LINEAR ALGEBRAIC GROUPS (MAT 1110, WINTER 2017) HOMEWORK 5, DUE APRIL 12, 2017

FLORIAN HERZIG

Problem 1. Suppose that $(X, \Phi, X^\vee, \Phi^\vee)$ is a root datum. Define a homomorphism $p : X \to X^\vee$ by $p(x) = \sum_{\alpha \in \Phi} \langle x, \alpha^\vee \rangle \alpha^\vee$. Show that

(i) $\langle x, p(x) \rangle \geq 0$ for all $x \in X$ and $\langle \beta, p(\beta) \rangle > 0$ for all $\beta \in \Phi$.
(ii) $\langle s_\beta x, p(s_\beta x) \rangle = \langle x, p(x) \rangle$ for all $x \in X$, $\beta \in \Phi$.
(iii) $\langle \beta, p(\beta) \rangle \beta^\vee = 2p(\beta)$, hence $\beta^\vee = \frac{2p(\beta)}{\langle \beta, p(\beta) \rangle}$, for all $\beta \in \Phi$. (Hint: when you expand the left-hand side, try to spot a term that can be expressed using $s_{\beta^\vee}$.)
(iv) Deduce that p induces an $\mathbb{Q}$-linear isomorphism $\mathbb{Q}\Phi \to \mathbb{Q}\Phi^\vee$ (obtained by restriction from $X \otimes \mathbb{Q} \to X^\vee \otimes \mathbb{Q}$). (Hint: use that the root datum is symmetric in X and $X^\vee$.)

Problem 2. Suppose that G is a connected reductive group with maximal torus T, roots Φ, and coroots $\Phi^\vee$ (with respect to T). Show that the following are equivalent.

(i) G is semisimple.
(ii) $\mathbb{Q}\Phi = X^*(T) \otimes \mathbb{Q}$.
(iii) $\mathbb{Q}\Phi^\vee = X_*(T) \otimes \mathbb{Q}$.
(iv) $G = \langle U_\alpha : \alpha \in \Phi \rangle$.
(v) $G = \mathcal{D}G$.

(A few hints: You should be able to show (i)⇒(ii)⇒(iii)⇒(iv)⇒(v)⇒(i). Try to express the radical of G in terms of the roots. Keep in mind the previous problem. Show that $\operatorname{im}(\alpha^\vee) \subset \langle U_\alpha, U_{-\alpha} \rangle$ by reducing to a calculation in SL_2. Show $U_\alpha \subset \mathcal{D}G$.)

Deduce that for G connected reductive we have $G = \mathcal{D}G \cdot RG$.

Problem 3. (Here's another way to characterise coroots, using the Weyl group.) Suppose that G is a connected reductive group with maximal torus T, roots Φ, and Weyl group W (with respect to T). For any $\alpha \in \Phi$ show that there exist precisely two elements $w \in W$ such that $w\mu - \mu \in \mathbb{Z}\alpha$ for all $\mu \in X^*(T)$, namely 1 and s_α. (Hint: show that $w(\alpha) = \pm\alpha$. Using s_α, assume WLOG that $w(\alpha) = \alpha$. Show that $w\mu = \mu - \langle \mu, \alpha' \rangle \alpha$ for some $\alpha' \dots$.)

Problem 4. Suppose the characteristic of k is not 2 and that $n \geq 2$. For any $d \geq 1$ let J_d denote the $d \times d$ antidiagonal matrix over k given by

$J_d = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ and define the orthogonal group $G := \mathrm{SO}_{2n} = \{g \in \mathrm{SL}_{2n} : {}^t g \cdot J_{2n} \cdot g = J_{2n}\}$. Recall that $T = D_{2n} \cap G$ is a maximal torus. You may assume anything you established for this group (or anything that you were asked to prove) last time. You may also assume that the Lie algebra equals $\{X \in M_{2n}(k) : {}^t X \cdot J_{2n} + J_{2n} \cdot X = 0\}$ as sub-Lie algebra of $\mathrm{Lie}\,\mathrm{GL}_{2n} = M_{2n}(k)$. (This may be proved like for the symplectic group in my online notes.)

(i) Determine the roots of G with respect to T.
(ii) Determine the coroot $\alpha^\vee$ for each root α. (It may be easiest to use the previous problem to find s_α and hence $\alpha^\vee$. Alternatively, just work out one of the groups G_α and coroots $\alpha^\vee$ by hand and then use the Weyl group to get the rest...)