THE CLASSIFICATION OF IRREDUCIBLE ADMISSIBLE MOD p REPRESENTATIONS OF A p-ADIC GL_n.

FLORIAN HERZIG

ABSTRACT. Let F be a finite extension of $\mathbb{Q}_p$. Using the mod p Satake transform, we define what it means for an irreducible admissible smooth representation of an F-split p-adic reductive group over $\overline{\mathbb{F}}_p$ to be supersingular. We then give the classification of irreducible admissible smooth $\mathrm{GL}_n(F)$-representations over $\overline{\mathbb{F}}_p$ in terms of supersingular representations. As a consequence we deduce that *supersingular* is the same as *supercuspidal*. These results generalise the work of Barthel–Livné for $n = 2$. For general split reductive groups we obtain similar results under stronger hypotheses.

1. INTRODUCTION

Let F be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k. The hypothetical mod p Langlands correspondence is expected to associate to an n-dimensional mod p Galois representation $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ an admissible smooth representation $\pi(\rho)$ of $\mathrm{GL}_n(F)$ over $\overline{\mathbb{F}}_p$ (or maybe an equivalence class of such representations). So far this is understood in the case $n = 2$ and $F = \mathbb{Q}_p$ (see [Bre03b], [Col10]), with some recent progress on the case $n = 2$ and $F/\mathbb{Q}_p$ unramified (see [BP]). While mod p Galois representations are relatively easy to understand, the theory of mod p smooth representations of p-adic reductive groups is only at its beginnings. It takes its origin with the fundamental work of Barthel–Livné [BL95], [BL94] that classifies irreducible representations of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_p$ that have a central character into four classes: (i) irreducible principal series, (ii) one-dimensional representations, (iii) twists of the Steinberg representation, and (iv) "supersingular" representations. While the first three classes are explicit, the classification of supersingular representations has only been completed so far for $F = \mathbb{Q}_p$ [Bre03a]. When $F \neq \mathbb{Q}_p$ it turned out to be much more complicated [BP], [Hu10].

In this paper we generalise the work of Barthel–Livné to GL_n, giving the classification of irreducible admissible representations of $\mathrm{GL}_n(F)$ over $\overline{\mathbb{F}}_p$ in terms of supersingular representations. (From now on "admissible" is short for "admissible smooth".) We first define what it means for an irreducible admissible representation to be supersingular (generalising the definition

for GL_2) by means of the mod p Satake isomorphism [Her11]. To state our main theorem we need to recall the definition of the generalised Steinberg representations. Let P denote any standard parabolic subgroup, that is a parabolic subgroup containing the Borel subgroup B of upper triangular matrices. Let $\overline{P}$ denote the opposite parabolic and let

$$\mathrm{Sp}_P = \frac{\mathrm{Ind}_{\overline{P}(F)}^{GL_n(F)} 1}{\sum_{Q \supsetneq P} \mathrm{Ind}_{\overline{Q}(F)}^{GL_n(F)} 1}.$$

It is known to be irreducible and admissible [GK].

Theorem 1.1. *The irreducible admissible $GL_n(F)$-representation are given by $\mathrm{Ind}_{\overline{P}(F)}^{GL_n(F)}(\sigma_1 \otimes \cdots \otimes \sigma_r)$, where*

(i) *P is a standard parabolic with Levi $\prod_{i=1}^r GL_{n_i}$;*
(ii) *σ_i is an irreducible admissible $GL_{n_i}(F)$-representation such that either*
   ○ *σ_i is supersingular and $n_i > 1$, or*
   ○ *$\sigma_i \cong \mathrm{Sp}_{Q_i} \otimes (\eta_i \circ \det)$ for some smooth character $\eta_i : F^\times \to \overline{\mathbb{F}}_p^\times$ and some standard parabolic $Q_i \subset GL_{n_i}$;*
(iii) *$\eta_i \neq \eta_{i+1}$ whenever both σ_i and σ_{i+1} fall into the second case.*

Moreover, P is uniquely determined and each σ_i is unique up to isomorphism.

Note that Ollivier [Oll06] proved the irreducibility of this representation when it is a principal series, i.e., when $n_1 = \cdots = n_r = 1$. In this case $\sigma_i = \eta_i$ for all i and consecutive characters are distinct. When $n = 2$ the result recovers the classification of Barthel–Livné: when $P = B$ one obtains the irreducible principal series; when $P = G$ one obtains twists of generalised Steinberg representations (either trivial or Steinberg in this case) and supersingular representations.

Corollary 1.2. *Suppose that π is an irreducible admissible $GL_n(F)$-representation. Suppose that $Q = LN'$ is a standard parabolic and that τ is an irreducible admissible $L(F)$-representation.*

(i) *$\mathrm{Ind}_{\overline{Q}(F)}^{GL_n(F)} \tau$ is of finite length, and all constituents occur with multiplicity one.*
(ii) *π is supersingular if and only if π is supercuspidal.*

Note that the constituents in part (i) can be described explicitly. By a *supercuspidal* representation in part (ii) we mean an irreducible admissible representation that does not occur among the constituents of any parabolic induction in part (i) of the corollary with $Q \neq GL_n$.

These results follow from the work of Barthel–Livné when $n = 2$. Part (i) was also known in the case of the trivial principal series $\mathrm{Ind}_{\overline{B}(F)}^{GL_n(F)} 1$ by [GK] (the constituents are the Sp_P) and for all principal series when $n = 3$ [Vig08].

It is not hard to determine the submodule structure of the representations in part (i) using the methods of this paper. See Section 10.

1.1. **Comparison with classical results.** It is interesting to compare the results with the results of Bernstein and Zelevinsky over the complex numbers [BZ77], [Zel80]. The most striking difference is the complete lack of intertwining operators in our context, by the uniqueness part of Theorem 1.1. Thus for example the principal series $\mathrm{Ind}_{\overline{B}}^{G}(\chi_1 \otimes \cdots \otimes \chi_n)$ and $\mathrm{Ind}_{\overline{B}}^{G}(\chi_1' \otimes \cdots \otimes \chi_n')$ do not share any common constituents unless $\chi_i = \chi_i'$ for all i. Note that there is no obvious way to produce intertwining operators over $\overline{\mathbb{F}}_p$ due to the lack of an $\overline{\mathbb{F}}_p$-valued Haar measure. One would however expect non-trivial extensions between the irreducible representations and the knowledge of these will play some role in the understanding of a mod p Langlands correspondence. (For $\mathrm{GL}_2(\mathbb{Q}_p)$ extensions between irreducibles were computed in [Col10], [Eme10b], [BP], [Paš10].)

Another difference, maybe related to the lack of intertwining operators, is that parabolic inductions tend to be irreducible. In some sense the only reducibilities arise for "obvious" reasons, namely when $\eta_i = \eta_{i+1}$ for some i in Theorem 1.1. And in those cases constituents occur with multiplicity one, unlike over the complex numbers (in fact Zelevinsky formulated an analogue of the Kazhdan–Lusztig conjecture to predict the multiplicities [Zel81]).

1.2. **Methods used.** We now want to explain in more detail what goes into the proofs. Most of our methods work for more general groups, so let G be a split connected reductive group over F. It is known that G extends to a connected reductive group over the ring of integers $\mathcal{O}$. Fix such an integral model $G_{/\mathcal{O}}$, and fix a maximal split torus $T_{/\mathcal{O}}$ and a Borel subgroup $B_{/\mathcal{O}}$ containing T. Then $K := G(\mathcal{O})$ is a hyperspecial maximal compact subgroup of $G(F)$ [Tit79, 3.8.1]. We usually write G for $G(F)$, etc. This should not lead to any confusion.

1.2.1. *Weights, Hecke eigenvalues, and supersingular representations.* Let π be any admissible smooth G-representation over $\overline{\mathbb{F}}_p$. It is easy to see that $\pi|_K$ contains an irreducible K-subrepresentation V. Since the kernel of $K = G(\mathcal{O}) \to G(k)$ is a pro-p group and the coefficient field has characteristic p, it follows that V factors through a representation of the finite group $G(k)$ over $\overline{\mathbb{F}}_p$. Such a representation is called a *K-weight*, or simply *weight*. The multiplicity space $\mathrm{Hom}_K(V, \pi)$ is finite-dimensional (as π is admissible). Let $\mathrm{c\text{-}Ind}_K^G V$ denote the representation compactly induced from V. By Frobenius reciprocity we have $\mathrm{Hom}_K(V, \pi) \cong \mathrm{Hom}_G(\mathrm{c\text{-}Ind}_K^G V, \pi)$, so that the *Hecke algebra* $\mathcal{H}_G(V) := \mathrm{End}_G(\mathrm{c\text{-}Ind}_K^G V)$ naturally acts on it. Let $X_*(T)_-$ denote the subset of antidominant coweights in $\mathrm{Hom}(\mathbb{G}_m, T)$. In [Her11] we established an analogue of the Satake isomorphism, showing that $\mathcal{H}_G(V) \cong \overline{\mathbb{F}}_p[X_*(T)_-]$. (The map depends on the choice of positive roots and on the choice of a uniformiser.) In particular, $\mathcal{H}_G(V)$ is commutative. Thus the $\mathcal{H}_G(V)$-module $\mathrm{Hom}_K(V, \pi)$ is a direct sum of generalised

eigenspaces, and there exists an eigenvector in each of them. Note that an algebra homomorphism $\chi : \mathcal{H}_G(V) \to \overline{\mathbb{F}}_p$ occurs as set of Hecke eigenvalues of $\mathcal{H}_G(V)$ on $\mathrm{Hom}_K(V, \pi)$ if and only if there is a non-zero G-linear map

$$(1.3) \qquad \qquad \text{c-Ind}_K^G V \otimes_{\mathcal{H}_G(V),\chi} \overline{\mathbb{F}}_p \to \pi.$$

Suppose that π is moreover irreducible. For each set of Hecke eigenvalues χ occurring on $\mathrm{Hom}_K(V, \pi)$ we get a monoid homomorphism $\chi' : X_*(T)_- \to \overline{\mathbb{F}}_p$ via the Satake isomorphism. Note that in our setup above we fixed a reductive integral structure $G_{/\mathcal{O}}$, giving rise to K. We say that π is *supersingular* if, for any reductive integral structure $G_{/\mathcal{O}}$, we have that χ' is zero on all non-invertible elements of $X_*(T)_-$ for all χ and all V. (When $G = \mathrm{GL}_n$, all hyperspecial maximal subgroups are conjugate, so that we can work with a fixed $G_{/\mathcal{O}}$ and K. Moreover, as a corollary to Theorem 1.1 we will see that χ' is independent of χ and V above, so that the condition only needs to be checked once.)

