cts}} \cong (\text{Ind}_B^G \chi)^{\text{cts}}$ we can regard h'_n as an element of $(\text{Ind}_B^G \chi)^{\text{cts}}$, namely the function $g \mapsto h'_n(g)(1)$. A concrete description of h'_n is as follows. First we have

$$\begin{aligned} \text{supp}(h'_n) &\subset \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathcal{O}_F & \mathcal{O}_F & 1 \end{pmatrix} P/B \subset I^{s_1 s_2} ((I^{s_1 s_2} \cap L)(B \cap L) \cup (I^{s_1 s_2} \cap L) \dot{s}_1 (B \cap L)) B/B \\ &\subset I^{s_1 s_2} B/B \cup I^{s_1 s_2} \dot{s}_1 B/B, \end{aligned}$$

where $I^{s_1 s_2} := (\dot{s}_1 \dot{s}_2)^{-1} I \dot{s}_1 \dot{s}_2$ is another Iwahori subgroup. On each orbit we have the following. If $a \in (\varpi_F), b, c \in \mathcal{O}_F$ then

$$\begin{aligned} & h'_n \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \\ &= h'_n \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b-ac & c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \\ &= \delta_{\text{val}(a) \geq c(\eta_1 \eta_2)} \begin{cases} h \left(\frac{b-ac}{\varpi_F^n}, -1 \right) \frac{q^{\text{val}(a)}}{\eta_2(a)} & (\text{val}(b-ac) \geq n), \\ \eta_2 \left(\frac{b-ac}{\varpi_F^n} \right) k \left(-\frac{\varpi_F^n}{b-ac}, 0, 1 \right) \frac{q^{\text{val}(a)}}{\eta_2(a)} & (\text{val}(b-ac) < n), \end{cases} \end{aligned}$$

where we interpret $\frac{q^{\text{val}(a)}}{\eta_2(a)}$ as 0 when $a = 0$. If $a, b, c \in \mathcal{O}_F$, then we have

$$\begin{aligned} & h'_n \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ c & b & 1 \end{pmatrix} \dot{s}_1 \right) \\ &= h'_n \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & b-ac & 1 \end{pmatrix} \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) \\ &= \delta_{c(\eta_1 \eta_2) = 0} \frac{q-1}{q(q^2 - \eta_1 \eta_2(\varpi_F))} \begin{cases} h \left(\frac{c}{\varpi_F^n}, -1 \right) & (\text{val}(c) \geq n), \\ \eta_2 \left(\frac{c}{\varpi_F^n} \right) k \left(-\frac{\varpi_F^n}{c}, 0, 1 \right) & (\text{val}(c) < n). \end{cases} \end{aligned}$$

Put $h_n := (\dot{s}_1 \dot{s}_2)^{-1} f_n \in \pi_0(g)$. The value on $I^{s_1 s_2} B/B$ is as follows. If $a \in (\varpi_F), b, c \in \mathcal{O}_F$, then

$$\begin{aligned} & h_n \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \\ &= f_n \left(\begin{pmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix} \dot{s}_1 \dot{s}_2 \right) \\ &= f_n \left(\begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b-ac & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix} \dot{s}_1 \dot{s}_2 \right) \\ &= f_n \left(\begin{pmatrix} 1 & b-ac & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix} \dot{s}_1 \dot{s}_2 \right) \\ &= \delta_{\text{val}(a) \geq c(\eta_1 \eta_2)} \begin{cases} \gamma^n h \left(\frac{b-ac}{\varpi_F^n}, -\frac{a}{\varpi_F^n} \right) & (\text{val}(b-ac) \geq n, \text{val}(a) \geq n), \\ \frac{q^{\text{val}(a)}}{\eta_2(a)} h \left(\frac{b-ac}{\varpi_F^n}, -1 \right) & (\text{val}(b-ac) \geq n, \text{val}(a) < n), \\ \frac{q^{\text{val}(a)}}{\eta_2(a)} \eta_2 \left(\frac{b-ac}{\varpi_F^n} \right) k \left(-\frac{\varpi_F^n}{b-ac}, 0, 1 \right) & (\text{val}(b-ac) < n, \text{val}(a) \leq \text{val}(b-ac)), \\ 0 & (\text{val}(b-ac) < n, \text{val}(a) > \text{val}(b-ac)). \end{cases} \end{aligned}$$

