LINEAR ALGEBRAIC GROUPS (MAT 1110, WINTER 2017) HOMEWORK 4, DUE MARCH 29, 2017

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Problem 1. Suppose that G is a connected linear algebraic group.

(i) Show that G has a unique Borel iff G is solvable.
(ii) Show that G has a unique maximal torus iff G is nilpotent.
(iii) Show that maximal tori of G are trivial iff G is unipotent.
(iv) Show that if every element of G is semisimple, then G is a torus.

Problem 2. Suppose the characteristic of k is not 2 and that $n \geq 1$. For any $d \geq 1$ let J_d denote the $d \times d$ antidiagonal matrix over k given by $J_d = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ and define the orthogonal group $G := \mathrm{SO}_{2n} = \{g \in \mathrm{SL}_{2n} : {}^t g \cdot J_{2n} \cdot g = J_{2n}\}$. You may assume that it is a connected closed subgroup of SL_{2n}.

(i) Let $T = D_{2n} \cap G$. Show that T is a maximal torus of G and that it has dimension n. (Hint: show that $Z_G(T) = T$.)
(ii) Let $B = B_{2n} \cap G$. Show that B is a Borel subgroup of G. Also compute $\dim B$. (Hint: first show that B is a semidirect product of the groups $\{\begin{pmatrix} b & \\ & J_n \cdot {}^t b^{-1} \cdot J_n \end{pmatrix} : b \in B_n\}$ and $\{\begin{pmatrix} 1 & J_n \cdot x \\ & 1 \end{pmatrix} : x + {}^t x = 0\}$ and deduce it is connected. Suppose $B \leq B'$, where B' is a Borel. Show that B' has to stabilise a line of $\mathbb{P}^{2n}$ (with its natural G-action). But e.g. since T only stabilises the lines $\langle e_i \rangle$ $(1 \leq i \leq 2n)$ spanned by the standard basis vectors deduce that B' stabilises $\langle e_1 \rangle$ and hence also the hyperplane $\langle e_1 \rangle^\perp$, which contains $\langle e_1 \rangle$. Now use induction to see that $B' \subset B_{2n}$.)
(iii) Deduce that G is reductive and that G is semisimple when $n > 1$. What group is G isomorphic to when $n = 1$?

Remark 1. A similar argument applies to SO_{2n+1} and Sp_{2n} $(n \geq 1)$.

Problem 3. Suppose $G = \mathrm{SL}_n$. Show that $D_n \cap G$ is a maximal torus of G and that $B_n \cap G$ is a Borel of G. Deduce that G is semisimple.

Problem 4. Solve the following exercises from Springer's book: 6.2.11(4), 6.3.7(4).