We now discuss two of the key tools that go into the proofs of the main results.

1.2.2. *Comparison of compact and parabolic inductions.* Suppose we are given a weight V and an algebra homomorphism $\chi : \mathcal{H}_G(V) \to \overline{\mathbb{F}}_p$. the support of $\chi' : X_*(T)_- \to \overline{\mathbb{F}}_p$ determines a facet of the antidominant Weyl chamber which in turn determines a standard parabolic P_χ. Let $P = MN$ be a standard parabolic containing P_χ. If V is sufficiently "regular" (depending on P), a condition that is satisfied by most weights, we show that there exists a natural isomorphism

$$(1.4) \qquad \qquad \text{c-Ind}_K^G V \otimes_{\mathcal{H}_G(V),\chi} \overline{\mathbb{F}}_p \xrightarrow{\sim} \mathrm{Ind}_P^G(\pi_M),$$

for a certain smooth M-representation π_M defined in terms of V and χ. Note that π_M is not admissible in general. It is a representation of the same form as the left-hand side. (See Theorem 3.1 for the precise statement.) When $P = G$, the isomorphism (1.4) is the identity map.

When $G = \mathrm{GL}_2$ and $P = B$ the isomorphism (1.4) was established by Barthel–Livné using a calculation with the Bruhat–Tits tree. In that case π_M is a character. In our proof we compare compact and parabolic inductions via a parahoric induction, building on the ideas of Schneider–Stuhler [SS91] and Vignéras [Vig04].

1.2.3. *Changing the weight.* In certain situations we construct isomorphisms of the form

$$\text{c-Ind}_K^G V_1 \otimes_{\mathcal{H}_G(V_1),\chi_1} \overline{\mathbb{F}}_p \xrightarrow{\sim} \text{c-Ind}_K^G V_2 \otimes_{\mathcal{H}_G(V_2),\chi_2} \overline{\mathbb{F}}_p,$$

where V_i are weights and $\chi_i : \mathcal{H}_G(V_i) \to \overline{\mathbb{F}}_p$ are algebra homomorphisms $(i = 1, 2)$. Suppose we have such an isomorphism. By (1.3), if V_1 occurs in π with eigenvalues χ_1, then V_2 occurs in π with eigenvalues χ_2.

While the method is general, for the moment we can obtain an explicit criterion mainly when $G = \mathrm{GL}_n$. The idea is to study the Hecke bimodule

$\mathrm{Hom}_G(\text{c-Ind}_K^G V_1, \text{c-Ind}_K^G V_2)$. Our proof requires an explicit version of the Satake transform, which we deduce from the corresponding result over the complex numbers (the formula of Lusztig and Kato).

1.2.4. *Irreducibility proof.* Let $G = \mathrm{GL}_n$. Suppose $\pi := \mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ as in Theorem 1.1. To prove irreducibility it is enough to show that any weight $V \subset \pi$ generates π as G-representation. Without loss of generality $V \hookrightarrow \pi$ is a Hecke eigenvector. If V is sufficiently regular we find a map $\pi_M \twoheadrightarrow \sigma_1 \otimes \cdots \otimes \sigma_r$ and use the isomorphism (1.4) to obtain a surjection c-$\mathrm{Ind}_K^G V \twoheadrightarrow \pi$. This shows that V generates π. Otherwise we use the criterion of §1.2.3 to change to a sufficiently regular weight and we are done as before. (To verify the criterion we use conditions (ii) and (iii) in Theorem 1.1.)

1.2.5. *Classification proof.* Let $G = \mathrm{GL}_n$. Suppose π is an irreducible admissible G-representation. Let V be a weight of π and let χ be a set of Hecke eigenvalues on $\mathrm{Hom}_K(V, \pi)$. Then π is a quotient of c-$\mathrm{Ind}_K^G V \otimes_{\mathcal{H}_G(V), \chi} \overline{\mathbb{F}}_p$ by (1.3). Suppose we can satisfy the conditions in §1.2.2 for some proper parabolic $P \neq G$. Then $\mathrm{Ind}_P^G(\pi_M) \twoheadrightarrow \pi$ by (1.4). Using Emerton's theory of ordinary parts [Eme10a] one gets a map $\mathrm{Ind}_P^G \sigma \twoheadrightarrow \pi$ for some irreducible admissible M-representation σ and one can proceed by induction. Otherwise one tries to change the weight using the criterion of §1.2.3. If both tactics fail, it turns out that π is either supersingular or π looks like a one-dimensional representation in the sense that it contains the same weight with the same Hecke eigenvalues as such a representation. A calculation with the Iwahori Hecke algebra allows us to conclude in the latter case.

Both the irreducibility and the classification proofs go through for general G provided one puts conditions on the weights of the representation so that all weights that occur in the arguments are sufficiently regular. See Theorems 8.8 and 9.17 for precise statements.

1.3. **Other results.** We complete the irreducibility proof of the generalised Steinberg representations Sp_P of split reductive groups. This was proved by Große-Klönne in case the root system is of type A, B, C, or D. In general we use his work to show that Sp_P contains a unique weight (with multiplicity one) and we determine the Hecke eigenvalues in that weight. We then use §1.2.2 to show that the unique weight generates the representation.

1.4. **Arrangement of the paper.** In §2–3 we discuss Hecke actions, variants of the mod p Satake transform, and the comparison result between compact and parabolic inductions. In §4 we give a parameterisation of Hecke eigenvalues which is analogous to the classical parameterisation by unramified characters of the torus. We also define what it means for an irreducible admissible representation to be supersingular. In §5 we adapt the result of Lusztig and Kato to give an explicit version of the mod p Satake isomorphism. In §6 we study maps between compact inductions and deduce

our "change of weight" criterion. In §7 we discuss generalised Steinberg representations. In §8–9 we deduce the main results. Finally in §10 we prove results on the submodule structure of parabolically induced representations.

1.5. **Notation.** Let F be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformiser ϖ, and residue field k of order q. Let G be a split connected reductive group over $\mathcal{O}$ and fix a maximal split torus T. Let $\Phi \subset X^*(T)$ denote the set of roots. For each root $\alpha \in \Phi$ denote by $\alpha^\vee$ the associated coroot. Choose a system of positive roots Φ^+ and let $B = TU$ denote the associated Borel subgroup. Let W be the Weyl group and let $K = G(\mathcal{O})$, a hyperspecial maximal compact subgroup of $G(F)$. For example, we could take $G = \mathrm{GL}_n$ with the diagonal maximal torus T and the Borel subgroup B consisting of upper triangular matrices.

The letters P, Q usually denote standard parabolic subgroups, i.e., parabolic subgroups containing the Borel B. A Levi decomposition $P = MN$ of a standard parabolic is implicitly assumed to be standard, i.e., M is the unique Levi subgroup of P that contains T. If $P = MN$ is a parabolic we denote by $\overline{P} = M\overline{N}$ the opposite parabolic (determined by our choice of Levi M). We will sometimes use that the multiplication map $\overline{N} \times M \times N \to G$ is injective on F-points (in fact it is even an open immersion on the level of $\mathcal{O}$-schemes [Jan03, §II.1.11]).

We denote by Z the *connected* centre of G. Similarly Z_M denotes the connected centre of a Levi subgroup M. We denote by $X_*(T)_-$ the set of antidominant coweights of T. The set of simple roots in Φ^+ is denoted by Δ. If $P = MN$ is a standard parabolic, we similarly define Δ_M (with respect to $\Phi_M^+ = \Phi_M \cap \Phi^+$).

We denote by $\mathrm{red} : K = G(\mathcal{O}) \to G(k)$ the reduction map and by $K(1)$ its kernel, which is a pro-p group. For a standard parabolic subgroup P, we let $\mathcal{P} := \mathrm{red}^{-1}(P(k))$ be the corresponding parahoric subgroup. We will also write I for the standard Iwahori subgroup $\mathcal{B}$ and define $I(1) := \mathrm{red}^{-1}(U(k))$ (the pro-p Sylow subgroup of I).

If H is a group and σ an H-representation, we denote by $\mathrm{soc}_H \sigma$ the H-socle, i.e., the largest semisimple subrepresentation.

We usually write G, P, M, ... when we really mean $G(F)$, $P(F)$, $M(F)$, ... This should cause no confusion.

We use the following abbreviated notations for the conjugation action: ${}^t K = tKt^{-1}$ and $K^t = t^{-1}Kt$.

All representations in this paper, unless otherwise stated, live on $\bar{k}$-vector spaces.

Contents

2. Hecke actions and the Satake transform

2.1. **Background.**

2.1.1. *Weights.*

Definition 2.1. A *weight* is an irreducible representation V of the finite group $G(k)$ over $\bar{k}$.

The set of q-restricted weights is defined to be:
$$X_q(T) = \{\lambda \in X^*(T) : 0 \leq \langle \lambda, \alpha^\vee \rangle < q \quad \forall \alpha \in \Delta\}.$$
We also define:
$$X^0(T) = \{\lambda \in X^*(T) : \langle \lambda, \alpha^\vee \rangle = 0 \quad \forall \alpha \in \Phi\}.$$
For $\nu \in X^*(T)$ dominant, let $F(\nu)$ denote the irreducible $G_{/\bar{k}}$-module of highest weight ν. Via the inclusion $G(k) \to G(\bar{k})$, we can consider $F(\nu)$ as $G(k)$-representation.