On $I^{s_1 s_2} \dot{s}_1 B/B$, it is as follows. Let $a, b, c \in \mathcal{O}_F$. Then

$$\begin{aligned} h_n \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ c & b & 1 \end{pmatrix} \dot{s}_1 \right) &= f_n \left(\begin{pmatrix} 1 & c & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \dot{w}_0 \right) \\ &= f_n \left(\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{w}_0 \right) \end{aligned}$$

$$\begin{aligned}
&= f_n \left(\begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{w}_0 \right) \\
&= \delta_{c(\eta_1\eta_2)=0} \frac{q-1}{q(q^2 - \eta_1\eta_2(\varpi_F))} \begin{cases} h\left(\frac{c}{\varpi_F^n}, -1\right) & (\text{val}(c) \geq n), \\ \eta_2\left(\frac{c}{\varpi_F^n}\right) k\left(-\frac{\varpi_F^n}{c}, 0, 1\right) & (\text{val}(c) < n), \end{cases}
\end{aligned}$$

which is the same as the value of h'_n .

Proof of Lemma 3.11. To prove that $\pi_0(g) = (\text{Ind}_B^G \chi)^{\text{cts}}$ we apply Corollary 2.3 with $\overline{N}_0 = \overline{N} \cap K$, where we think of $(\text{Ind}_B^G \chi)^{\text{cts}}$ as $(\text{Ind}_P^G(\text{Ind}_{B \cap L}^L \chi)^{\text{cts}})^{\text{cts}}$. As indicated in §1.1, our choice of norm on C induces an $L \cap K$ -invariant norm on $(\text{Ind}_{B \cap L}^L \chi)^{\text{cts}}$, which in turn induces a K -invariant norm $\|\cdot\|$ on $(\text{Ind}_P^G(\text{Ind}_{B \cap L}^L \chi)^{\text{cts}})^{\text{cts}}$. This norm $\|\cdot\|$ is nothing but the K -invariant norm induced on $(\text{Ind}_B^G \chi)^{\text{cts}}$ by our norm on C .

We have

$$\inf_n \|h'_n\| \geq \inf_n \left| h'_n \begin{pmatrix} 1 & 0 & 0 \\ \varpi_F^{c(\eta_1\eta_2)+1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = |h(0, -1)| |\gamma|^{c(\eta_1\eta_2)+1} > 0,$$

as $h(0, -1) \neq 0$. Hence it is sufficient to prove $\lim_{n \rightarrow \infty} (h_n - h'_n) = 0$. On $I^{s_1 s_2} \dot{s}_1 \dot{s}_2 B/B \cup I^{s_1 s_2} \dot{s}_2 \dot{s}_1 B/B \cup I^{s_1 s_2} \dot{w}_0 B/B$, $h_n = h'_n = 0$. On $I^{s_1 s_2} \dot{s}_1 B/B$, we have $h_n = h'_n$. On $I^{s_1 s_2} \dot{s}_2 B/B$, we have $h'_n = 0$ and it is sufficient to prove $\lim_{n \rightarrow \infty} \|h_n\| = 0$. This is equivalent to $\lim_{n \rightarrow \infty} \|f_n\| = 0$ on $I \dot{s}_1 B/B$, and this is true since the function k is bounded and $|\gamma| < 1$.

Finally we estimate

$$(3.6) \quad \left| h_n \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} - h'_n \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \right|.$$

for $a \in (\varpi_F)$ and $b, c \in \mathcal{O}_F$. If $\text{val}(a) \leq \text{val}(b - ac) < n$ or $\text{val}(a) < n \leq \text{val}(b - ac)$, then this is zero. If $\text{val}(b - ac) < n$ and $\text{val}(a) > \text{val}(b - ac)$, then (3.6) equals

$$\left| \frac{q^{\text{val}(a)}}{\eta_2(a)} \eta_2 \left(\frac{b - ac}{\varpi_F^n} \right) k \left(-\frac{\varpi_F^n}{b - ac}, 0, 1 \right) \right| = |\gamma|^{\text{val}(a)} \left| \eta_2 \left(\frac{b - ac}{\varpi_F^n} \right) k \left(-\frac{\varpi_F^n}{b - ac}, 0, 1 \right) \right|.$$

This is zero if $n \geq \text{val}(b - ac) + c(\eta_2)$ by Lemma 3.12(iii). Assume $n < \text{val}(b - ac) + c(\eta_2)$. Then $\text{val}(a) > n - c(\eta_2)$. As $|\gamma| < 1$, we get at most

$$\begin{aligned}
&\leq |\gamma|^{n-c(\eta_2)} |\eta_2(\varpi_F)|^{\text{val}((b-ac)/\varpi_F^n)} \left| k \left(-\frac{\varpi_F^n}{b - ac}, 0, 1 \right) \right| \\
&\leq |\gamma|^{n-c(\eta_2)} |\eta_2(\varpi_F)|^{\text{val}((b-ac)/\varpi_F^n)} \sup_{x \in (\varpi_F)} |k(x, 0, 1)|.
\end{aligned}$$

We have $-c(\eta_2) < \text{val}((b - ac)/\varpi_F^n) < 0$. Therefore there exists $r > 0$ which does not depend on a, b, c, n such that (3.6) is less than or equal to $r|\gamma|^n$ in this case.

