LINEAR ALGEBRAIC GROUPS (MAT 1110, WINTER 2017) HOMEWORK 3, DUE MARCH 15, 2017

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Problem 1. Show that if $H \leq G$ is a closed subgroup such that both H and G/H are irreducible (or equivalently connected), then G is irreducible. (Hint: you could use Springer exercise 5.5.9(1).)

Problem 1b (Optional)**.** Show that if $\varphi : G \to H$ is a homomorphism such that $\ker \varphi$ and $\operatorname{im} \varphi$ are connected, then G is connected. This doesn't quite follow from the previous problem, since the natural map $G/\ker \varphi \to \operatorname{im} \varphi$ might not be an isomorphism. (Hint: show that $\varphi(G^0) = \operatorname{im} \varphi$.)

Problem 2. If $N \leq H \leq G$ are closed subgroups with $N \lhd G$, then the natural map $H/N \to G/N$ is a closed immersion (so we can think of H/N as a closed subgroup of G/N) and we have a canonical isomorphism $(G/N)/(H/N) \xrightarrow{\sim} G/H$ of homogeneous G-spaces.

Problem 3. Assume that $\operatorname{char} k = 0$. Suppose $N, H \leq G$ are closed subgroups such that H normalises N. Show that HN is a closed subgroup of G and that we have a canonical isomorphism $HN/N \cong H/(H \cap N)$ of algebraic groups. Find a counterexample when $\operatorname{char} k > 0$.

Problem 4. Assume that $\operatorname{char} k = 0$. Suppose $\phi : G \twoheadrightarrow H$ is a surjective morphism of algebraic groups. If $H_1 \leq H_2 \leq H$ are closed subgroups, show that the map ϕ induces a canonical isomorphism $\phi^{-1}(H_2)/\phi^{-1}(H_1) \xrightarrow{\sim} H_2/H_1$. Find a counterexample when $\operatorname{char} k > 0$. **Update:** Show however that this is true when ϕ is separable.

Problem 5. Solve the following exercises from Springer's book: 5.3.5(2), 5.5.9(2) (Hint: use Chevalley's theorem), 5.5.11(2)