Proposition 2.2. *Suppose that the derived subgroup of G is simply connected. Then all weights are of the form $F(\nu)$, where $\nu \in X_q(T)$. Moreover for ν, $\nu' \in X_q(T)$, we have $F(\nu) \cong F(\nu')$ as $G(k)$-representations if and only if $\nu - \nu' \in (q-1)X^0(T)$.*

This goes back to Steinberg if G is semisimple; in general, see Prop. 1.3 in the appendix to [Her09]. If the derived subgroup of G fails to be simply connected, weights can be described using a z-extension of $G_{/k}$ (as in the proof of [Her11, Lemma 2.5]).

We will sometimes denote by 1 the trivial weight. If $P = MN$ is a standard parabolic, we denote by $X_M^0(T)$ and $F^M(\nu)$ the analogues of $X^0(T)$ and $F(\nu)$ for the Levi M.

For the following, very useful Lemma see [Her11, Lemma 2.5]. It was first proved by Smith and Cabanes.

Lemma 2.3. *Suppose that V is a weight and that $P = MN$ is a standard parabolic. Then $V^{N(k)}$ and $V_{\overline{N}(k)}$ are weights for M and the natural, $M(k)$-linear map $V^{N(k)} \to V_{\overline{N}(k)}$ is an isomorphism. In particular, $V^{U(k)} \cong V_{\overline{U}(k)}$ is one-dimensional.*

Suppose the derived subgroup of G is simply connected. If $V \cong F(\nu)$ for some $\nu \in X_q(T)$, then $V^{N(k)} \cong F^M(\nu)$. Moreover, as a subspace of $F(\nu)$, $V^{N(k)}$ is the sum of all weight spaces $F(\nu)_{\nu'}$ with $\nu - \nu' \in \mathbb{Z}\Phi_M$.

Definition 2.4. A weight V is said to be *M-regular* if $\mathrm{Stab}_W(V^{U(k)}) \subset W_M$.

Here $\mathrm{Stab}_W(V^{U(k)})$ denotes the set of $w \in W$ that preserve the one-dimensional, $T(k)$-stable subspace $V^{U(k)} \subset V$. Let us make this condition more explicit in case $V = F(\nu)$ for some $\nu \in X_q(T)$. Note that $\mathrm{Stab}_W(V^{U(k)}) = \mathrm{Stab}_W(\nu)$, which is generated by the simple reflections s_α for $\alpha \in \Delta$ such that $\langle \nu, \alpha^\vee \rangle = 0$. Thus $V = F(\nu)$ is M-regular if and only if $0 < \langle \nu, \alpha^\vee \rangle \le q - 1$ for all $\alpha \in \Delta - \Delta_M$.

Lemma 2.5. *The map $V \mapsto V^{N(k)}$ from M-regular weights for G to weights for M is a bijection.*

Proof. Suppose first that the derived subgroup of G is simply connected. Then the same is true for M. Let $\overline{V}$ be a weight for M. Then $\overline{V} \cong F^M(\nu)$ for some $\nu \in X^*(T)$ such that $0 \le \langle \nu, \alpha^\vee \rangle \le q - 1$ for all $\alpha \in \Delta_M$. We need to find $\nu' \in X^*(T)$ such that (a) $0 \le \langle \nu', \alpha^\vee \rangle \le q - 1$ for all $\alpha \in \Delta_M$, (b) $0 < \langle \nu', \alpha^\vee \rangle \le q - 1$ for all $\alpha \in \Delta - \Delta_M$, and (c) $\nu - \nu' \in (q-1)X_M^0(T)$. The first two conditions express that ν' is q-restricted and that the weight $V \cong F(\nu')$ is M-regular. Condition (c), which in fact implies (a), expresses that $V^{N(k)} \cong \overline{V}$. Clearly there is such a ν' and it is uniquely determined up to $(q-1)X^0(T)$. This completes the proof.

In the general case, pick a z-extension $1 \to R \to \widetilde{G} \to G \to 1$ in the special fibre, just as in [Her11, Lemma 2.5]. We know there is a unique weight V for $\widetilde{G}$ such that $V^{\widetilde{N}(k)} \cong \overline{V}$ as $\widetilde{M}(k)$-representation. Since $R(k)$ acts trivially on $\overline{V}$ and since it acts on V via the central character, we see that V descends to a $G(k)$-representation. The uniqueness of V is even easier. $\qquad\square$

2.1.2. *Smooth representations.* We recall that a smooth G-representation π is said to be *admissible* if π^H is finite-dimensional for all open subgroups H of G. It is sufficient to verify this condition for one open pro-p subgroup H. An irreducible admissible G-representation π has a central character, which we denote by $\omega_\pi : Z \to \bar{k}^\times$.

Any irreducible smooth K-representation factors through $G(k)$ (as $K(1)$ is pro-p); it is thus a weight. If π is admissible, $\mathrm{soc}_K \pi \subset \pi^{K(1)}$ is finite-dimensional and non-zero, so π contains a weight V. (It is clear that even any smooth G-representation contains a weight.) We will also say that V is a weight of π.

For a closed subgroup $H \subset G$ and a smooth H-representation σ we denote by $\mathrm{Ind}_H^G \sigma$ (resp., c-$\mathrm{Ind}_H^G \sigma$) the representation that is induced (resp., compactly induced) from σ. If H is open, then c-$\mathrm{Ind}_H^G \sigma \cong \bar{k}[G] \otimes_{\bar{k}[H]} \sigma$, so c-$\mathrm{Ind}_H^G$ is a left adjoint to the forgetful functor. In this case, we will denote by $[g, x] \in$ c-$\mathrm{Ind}_H^G \sigma$ the element that is supported on Hg^{-1} and sends g^{-1} to $x \in \sigma$. If P is a parabolic subgroup, then Ind_P^G is *exact*. (This is because the map $G \to G/P$ has continuous sections; see [Eme10a, Prop. 4.1.5].)

Suppose that H is a compact open subgroup and that V is a finite-dimensional smooth H-representation. We define the *Hecke algebra* of V to be $\mathcal{H}_H(V) := \mathrm{End}_G(\text{c-}\mathrm{Ind}_H^G V)$. Using the above adjunction, we can and usually will think of it as the $\bar{k}$-algebra of compactly supported functions $\varphi : G \to \mathrm{End}_{\bar{k}} V$ satisfying $\varphi(h_1 g h_2) = h_1 \circ \varphi(g) \circ h_2$ for all h_1, $h_2 \in H$, $g \in G$, where the multiplication is given by convolution. Explicitly, for $\varphi \in \mathcal{H}_H(V)$ and $f \in$ c-$\mathrm{Ind}_H^G V$, we have

$$\varphi(f)(g) = \sum_{\gamma \in G/H} \varphi(g\gamma) f(\gamma^{-1}).$$

Note that if π is a smooth G-representation, $\mathcal{H}_H(V)$ naturally acts on the multiplicity space $\mathrm{Hom}_H(V, \pi) \cong \mathrm{Hom}_G(\text{c-}\mathrm{Ind}_H^G V, \pi)$. (This is a right action.) Explicitly, if $\varphi \in \mathcal{H}_H(V)$ and $f : V \to \pi$ is H-linear, then

$$(f * \varphi)(v) = \sum_{\gamma \in H \backslash G} \gamma^{-1} f(\varphi(\gamma) v).$$

Suppose now that V is a weight. We will usually write $\mathcal{H}_G(V)$ instead of $\mathcal{H}_K(V)$. We will see just below that $\mathcal{H}_G(V)$ is commutative. If $V = 1$ then $\mathcal{H}_G(V)$ is the usual unramified Hecke algebra $\bar{k}[K \backslash G / K]$.

2.2. **The mod p Satake transform.** We begin by recalling some results of [Her11]. Let T^- denote the submonoid of T,

$$T^- = \{t \in T : \mathrm{ord}_F(\alpha(t)) \leq 0 \quad \forall \alpha \in \Delta\},$$

and let $\mathcal{H}_T^-(V^{U(k)})$ denote the subalgebra of $\mathcal{H}_T(V^{U(k)})$ consisting of those functions $\varphi : T \to \bar{k}$ that are supported on T^-.

Theorem 2.6. *Suppose that V is a weight. Then*

$$\mathcal{S}_G : \mathcal{H}_G(V) \to \mathcal{H}_T(V^{U(k)})$$

$$\varphi \mapsto \left(t \mapsto \sum_{u \in U/U(\mathcal{O})} \varphi(tu)\Big|_{V^{U(k)}} \right)$$

is an injective $\bar{k}$-algebra homomorphism with image $\mathcal{H}_T^-(V^{U(k)})$.

In particular, $\mathcal{H}_G(V) \cong \bar{k}[X_*(T)_-]$ is commutative and noetherian (Gordan's lemma shows that $X_*(T)_-$ is finitely generated). We recall that $G = \coprod K\lambda(\varpi)K$, where λ ranges over $X_*(T)_-$ (refined Cartan decomposition). Moreover, $\mathcal{H}_G(V)$ has a basis given by T_λ for $\lambda \in X_*(T)_-$, where T_λ

has support $K\lambda(\varpi)K$ and sends $\lambda(\varpi)$ to the endomorphism $V \twoheadrightarrow V_{N_\lambda(k)} \xleftarrow{\sim} V^{N_{-\lambda}(k)} \hookrightarrow V$ (see §2.4 for the definition of $P_\lambda = M_\lambda N_\lambda$). We also denote by $\tau_\lambda \in \mathcal{H}_T(V^{U(k)})$ the element supported on $\lambda(\varpi)T(\mathcal{O})$ that sends $\lambda(\varpi)$ to 1. We have:

$$(2.7) \qquad \mathcal{S}_G(T_\lambda) = \sum_{\substack{\mu \in X_*(T)_- \\ \mu \geq_\mathbb{R} \lambda}} a_\lambda(\mu)\tau_\mu, \quad \text{with } a_\lambda(\mu) \in \bar{k} \text{ and } a_\lambda(\lambda) = 1.$$

Here, $\mu \geq_\mathbb{R} \lambda$ means that $\mu - \lambda$ is a non-negative real linear combination of the simple coroots.