If $\text{val}(b - ac) \geq n$ and $\text{val}(a) \geq n$, then

$$\left| h_n \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \right| \leq |\gamma|^n \sup_{x, y \in \mathcal{O}_F} |h(x, y)|$$

and

$$\left| h'_n \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \right| \leq \left| \frac{q^{\text{val}(a)}}{\eta_2(a)} \right| \sup_{x \in \mathcal{O}_F} |h(x, -1)|.$$

Since $\text{val}(a) \geq n$ and $|\gamma| < 1$, we have $|q^{\text{val}(a)}/\eta_2(a)| = |\gamma|^{\text{val}(a)} \leq |\gamma|^n$. Hence (3.6) is less than or equal to $|\gamma|^n \sup_{x, y \in \mathcal{O}_F} |h(x, y)|$ in this case.

In summary, (3.6) is less than or equal to $r_1 |\gamma|^n$ for some $r_1 > 0$ which does not depend on a, b, c, n . Hence it converges to zero uniformly. \square

3.4. Proof of Proposition 3.5. Let $\pi \subset (\text{Ind}_B^G \chi)^{\text{sm}}$ be a non-zero subrepresentation. We have a non-zero map $\pi \hookrightarrow (\text{Ind}_B^G \chi)^{\text{sm}}$. Hence by Frobenius reciprocity, we have a non-zero T -equivariant map $\pi_U \rightarrow \chi$. The composition $\pi_N \rightarrow \pi_U \rightarrow \chi$ is non-zero Z_L -equivariant map. Therefore $\pi_N^{Z_L=\chi}$ is non-zero and it is a subrepresentation of $((\text{Ind}_B^G \chi)^{\text{sm}})_N^{Z_L=\chi} \cong (\text{Ind}_{B \cap L}^L \chi)^{\text{sm}}$.

By our assumption we have $\chi_1 \neq \chi_2$. Therefore, $(\text{Ind}_{B \cap L}^L \chi)^{\text{sm}}$ is irreducible or has as socle a twist of the Steinberg representation. If $(\text{Ind}_{B \cap L}^L \chi)^{\text{sm}}$ is irreducible, then $\pi_N^{Z_L=\chi} = (\text{Ind}_{B \cap L}^L \chi)^{\text{sm}}$. Therefore π contains f_n for all $n \geq 1$ and so π is dense in $(\text{Ind}_B^G \chi)^{\text{cts}}$ by Lemma 3.11. Hence $(\text{Ind}_B^G \sigma)^{\text{cts}}$ is irreducible by Theorem 2.1.

Assume that $(\text{Ind}_{B \cap L}^L \chi)^{\text{sm}}$ is reducible with socle a twist of the Steinberg representation. Hence $\pi_N^{Z_L=\chi}$ also contains a twist of the Steinberg representation. The socle of $(\text{Ind}_{B \cap L}^L \chi)^{\text{sm}}$ is the kernel of a surjective morphism $\varphi: (\text{Ind}_{B \cap L}^L \chi)^{\text{sm}} \rightarrow (\chi_1 | \cdot|_F^{-1} \circ \det) \boxtimes \chi_3$. If the support of $f \in (\text{Ind}_{B \cap L}^L \chi)^{\text{sm}}$ is contained in $(U \cap L) \dot{s}_1 (B \cap L)$, then we can normalize φ so that $\varphi(f) = \int_{U \cap L} f(x \dot{s}_1) dx$. Hence we have

$$\begin{aligned} \varphi(f_n|_L) &= \int_{U \cap L} f_n(x \dot{s}_1) dx \\ &= \int_F f_n \left(\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{s}_1 \right) dx \\ &= \gamma^n \int_{(\varpi_F^n)} g \left(\frac{x}{\varpi_F^n} \right) dx \\ &= \gamma^n |\varpi_F^n|_F \int_{\mathcal{O}_F} g(x) dx = 0. \end{aligned}$$

Therefore $f_n|_L \in \ker(\varphi) \subset \pi_N^{Z_L=\chi}$ and we have $f_n \in \pi$. By Lemma 3.11 π is dense in $(\text{Ind}_B^G \chi)^{\text{cts}}$ and hence $(\text{Ind}_B^G \chi)^{\text{cts}}$ is irreducible by Theorem 2.1.

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