LINEAR ALGEBRAIC GROUPS (MAT 1110, WINTER 2017) HOMEWORK 1, DUE FEBRUARY 8, 2017

FLORIAN HERZIG

Problem 1. Let $M \leq \mathrm{GL}_n$ be the subgroup of monomial matrices (having precisely one non-zero entry in each row and each column).

(i) Prove that M is a closed subgroup of GL_n, and that its identity component M° is the group D_n of invertible diagonal matrices.
(ii) Show that M is the normaliser of D_n in GL_n.

Problem 2. For each of the following groups determine the semisimple and unipotent elements. (i) $\mathbb{G}_a$, (ii) any finite-dimensional vector space V (with addition), (iii) any finite group (considered as algebraic group), (iv) the semidirect product $\mathbb{G}_a \rtimes \mathbb{G}_m$, where $x \in \mathbb{G}_m$ acts on $y \in \mathbb{G}_a$ as $x^n y$. (You should allow the field $k = \overline{k}$ to be of arbitrary characteristic!)

Problem 3.

(i) Consider the action of $\mathrm{GL}(V)$ on $\mathbb{P}V$ and on $\mathbb{P}V \times \mathbb{P}V$, where V is a finite-dimensional vector space. Determine the orbits in each case.
(ii) Suppose that the algebraic group G acts on the variety X. If the action has only finitely many orbits, show that there exists an open orbit. Show that this is false in general.

Problem 4. Solve the following exercises from Springer's book: 2.2.2(2), 2.2.2(4), 2.4.10(3).

Update: The last part of Problem 2.2.2(2) is harder than I thought. I posted a solution at

`http://mathoverflow.net/questions/98881/connectedness-of-the-linear-algebraic-group-so-n/.`