For $z \in Z$ we also define $T_z \in \mathcal{H}_G(V)$ such that $\mathrm{supp}(T_z) = Kz$ and $T_z(z) = \mathrm{id}_V$. Thus $T_{\lambda(\varpi)} = T_\lambda$ whenever $\lambda(\varpi) \in Z$. The following formulae for $z, z_1, z_2 \in Z$, $z_0 \in Z(\mathcal{O})$ will be useful later.

$$(2.8) \qquad\qquad\qquad T_{z_1 z_2} = T_{z_1}T_{z_2}, \; T_1 = 1,$$

$$(2.9) \qquad\qquad\qquad T_{z_0 z} = \omega_V(z_0)^{-1}T_z,$$

$$(2.10) \qquad\qquad\qquad T_z = z^{-1} \quad \text{on c-Ind}_K^G V,$$

where $\omega_V : Z(k) \to \bar{k}^\times$ denotes the central character of V.

2.3. **Variants of the Satake transform.** Let $P = MN$ be a standard parabolic subgroup. There is a "partial" Satake homomorphism $'\mathcal{S}_G^M : \mathcal{H}_G(V) \to \mathcal{H}_M(V_{\overline{N}(k)})$. It is defined by

$$'\mathcal{S}_G^M : \mathcal{H}_G(V) \to \mathcal{H}_M(V_{\overline{N}(k)})$$

$$\varphi \mapsto \left(m \mapsto p_{\overline{N}} \sum_{\overline{N}(\mathcal{O}) \backslash \overline{N}} \varphi(\overline{n}m) \right),$$

where $p_{\overline{N}}$ denotes the projection $V \twoheadrightarrow V_{\overline{N}(k)}$. It is easy to check that this map is well defined (just as for $\mathcal{S}_G$ [Her11]). We also have $\mathcal{S}_G^M : \mathcal{H}_G(V) \to \mathcal{H}_M(V^{N(k)})$, defined in the same way as $\mathcal{S}_G$.

We have a simple compatibility between $'\mathcal{S}_G^M$ and $\mathcal{S}_G^M$. There is an algebra isomorphism $\mathcal{H}_G(V) \xrightarrow{\sim} \mathcal{H}_G(V^*)$ which sends φ to φ' with $\varphi'(g) = \varphi(g^{-1})^*$. (Recall that $\mathcal{H}_G(V)$ is commutative.) Similarly we have an isomorphism $\mathcal{H}_M(V^{\overline{N}(k)}) \xrightarrow{\sim} \mathcal{H}_M((V^*)_{\overline{N}(k)})$. In the following lemma we denote by $\mathcal{S}_G^{M,\mathrm{op}}$ the analogue of $\mathcal{S}_G^M$ that is computed with respect to $\overline{P} = M\overline{N}$ (a standard parabolic for Φ^-).

Lemma 2.11. *We have the following commutative diagram.*

$$
\begin{array}{ccc}
\mathcal{H}_G(V) & \xrightarrow{\;\mathcal{S}_G^{M,\mathrm{op}}\;} & \mathcal{H}_M(V^{\overline{N}(k)}) \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\
\mathcal{H}_G(V^*) & \xrightarrow{\;'\mathcal{S}_G^M\;} & \mathcal{H}_M((V^*)_{\overline{N}(k)})
\end{array}
$$

In particular $'S_G := 'S_G^T$ is injective. Its image consists of those elements of $\mathcal{H}_T((V^*)_{\overline{U}(k)})$ that are supported on T^- and is isomorphic to $\bar{k}[X_*(T)_-]$.

Proof. This is straightforward, noting that $V^{\overline{N}(k)} \to V$ is dual to $V^* \to (V^*)_{\overline{N}(k)}$. $\qquad\square$

We will see in Cor. 2.19 that S_G^M and $'S_G^M$ are actually the same, under the natural identification of $V^{N(k)}$ and $V_{\overline{N}(k)}$ (at least when the derived subgroup of G is simply connected). This does not seem to be obvious from the definition.

Proposition 2.12. *The map $'S_G^M$ is an injective algebra homomorphism. It is a localisation map of integral domains. Also $'S_G = 'S_M \circ 'S_G^M$.*

The same of course also holds for S_G^M.

Proof. It is easy to see that $'S_G^M$ is an algebra homomorphism (see also [Her11]). A direct calculation shows that $'S_G = 'S_M \circ 'S_G^M$. Since $'S_G$ is injective, so is $'S_G^M$. As we have fixed a uniformiser ϖ, we can identify $\mathcal{H}_T(V_{\overline{U}(k)})$ with $\bar{k}[X_*(T)]$. By Lemma 2.11 the image of $\mathcal{H}_G(V)$ in $\bar{k}[X_*(T)]$ is $\bar{k}[X_*(T)_-]$ and the image of $\mathcal{H}_M(V_{\overline{N}(k)})$ in $\bar{k}[X_*(T)]$ is $\bar{k}[X_*^M(T)_-]$, where $X_*^M(T)_-$ is the monoid of antidominant coweights for T in M. Finally note that $\bar{k}[X_*(T)_-] \hookrightarrow \bar{k}[X_*^M(T)_-]$ is a localisation map of integral domains: it suffices to invert any $\lambda \in X_*(T)_-$ such that for $\alpha \in \Delta$, $\langle \lambda, \alpha \rangle = 0$ if and only if $\alpha \in \Delta_M$. $\qquad\square$

Suppose now that V is a weight for G and σ a smooth M-representation. By the Iwasawa decomposition we have $(\mathrm{Ind}_P^G \sigma)|_K \cong \mathrm{Ind}_{P(\mathcal{O})}^K \sigma$, so the natural map

$$(2.13) \qquad \mathrm{Hom}_K(V, \mathrm{Ind}_P^G \sigma) \to \mathrm{Hom}_{M(\mathcal{O})}(V_{\overline{N}(k)}, \sigma)$$

$$f \mapsto \overline{f} = \big(\overline{v} \mapsto f(v)(1)\big)$$

is a bijection (here v is any lift of $\overline{v}$). Note that $\mathcal{H}_G(V)$ naturally acts on the left-hand side, and on the right-hand side via $'S_G^M$.

Lemma 2.14. *The natural map in (2.13) is $\mathcal{H}_G(V)$-equivariant.*

Proof. For $\varphi \in \mathcal{H}_G(V)$ let $\varphi_M = 'S_G^M(\varphi)$. We compute

$$(\overline{f} * \varphi_M)(\overline{v}) = \sum_{M(\mathcal{O})\backslash M} m^{-1} f\bigg(\sum_{\overline{N}(\mathcal{O})\backslash \overline{N}} \varphi(\overline{n}m)v \bigg)(1)$$

$$= \sum_{M(\mathcal{O})\backslash M} \sum_{\overline{N}(\mathcal{O})\backslash \overline{N}} f(\varphi(\overline{n}m)v)(m^{-1}\overline{n}^{-1})$$

and

$$(\overline{f * \varphi})(\overline{v}) = (f * \varphi)(v)(1) = \sum_{K\backslash G} f(\varphi(g)v)(g^{-1}).$$

By the Iwasawa decomposition $G = K\overline{P}$ these two expressions are equal. $\quad\square$

2.4. **Various lemmas.** Suppose that $\lambda \in X_*(T)$. Let $P_\lambda = M_\lambda N_\lambda$ denote the parabolic subgroup of G defined by λ. For the following proposition, see [Her11, Prop. 3.8].

Proposition 2.15. *Let* $t = \lambda(\varpi)$. *Then* $\mathrm{red}(K \cap K^t) = P_\lambda(k)$.

Define $Z_M^- := Z_M \cap T^-$ and $Z_M^+ := Z_M \cap (T^-)^{-1}$. We also need
$$Z_M^{--} := \{z \in Z_M : \mathrm{ord}_F(\alpha(z)) < 0 \quad \forall \alpha \in \Delta - \Delta_M\}.$$

Lemma 2.16. *Suppose* $P = MN$ *is a standard parabolic subgroup. We let* $\overline{\mathcal{P}}^- = \overline{N} \cap \overline{\mathcal{P}}$, $\overline{\mathcal{P}}^0 = M \cap \overline{\mathcal{P}}$, $\overline{\mathcal{P}}^+ = N \cap \overline{\mathcal{P}}$. *Then* $\overline{\mathcal{P}} = \overline{\mathcal{P}}^- \overline{\mathcal{P}}^0 \overline{\mathcal{P}}^+$ *(in any order).* *Moreover, conjugation by* T^- *contracts* $\overline{\mathcal{P}}^-$ *and expands* $\overline{\mathcal{P}}^+$.
If $h_0 \in Z_M^-$ *then the sets* $(\overline{\mathcal{P}}^+)^{h_0^n}$ *for* $n \geq 0$ *form a neighbourhood basis of the identity in* N.

By interchanging positive and negative roots we obtain similarly that $\mathcal{P} = \mathcal{P}^- \mathcal{P}^0 \mathcal{P}^+$ in any order with T^- contracting $\mathcal{P}^-$ and expanding $\mathcal{P}^+$.

Proof. Note that $\overline{\mathcal{P}}^- = \overline{N}(\mathcal{O})$, $\overline{\mathcal{P}}^0 = M(\mathcal{O})$, and $\overline{\mathcal{P}}^+ = \ker(N(\mathcal{O}) \to N(k))$. For any $\mathcal{O}$-group scheme H let us write $^1H(\mathcal{O})$ for $\ker(H(\mathcal{O}) \to H(k))$. Assume for now that $K(1) = {}^1\overline{U}(\mathcal{O})\,{}^1T(\mathcal{O})\,{}^1U(\mathcal{O})$. It is not hard to prove this by an elementary argument when $G = \mathrm{GL}_n$; we will justify it in general below.

From our assumption, $K(1) \subset \overline{P}(\mathcal{O})^1 P(\mathcal{O})$. From $\overline{P}(\mathcal{O}) = \overline{N}(\mathcal{O})M(\mathcal{O})$ and $P(\mathcal{O}) = M(\mathcal{O})N(\mathcal{O})$ it follows that $K(1) \subset \overline{\mathcal{P}}^- \overline{\mathcal{P}}^0 \overline{\mathcal{P}}^+$. Thus $\overline{\mathcal{P}} = \overline{P}(\mathcal{O})K(1) \subset \overline{\mathcal{P}}^- \overline{\mathcal{P}}^0 \overline{\mathcal{P}}^+$. The reverse inclusion is obvious. Thus $\overline{\mathcal{P}} = \overline{\mathcal{P}}^- \overline{\mathcal{P}}^0 \overline{\mathcal{P}}^+$. Noting that $\overline{\mathcal{P}}^0$ normalises both $\overline{\mathcal{P}}^-$ and $\overline{\mathcal{P}}^+$, and by using the inverse, we see that the order does not matter. (Moreover, it is a direct product decomposition because multiplication $\overline{N} \times M \times N \to G$ is injective.)

Let us show that T^- contracts $\overline{\mathcal{P}}^-$ and expands $\overline{\mathcal{P}}^+$. Choose root homomorphisms $x_\alpha : \mathbb{G}_a \xrightarrow{\sim} U_\alpha$ [Jan03, II.1.2]. Fix an ordering of the set of roots Φ. Note that $\overline{N} = \prod_{\Psi^-} U_\alpha$ and $N = \prod_{\Psi^+} U_\alpha$ (as $\mathcal{O}$-schemes), where $\Psi^\pm = \Phi^\pm - \Phi_M^\pm$. Thus $\overline{\mathcal{P}}^- = \prod_{\Psi^-} x_\alpha(\mathcal{O})$ and $\overline{\mathcal{P}}^+ = \prod_{\Psi^+} x_\alpha(\varpi\mathcal{O})$. The claim follows since $t x_\alpha(u) t^{-1} = x_\alpha(\alpha(t)u)$ for $t \in T(F)$ and $u \in F$.

Now fix an element $h_0 \in Z_M^{--}$. The multiplication map $\prod x_\alpha : \prod_{\Psi^+} \mathbb{G}_a \to N$ induces an isomorphism of $\mathcal{O}$-schemes, thus also a homeomorphism $\prod_{\Psi^+} F \to N(F)$. Write again $\overline{\mathcal{P}}^+ = \prod_{\Psi^+} x_\alpha(\varpi\mathcal{O})$, so that $(\overline{\mathcal{P}}^+)^t = \prod_{\Psi^+} x_\alpha(\alpha(t)^{-1}\varpi\mathcal{O})$ for any $t \in T(F)$. By the definitions, $\mathrm{ord}_F(\alpha(h_0^n)) \leq -n$ for all $\alpha \in \Psi^+$ and all $n \geq 0$. Thus $(\overline{\mathcal{P}}^+)^{h_0^n}$ for $n \geq 0$ form a neighbourhood basis of the identity in $N(F)$.

Finally we justify that $K(1) = {}^1\overline{U}(\mathcal{O})\,{}^1T(\mathcal{O})\,{}^1U(\mathcal{O})$ using Bruhat–Tits theory. Fraktur letters denote the $\mathcal{O}$-group schemes defined in [BT84]. Since G is a reductive group over $\mathcal{O}$ with special fibre $G_{/F}$, there is a (hyper-)special

point x in the reduced building of $G_{/F}$ such that $G \cong \mathfrak{G}_x^0$ [BT84, 5.1.40]. We identify these two group schemes. Let $f := f'_x$ be the corresponding "quasi-concave" function on the set of roots [BT84, 4.6.24], so that $G = \mathfrak{G}_f^0$. Now this $\mathcal{O}$-group scheme has two split maximal tori, T and $\mathfrak{T}$. By [G$^+$70, Exp. XXVI, Prop. 6.16] there is an element $\gamma \in G(\mathcal{O})$ that conjugates $\mathfrak{T}$ to T. In the generic fibre the two unipotent subgroups, $U_{/F}$ and ${}^\gamma(\mathfrak{U}_{/F}^+)$ might correspond to two different choices of positive roots. By multiplying γ by a suitable element of $N(T)(\mathcal{O})$ we can assure that U and ${}^\gamma\mathfrak{U}^+$ agree on the generic fibre. But then they are the same (see for example [BT84, 1.2.6]). By [BT84, 4.6.8] we have ${}^1\mathfrak{G}_f^0(\mathcal{O}) = {}^1\mathfrak{U}^-(\mathcal{O}){}^1\mathfrak{T}(\mathcal{O}){}^1\mathfrak{U}^+(\mathcal{O})$. Conjugating by γ we obtain $K(1) = {}^1G(\mathcal{O}) = {}^1\overline{U}(\mathcal{O}){}^1T(\mathcal{O}){}^1U(\mathcal{O})$. $\qquad\square$

Lemma 2.17. *Suppose that $P = MN$ and $Q = LN'$ are standard parabolics. Suppose that V is both M and L-regular. Then $p_{\overline{N'}}(\kappa V^{N(k)}) \neq 0$ for $\kappa \in K$ implies $\kappa \in \overline{Q}P$.*

If $\mathrm{Stab}_W(V^{U(k)})$ equals W_M (resp., W_L), we may drop the assumption that V be L-regular (resp., M-regular).

Proof. It is easy to reduce to the case that the derived subgroup of G is simply connected, just as in [Her11, Lemma 2.5]. (Note that the statements we want to prove just concern the finite group $G(k)$.) In this case $V = F(\nu)$ for some $\nu \in X_q(T)$, moreover $V^{N(k)} = F(\nu)^N$.

We claim that for all weights μ of $F(\nu)^N$ and all $\alpha \in \Phi^+ - \Phi_M^+$ we have $\langle \mu, \alpha^\vee \rangle > 0$. We know that $F(\nu)^N$ is the irreducible M-representation of highest weight ν, so its weights lie in the convex hull of $w\nu$ ($w \in W_M$). Note that any W_M preserves $\Phi^+ - \Phi_M^+$. (Reduce to the case of a simple reflection s_β with $\beta \in \Phi_M$. It preserves Φ_M and $\Phi^+ - \{\beta\}$.) Therefore $\langle w\nu, \alpha^\vee \rangle = \langle \nu, w^{-1}\alpha^\vee \rangle \geq 0$ for all $w \in W_M$. If equality holds, then $s_{w^{-1}\alpha}(\nu) = \nu$. Thus $s_{w^{-1}\alpha} \in W_M$ (as V is M-regular), which implies that $\alpha \in \Phi_M$, a contradiction. The claim follows.

By the rational Bruhat decomposition, $G(k) = \coprod_{W_L \backslash W / W_M} \overline{Q}(k)\dot{\sigma}P(k)$. It will thus suffice to show that $p_{\overline{N'}}(\dot{\sigma}V^{N(k)}) \neq 0$ for $\sigma \in W$ implies $\sigma \in W_L W_M$. By Lemma 2.3 we see that $V = V^{N'(k)} \oplus \ker p_{\overline{N'}}$ and that these two subspaces are preserved by the torus T acting on $V = F(\nu)$, sharing no common weight. (Note that $p_{\overline{N'}}$ is identified with $F(\nu) \to F(\nu)_{\overline{N'}}$. This follows, for example, since the natural surjection $V_{\overline{N'}(k)} \to F(\nu)_{\overline{N'}}$ is an isomorphism, since the domain is irreducible as $L(k)$-representation.) So there is a weight μ of $F(\nu)^N$ such that $\sigma\mu$ is a weight of $F(\nu)_{\overline{N'}} \cong F(\nu)^{N'}$. By the previous paragraph,

$$\langle \mu, \alpha^\vee \rangle > 0 \quad \forall \alpha \in \Phi^+ - \Phi_M^+, \quad \langle \sigma\mu, \beta^\vee \rangle > 0 \quad \forall \beta \in \Phi^+ - \Phi_L^+.$$

It follows that $\sigma(\Phi^+ - \Phi_M^+) \subset \Phi^+ \cup \Phi_L^-$.

We claim that there is a $w \in W_M$ such that $\sigma w(\Phi^+) \subset \Phi^+ \cup \Phi_L^-$. Suppose there is a simple root α of M such that $\sigma(\alpha) < 0$. Then s_α preserves $\Phi^+ - \Phi_M^+$, so $\sigma s_\alpha(\Phi^+ - \Phi_M^+) \subset \Phi^+ \cup \Phi_L^-$ while σs_α maps one fewer simple

root of M to a negative root. By induction we find a $w \in W_M$ such that $\sigma w(\Phi^+ - \Phi_M^+) \subset \Phi^+ \cup \Phi_L^-$ and σw maps all simple roots of M to positive roots. This implies the claim.

Equivalently, $w^{-1}\sigma^{-1}(\Phi^- - \Phi_L^-) \subset \Phi^-$. The same argument as in the previous paragraph shows that there is a $w' \in W_L$ such that $w^{-1}\sigma^{-1}w'(\Phi^-) \subset \Phi^-$. This shows that $\sigma = w'w^{-1} \in W_L W_M$, which completes the proof.

To justify the final statement, suppose that $\mathrm{Stab}_W(V^{U(k)}) = W_M$. Then $V^{N(k)} = V^{U(k)}$ is one-dimensional. (One can see this directly using the Bruhat decomposition; alternatively note that $\nu \in X_M^0(T)$.) Thus if $p_{\overline{N'}}(\dot{\sigma} V^{N(k)}) \neq 0$ then $\sigma\nu$ is a weight of $F(\nu)^{N'}$. Pick simple roots $\alpha_i \in \Delta$ with corresponding simple reflections $s_i \in W$ such that

$$\nu \gneqq s_1\nu \gneqq \cdots \gneqq s_r \cdots s_2 s_1 \nu = \sigma\nu.$$

By Lemma 2.3 it follows that $\nu - \sigma\nu \in \mathbb{Z}\Phi_L$. So each α_i is in Φ_L, hence $\sigma \in s_r \cdots s_2 s_1 \mathrm{Stab}_W(\nu) \subset W_L W_M$. The argument in case $\mathrm{Stab}_W(V^{U(k)}) = W_L$ is similar, or follows by duality. $\square$

Corollary 2.18. *Suppose that $P = MN$ is a standard parabolic and that $\lambda \in X_*(T)_-$. Suppose that V is M-regular, and suppose that either $\mathrm{Stab}_W(V^{U(k)}) = W_M$ or that V is also M_λ-regular. Let $t = \lambda(\varpi)$.*

   (i) *If $T_\lambda(g)|_{V^{N(k)}} \neq 0$ then $g \in Kt\mathcal{P}$. If $p_{\overline{N}} \circ T_\lambda(g) \neq 0$ then $g \in \overline{\mathcal{P}}tK$.*
   (ii) *We have $\mathcal{S}_G^M(T_\lambda) = T_\lambda^M$ and $'\mathcal{S}_G^M(T_\lambda) = T_\lambda^M$.*

This generalises [Her11, Proposition 1.4]. Note that if $\lambda(\varpi) \in Z_M^-$, then an M-regular V is also M_λ-regular.

Proof. (i) Writing $g = \kappa' t\kappa$ with $\kappa', \kappa \in K$, we see that $T_\lambda(t)\kappa|_{V^{N(k)}} \neq 0$, so $p_{N_\lambda}(\kappa V^{N(k)}) \neq 0$. Lemma 2.17 shows that $\kappa \in \mathcal{P}_\lambda \cdot \mathcal{P}$. By Prop. 2.15, $\mathrm{red}(\mathcal{P}_\lambda) = \mathrm{red}(K \cap K^t)$, so $\kappa \in (K \cap K^t)\mathcal{P}$ and $g = \kappa' t\kappa \in Kt\mathcal{P}$.

The other part is similar, or follows by duality.

(ii) Take any $m \in M$ and suppose that $T_\lambda(mn)|_{V^{N(k)}} \neq 0$ (some $n \in N$) is a non-zero term contributing to $(\mathcal{S}_G^M T_\lambda)(m)$. By part (i) and by Lemma 2.16 (twice), we have

$$mn \in Kt\mathcal{P} \cap P = KtP(\mathcal{O}) \cap P = P(\mathcal{O})tP(\mathcal{O}) = M(\mathcal{O})tM(\mathcal{O})N(\mathcal{O}).$$

Thus $m \in M(\mathcal{O})tM(\mathcal{O})$ and $n \in N(\mathcal{O})$. This shows that $\mathcal{S}_G^M(T_\lambda)$ is supported on $M(\mathcal{O})tM(\mathcal{O})$ and that $(\mathcal{S}_G^M T_\lambda)(t) = T_\lambda(t)|_{V^{N(k)}}$ is a linear projection (which is non-zero since $V^{U(k)} \subset V^{N(k)}$), so $\mathcal{S}_G^M(T_\lambda) = T_\lambda^M$. The other statement follows similarly or by using duality. $\square$

Corollary 2.19. *Suppose that the derived subgroup of G is simply connected. Let $P = MN$ be a standard parabolic subgroup. Under the natural identification of $\mathcal{H}_M(V^{N(k)})$ and $\mathcal{H}_M(V_{\overline{N}(k)})$ we have $\mathcal{S}_G^M = '\mathcal{S}_G^M$.*

Proof. We first show that $\mathcal{S}_G = '\mathcal{S}_G$ when $\dim V = 1$. We will make use of the "Gelfand involution" on $\mathcal{H}_G(V)$ (see [Her11], end of Section 2.1). Let $\tau : G \to G$ denote a "transpose" involution as in [Jan03, II.1.16]; it

fixes T pointwise and interchanges U_α and $U_{-\alpha}$ for all $\alpha \in \Phi$. Let ${}^\tau V$ denote $\mathrm{Hom}(V, \bar{k})$ with $G(k)$ acting via τ. As the derived subgroup of G is simply connected, there is a $G(k)$-linear isomorphism $\varsigma : V \to {}^\tau V$. A linear map $\varphi \in \mathrm{End}_{\bar{k}} V$ induces ${}^\tau\varphi \in \mathrm{End}_{\bar{k}} V$ by dualising and applying ς. We verified in [Her11] that ${}^\tau f({}^\tau g) = f(g)$ for all $f \in \mathcal{H}_G(V)$ and all $g \in G$. As $\dim V = 1$, we have $\mathrm{End}_{\bar{k}} V = \bar{k}$ and thus ${}^\tau\varphi = \varphi$. It follows that

$$\mathcal{S}_G(f)(t) = \sum_{U/U(\mathcal{O})} f(tu) = \sum_{U/U(\mathcal{O})} f({}^\tau(tu)) = \sum_{\overline{U}(\mathcal{O})\backslash\overline{U}} f(\overline{u}t) = {}'\mathcal{S}_G(f)(t).$$

Next we show that $\mathcal{S}_G = {}'\mathcal{S}_G$ in general. Let M be the standard Levi with $\mathrm{Stab}_W(V^{U(k)}) = W_M$. Under the natural identification, we have by Cor. 2.18 that $\mathcal{S}_G^M(T_\lambda) = T_\lambda^M = {}'\mathcal{S}_G^M(T_\lambda)$, for any $\lambda \in X_*(T)_-$. Note that $V^{N(k)}$ is one-dimensional (for example, by the Bruhat decomposition), so by the above, $\mathcal{S}_M = {}'\mathcal{S}_M$ for this weight. By transitivity, $\mathcal{S}_G = {}'\mathcal{S}_G$.

Finally if M is arbitrary we use that ${}'\mathcal{S}_G = {}'\mathcal{S}_M \circ {}'\mathcal{S}_G^M$ and $\mathcal{S}_G = \mathcal{S}_M \circ \mathcal{S}_G^M$. By the above we already know that $\mathcal{S}_G = {}'\mathcal{S}_G$ and $\mathcal{S}_M = {}'\mathcal{S}_M$. The claim follows by the injectivity of the latter map. $\qquad\square$

Lemma 2.20. *Suppose $\overline{P} = M\overline{N}$. Then $G = \overline{\mathcal{P}}TK$.*

Suppose the subset $X \subset G$ has finite image in $\overline{\mathcal{P}}\backslash G$. Then there exists $h \in Z_M^-$ such that $hX \subset \overline{\mathcal{P}}T^-K$.

See also [SS91, Lemma 12].

Proof. For the first claim, suppose $g \in G$. By the Cartan decomposition, there is a $\lambda \in X_*(T)_-$ such that $g \in K\lambda(\varpi)K$. By Prop. 2.15, we have

$$\overline{\mathcal{P}}\backslash K\lambda(\varpi)K/K \cong \overline{P}(k)\backslash G(k)/P_{-\lambda}(k).$$

The rational Bruhat decomposition shows that $g \in \overline{\mathcal{P}}\dot{w}\lambda(\varpi)K = \overline{\mathcal{P}}({}^w\lambda(\varpi))K$, for some $w \in W$. Alternatively one could use the Bruhat–Tits decomposition $G = \overline{\mathcal{B}}N(T)\overline{\mathcal{B}}$.

For the second claim, say $X \subset \bigcup \overline{\mathcal{P}}t_i k_i$ (finite union). We now use that $(G, \overline{\mathcal{B}}, N(T))$ is a generalised Tits system [Iwa66, §1]. The relevant axioms are as follows: (i) $H := N(T) \cap \overline{\mathcal{B}}$ is normal in $N(T)$, (ii) $N(T)/H$ is a semidirect product of a subgroup Ω and a normal subgroup W', (iii) there exists a generating set $S = \{w_i : i \in I\}$ of W' each of whose elements has order 2, (iv) for any $\sigma \in \Omega W'$ and any $i \in I$, $\sigma\overline{\mathcal{B}}w_i \subset \overline{\mathcal{B}}\sigma w_i\overline{\mathcal{B}} \cup \overline{\mathcal{B}}\sigma\overline{\mathcal{B}}$, (v) any element of Ω normalises $\overline{\mathcal{B}}$. Note that $H = T(\mathcal{O})$ in our case. We remark that it is in fact not hard to justify using Bruhat–Tits theory that $(G, \overline{\mathcal{B}}, N(T))$ is a generalised Tits system, namely by applying [BT72, Thm. 6.5]; so S consists of the reflections in the walls of the chamber C corresponding to $\overline{\mathcal{B}}$ and Ω is the stabiliser of C, as a subset of the apartment of T.

For $G = \mathrm{GL}_n$ it is not hard to write down Ω and S explicitly [Iwa66, §2].

It follows easily from the above axioms that for all $n' \in N(T)$ there are $n_1, \ldots, n_r \in N(T)$ such that

$$n\overline{\mathcal{B}}n' \subset \bigcup \overline{\mathcal{B}}nn_j\overline{\mathcal{B}} \quad \forall n \in N(T).$$

(Just write the image of n' in $N(T)/H$ as a product of elements in $S \cup \Omega$ and induct.) Thus there are finitely many $n_{ij} \in N(T)$ such that $h\overline{\mathcal{B}}t_i \subset \bigcup_j \overline{\mathcal{B}}hn_{ij}\overline{\mathcal{B}}$ for all i and for all $h \in Z_M^-$. It follows using Lemma 2.16 that

$$h\overline{\mathcal{P}}t_i k_i = \overline{\mathcal{P}}^- \overline{\mathcal{P}}^0 h\overline{\mathcal{P}}^+ t_i k_i \subset \bigcup_j \overline{\mathcal{P}}hn_{ij}K,$$

as $\overline{\mathcal{P}}^+ \subset \overline{\mathcal{B}} \subset \overline{\mathcal{P}}$. Writing $n_{ij} = t_{ij}\dot{w}_{ij}$ with $t_{ij} \in T$ and $w_{ij} \in W$ we see that the right-hand side equals $\bigcup_j \overline{\mathcal{P}}ht_{ij}K$. Since $M(\mathcal{O}) \subset \overline{\mathcal{P}}$ we may replace t_{ij} with any W_M-conjugate without changing its double coset. In this way we can ensure that t_{ij} is antidominant as element of M. Then it is possible to find an $h \in Z_M^-$ such that $ht_{ij} \in T^-$ for all i, j. $\qquad\square$

2.5. Compatibilities between Hecke actions.

Let $\overline{V}$ be a weight for M. Consider the following subspaces of Hecke algebras:

$$\mathcal{H}'_M(\overline{V}) := \{\varphi : \operatorname{supp}(\varphi) \subset M(\mathcal{O})Z_M^-M(\mathcal{O})\} \subset \mathcal{H}_M(\overline{V}),$$
$$\mathcal{H}'_{\mathcal{P}}(\overline{V}) := \{\varphi : \operatorname{supp}(\varphi) \subset \mathcal{P}Z_M^-\mathcal{P}\} \subset \mathcal{H}_{\mathcal{P}}(\overline{V}),$$
$$\mathcal{H}'_{\overline{\mathcal{P}}}(\overline{V}) := \{\varphi : \operatorname{supp}(\varphi) \subset \overline{\mathcal{P}}Z_M^-\overline{\mathcal{P}}\} \subset \mathcal{H}_{\overline{\mathcal{P}}}(\overline{V}).$$

The following lemma shows that each of them is a (commutative) subalgebra and that we can naturally identify them.

Lemma 2.21. *The subspaces $\mathcal{H}'_M(\overline{V})$, $\mathcal{H}'_{\mathcal{P}}(\overline{V})$, $\mathcal{H}'_{\overline{\mathcal{P}}}(\overline{V})$ are subalgebras. The map $\varphi \mapsto \varphi|_M$ from $\mathcal{H}'_{\mathcal{P}}(\overline{V})$ to $\mathcal{H}'_M(\overline{V})$ (resp., from $\mathcal{H}'_{\overline{\mathcal{P}}}(\overline{V})$ to $\mathcal{H}'_M(\overline{V})$) is an algebra isomorphism.*

See also Vignéras [Vig04, §A.7] for the subalgebras $\mathcal{H}'_{\mathcal{P}}(\overline{V})$ and $\mathcal{H}'_{\overline{\mathcal{P}}}(\overline{V})$.

Proof. Clearly $\mathcal{H}'_M(\overline{V})$ is a subalgebra. Once we show that the two maps are bijective and compatible with the algebra structures, it will follow that the other two subspaces are subalgebras. We concentrate on the first map. The verification for the second map is similar or may be deduced by duality.

The map is clearly injective. To check surjectivity, note that the T_h^M for $h \in Z_M^-$ span $\mathcal{H}'_M(\overline{V})$. For $h \in Z_M^-$ define $E_h : G \to \operatorname{End}(\overline{V})$ in $\mathcal{H}'_{\mathcal{P}}(\overline{V})$ with support $\mathcal{P}h\mathcal{P}$ and such that $E_h(h) = \operatorname{id}_{\overline{V}}$. Let us check that it is well defined. If $p_1 h = h p_2$ for $p_i \in \mathcal{P}$, write $p_i = p_i^- p_i^0 p_i^+$ according to the Iwahori decomposition $\mathcal{P} = \mathcal{P}^-\mathcal{P}^0\mathcal{P}^+$ (Lemma 2.16). It follows that $p_1^-(p_1^0 h)(p_1^+)^h = {}^h p_2^-(hp_2^0)p_2^+$ in $\overline{N}MN$, so $p_1^0 = p_2^0$. Thus $p_1 E_h(h) = p_1^0 = p_2^0 = E_h(h)p_2 \in \operatorname{End}(\overline{V})$ and E_h is well defined. As $\mathcal{P}h\mathcal{P} = \mathcal{P}^-(\mathcal{P}^0 h)\mathcal{P}^+$, we see that $E_h|_M = T_h^M$.

To check that the map is compatible with the algebra structures, by (2.8) it suffices to show that $E_{h_1} * E_{h_2} = E_{h_1 h_2}$ $(h_i \in Z_M^-)$. We have

$$\mathcal{P}h_1\mathcal{P}h_2\mathcal{P} = \mathcal{P}(h_1\mathcal{P}^-\mathcal{P}^0)(\mathcal{P}^+ h_2)\mathcal{P} = \mathcal{P}h_1 h_2\mathcal{P}$$

by Lemma 2.16 and

$$(E_{h_1} * E_{h_2})(h_1 h_2) = \sum_{\mathcal{P}/(\mathcal{P} \cap {}^{h_1}\mathcal{P})} E_{h_1}(p h_1) E_{h_2}(h_1^{-1} p^{-1} h_1 h_2).$$

Since $\mathcal{P} \cap {}^{h_1}\mathcal{P} = ({}^{h_1}\mathcal{P}^-)\mathcal{P}^0\mathcal{P}^+$, we may replace the index set by $\mathcal{P}^-/{}^{h_1}\mathcal{P}^-$. To obtain a non-zero term for $p \in \mathcal{P}^-$ we also need $(h_1^{-1} p^{-1} h_1)h_2 \in \mathcal{P}h_2\mathcal{P} = \mathcal{P}^-(\mathcal{P}^0 h_2)\mathcal{P}^+$. As the product map $\overline{N} \times M \times N \to G$ is injective, $p \in {}^{h_1}\mathcal{P}^-$. Thus $(E_{h_1} * E_{h_2})(h_1 h_2) = \mathrm{id}_{\overline{V}}$. $\qquad\qquad\qquad\qquad\qquad\square$

Suppose that V is an M-regular weight. Consider the subspace

$$\mathcal{H}'_G(V) := \{\varphi : \mathrm{supp}(\varphi) \subset KZ_M^- K\} \subset \mathcal{H}_G(V).$$

The following lemma shows that it is a subalgebra and that we can identify it with $\mathcal{H}'_{\mathcal{P}}(V^{N(k)})$ and with $\mathcal{H}'_{\overline{\mathcal{P}}}(V_{\overline{N}(k)})$. Moreover, the resulting identification $\mathcal{H}'_{\mathcal{P}}(V^{N(k)}) \cong \mathcal{H}'_G(V) \cong \mathcal{H}'_{\overline{\mathcal{P}}}(V_{\overline{N}(k)})$ is the one obtained in Lemma 2.21. (Recall that $V^{N(k)} \xrightarrow{\sim} V_{\overline{N}(k)}$ is a weight for M.)

Lemma 2.22. *Assume that V is M-regular. The subspace $\mathcal{H}'_G(V)$ is a subalgebra. There is an algebra isomorphism $i^{\mathcal{P}} : \mathcal{H}'_G(V) \to \mathcal{H}'_{\mathcal{P}}(V^{N(k)})$, which is characterised as follows. For any $\varphi \in \mathcal{H}'_G(V)$ the map $i^{\mathcal{P}}(\varphi)$ is supported on $\mathcal{P}Z_M^- \mathcal{P}$ and for $g \in \mathcal{P}Z_M^- \mathcal{P}$, $i^{\mathcal{P}}(\varphi)(g) \in \mathrm{End}(V^{N(k)})$ is the restriction of $\varphi(g) \in \mathrm{End}(V)$ to $V^{N(k)}$.*

Similarly there is an algebra isomorphism $i_{\overline{\mathcal{P}}} : \mathcal{H}'_G(V) \to \mathcal{H}'_{\overline{\mathcal{P}}}(V_{\overline{N}(k)})$ such that for any $\varphi \in \mathcal{H}'_G(V)$ the map $i_{\overline{\mathcal{P}}}(\varphi)$ is supported on $\overline{\mathcal{P}}Z_M^- \overline{\mathcal{P}}$ and for $g \in \overline{\mathcal{P}}Z_M^- \overline{\mathcal{P}}$, $i_{\overline{\mathcal{P}}}(\varphi)(g) \in \mathrm{End}(V_{\overline{N}(k)})$ is induced by $\varphi(g) \in \mathrm{End}(V)$.

The resulting identification of $\mathcal{H}'_{\mathcal{P}}(V^{N(k)})$ with $\mathcal{H}'_{\overline{\mathcal{P}}}(V_{\overline{N}(k)})$ is the one obtained in Lemma 2.21. The resulting identification of $\mathcal{H}'_G(V)$ with $\mathcal{H}'_M(V^{N(k)}) \cong \mathcal{H}'_M(V_{\overline{N}(k)})$ is given by $\mathcal{S}_G^M = {}'\mathcal{S}_G^M$.

Proof. Let $\varphi \in \mathcal{H}'_G(V)$. We verify that $\varphi(g)$ induces an endomorphism of $V^{N(k)}$ for $g \in \mathcal{P}Z_M^- \mathcal{P}$ by checking it for $g = h \in Z_M^-$. Take $n \in N(\mathcal{O})$ and note that $n\varphi(h) = \varphi(h)n^h$, where $n^h \in N(\mathcal{O})$ since h^{-1} contracts $N(\mathcal{O}) = \mathcal{P}^+$ (Lemma 2.16). Thus $i^{\mathcal{P}}(\varphi)$ is well defined.

We now use Lemma 2.21 and its proof. Note that $\mathcal{H}'_G(V)$ is spanned by the T_λ with $\lambda(\varpi) \in Z_M^-$. Fix such a λ and let $h := \lambda(\varpi)$. By definition, $T_\lambda(h) \in \mathrm{End}(V)$ induces the identity map on $V^{N_{-\lambda}(k)}$. But $N_{-\lambda} \subset N$ since $\lambda(\varpi) \in Z_M^-$, so $T_\lambda(h)$ induces the identity map on $V^{N(k)}$. As a consequence, $i^{\mathcal{P}}(T_\lambda) = E_h$. So T_λ in $\mathcal{H}'_G(V)$ is mapped to $E_h|_M = T_h^M = T_\lambda^M$ in $\mathcal{H}'_M(\overline{V})$. By Cor. 2.18, the composite map $\mathcal{H}'_G(V) \to \mathcal{H}'_M(\overline{V})$ is given by $\mathcal{S}_G^M$. As the T_λ^M with $\lambda(\varpi) \in Z_M^-$ span $\mathcal{H}'_M(\overline{V})$, the map is surjective. But $\mathcal{S}_G^M$ is injective on $\mathcal{H}_G(V)$, so $\mathcal{H}'_G(V)$ is the inverse image under $\mathcal{S}_G^M$ of $\mathcal{H}'_M(\overline{V})$, in particular it is a subalgebra. Moreover $i^{\mathcal{P}}$ is an algebra homomorphism as $\mathcal{S}_G^M$ is.

The algebra isomorphism $i_{\overline{\mathcal{P}}}$ is obtained by duality from $i^{\mathcal{P}}$ (interchanging positive and negative roots and V with V^*).

The two identifications coincide since for $m \in M(\mathcal{O})Z_M^-$, the endomorphisms of $V^{N(k)}$ and $V_{\overline{N}(k)}$ induced by $\varphi(m)$ are identified via $V^{N(k)} \xrightarrow{\sim} V_{\overline{N}(k)}$. $\square$

After this subsection we will no longer use the primed notation. If $\overline{V}$ is a weight for M, we will denote by $\mathcal{H}(\overline{V})$ the isomorphic Hecke algebras of Lemma 2.21. Thus $\mathcal{H}(\overline{V})$ can be thought of as subalgebra of $\mathcal{H}_M(\overline{V})$, $\mathcal{H}_{\mathcal{P}}(\overline{V})$, and $\mathcal{H}_{\overline{\mathcal{P}}}(\overline{V})$. Similarly, suppose that V is an M-regular weight for G. Then $V^{N(k)} \xrightarrow{\sim} V_{\overline{N}(k)}$ is a weight for M and we denote it by $\overline{V}$. We will denote by $\mathcal{H}(V)$ the isomorphic Hecke algebras of Lemma 2.22. Thus $\mathcal{H}(V)$ can be thought of as subalgebra of $\mathcal{H}_G(V)$, $\mathcal{H}_M(\overline{V})$, $\mathcal{H}_{\mathcal{P}}(\overline{V})$, and $\mathcal{H}_{\overline{\mathcal{P}}}(\overline{V})$. Note that $\mathcal{H}(V) \cong \mathcal{H}(\overline{V})$. There can be no confusion since M-regular weights for G are in natural bijection with weights for M (by Lemma 2.5).

We will even write $\mathcal{H}$ when it is clear from the context what V (or $\overline{V}$) is.

In the following proposition we will use natural maps $\text{c-Ind}_{\mathcal{P}}^G V^{N(k)} \to \text{c-Ind}_K^G V \to \text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)}$. They are obtained by Frobenius reciprocity from $V^{N(k)} \to V \subset \text{c-Ind}_K^G V$, respectively $V \to \text{Ind}_{\overline{\mathcal{P}}}^K V_{\overline{N}(k)} \subset \text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)}$. Alternatively, as Hecke operators they are supported on K and map the identity of K to the natural map $V^{N(k)} \to V$, respectively $V \to V_{\overline{N}(k)}$.

Proposition 2.23. *Assume that V is M-regular. We have the following diagram of Hecke operators.*

For all $\varphi \in \mathcal{H} \subset \mathcal{H}_G(V)$ the two squares commutes. If moreover $\text{supp}(\varphi) \subset KZ_M^{--}K$ then there are diagonal arrows making the whole diagram commute.

Proof. First note that the horizontal map on the top left is surjective: this is because the map is obtained by compactly inducing the K-linear surjection $\text{c-Ind}_{\mathcal{P}}^K V^{N(k)} \to V$ from K to G. Similarly, the horizontal map on the top right is injective.

To check commutativity, we first deal with the left half of the diagram. Without loss of generality, φ is supported on a single double coset KhK with $h = \lambda(\varpi) \in Z_M^-$. By Frobenius reciprocity it suffices to check that the two maps around the left square agree on $V^{N(k)} \subset \text{c-Ind}_{\mathcal{P}}^G V^{N(k)}$. A vector $v \in V^{N(k)}$ is mapped to $[1, v] \in \text{c-Ind}_K^G V$ which in turn is mapped to $\sum_{K \backslash KhK} [g^{-1}, \varphi(g)v] \in \text{c-Ind}_K^G V$ under φ. If $\varphi(g)v \neq 0$ then $g \in Kh\mathcal{P}$ by Cor. 2.18. We verify that the natural map $\mathcal{P} \backslash \mathcal{P}h\mathcal{P} \to K \backslash Kh\mathcal{P}$ is a

bijection. It is enough to show that $\mathcal{P} \cap K^h = \mathcal{P} \cap \mathcal{P}^h$ or equivalently that $^h\mathcal{P} \cap K = {}^h\mathcal{P} \cap \mathcal{P}$. Now note that $^h\mathcal{P} \cap K = ({}^h\mathcal{P}^-)\mathcal{P}^0\mathcal{P}^+ \subset \mathcal{P}$ by Lemma 2.16.

Thus we see that

$$(2.24) \qquad \sum_{K \backslash KhK} [g^{-1}, \varphi(g)v] = \sum_{\mathcal{P} \backslash \mathcal{P}h\mathcal{P}} [g^{-1}, i^{\mathcal{P}}(\varphi)(g)v],$$

which shows that the left square commutes.

Now suppose that $\mathrm{supp}(\varphi) = KhK$ for some $h = \lambda(\varpi) \in Z_M^{--}$. In this case $P_{-\lambda} = P$. We will define the diagonal map $\varphi_\delta \in \mathcal{H}_{K,\mathcal{P}}(V, V^{N(k)})$ as follows (see §6.1 for the notation). We let φ_δ agree with φ on $\mathcal{P}hK$ and we let it vanish outside. To see that it is well defined, note that $\varphi(h)$ maps V to $V^{N_{-\lambda}(k)} = V^{N(k)}$ and that $\mathcal{P}$ preserves $V^{N(k)}$. We check that the top left triangle commutes. For $v \in V^{N(k)}$ the arrow to the right maps it to $[1, v]$ as before and φ_δ further maps it to $\sum_{\mathcal{P} \backslash \mathcal{P}hK}[g^{-1}, \varphi(g)v]$. As noted above, if $\varphi(g)v \neq 0$ then $g \in Kh\mathcal{P}$. But $\mathcal{P}hK \cap Kh\mathcal{P} = \mathcal{P}h\mathcal{P}$ since $K \cap {}^hK \subset \mathcal{P}_{-\lambda} = \mathcal{P}$ by Prop. 2.15, so we are done as in (2.24). Since the map at the top is surjective, the bottom triangle also commutes.

We now dualise the left half of the diagram, in the following sense. We think of maps between compact inductions as Hecke operators and then apply the duality $\varphi \mapsto \varphi'$ considered in §2.3. (Strictly speaking, we also use this duality for maps between different compact inductions as in §6.1.) We obtain

$$\begin{array}{ccc} \text{c-Ind}_K^G V^* & \longrightarrow & \text{c-Ind}_{\mathcal{P}}^G (V^*)_{N(k)} \\ \varphi' \downarrow & & \downarrow i^{\mathcal{P}}(\varphi)' \\ \text{c-Ind}_K^G V^* & \longrightarrow & \text{c-Ind}_{\mathcal{P}}^G (V^*)_{N(k)} \end{array}$$

Since the natural maps $V^{N(k)} \to V$ and $V^* \to (V^*)_{N(k)}$ are dual, the top and bottom maps are the natural ones. By construction of $i_{\mathcal{P}}$, $i^{\mathcal{P}}(\varphi)' = i_{\mathcal{P}}(\varphi')$. By replacing V by V^* and by interchanging positive and negative roots, we obtain the right half of the diagram. $\square$

Corollary 2.25. *Suppose that V is M-regular and that $\chi : \mathcal{H}_M(V^{N(k)}) \to \bar{k}$ is an algebra homomorphism. Then the maps of Prop. 2.23 induce isomorphisms*

$$\text{c-Ind}_{\mathcal{P}}^G V^{N(k)} \otimes_{\mathcal{H},\chi} \bar{k} \xrightarrow{\sim} \text{c-Ind}_K^G V \otimes_{\mathcal{H},\chi} \bar{k} \xrightarrow{\sim} \text{c-Ind}_{\bar{\mathcal{P}}}^G V_{\overline{N}(k)} \otimes_{\mathcal{H},\chi} \bar{k},$$

where $\mathcal{H} = \mathcal{H}(V)$ as above.

Proof. Pick any $\lambda \in X_*(T)_-$ such that $h := \lambda(\varpi) \in Z_M^{--}$. Since $T_h^M T_{h^{-1}}^M = 1$, we have $\chi(T_h^M) \neq 0$. Let $\mathfrak{m} = \ker(\chi)$, an ideal of $\mathcal{H}$. We will use that $\mathcal{H}$ is commutative.

The first map is clearly surjective. Let $\varphi = T_\lambda \in \mathcal{H}$; it is identified with T_h^M so $\chi(\varphi) \neq 0$. Recall that we denoted the diagonal map corresponding to φ on the left-hand side of the diagram in Prop. 2.23 by φ_δ. We now show

that φ_δ is $\mathcal{H}$-linear. Let $\xi : \text{c-Ind}_{\mathcal{P}}^G V^{N(k)} \to \text{c-Ind}_K^G V$ denote the natural map, and suppose that $\psi \in \mathcal{H} \subset \mathcal{H}_G(V)$. Then

$$\varphi_\delta \circ \psi \circ \xi = i^{\mathcal{P}}(\varphi) \circ i^{\mathcal{P}}(\psi) = i^{\mathcal{P}}(\psi) \circ i^{\mathcal{P}}(\varphi) = i^{\mathcal{P}}(\psi) \circ \varphi_\delta \circ \xi,$$

and the claim follows, since ξ is surjective.

If $f \in \text{c-Ind}_{\mathcal{P}}^G V^{N(k)}$ and $\xi(f) \otimes 1 = 0$ then $\xi(f) \in \mathfrak{m}(\text{c-Ind}_K^G V)$. Then

$$\varphi(f) = \varphi_\delta(\xi(f)) \in \varphi_\delta(\mathfrak{m}(\text{c-Ind}_K^G V)) = \mathfrak{m}\,\varphi_\delta(\text{c-Ind}_K^G V),$$

so $\varphi(f) \otimes 1 = 0$. Thus $f \otimes 1 = \chi(\varphi)^{-1}(\varphi(f) \otimes 1) = 0$, so the first map is injective.

Suppose $y \otimes 1 \in \text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)} \otimes_{\mathcal{H},\chi} \bar{k}$. Then $y \otimes 1 = \chi(\varphi)^{-1}(\varphi(y) \otimes 1)$ comes from $\text{c-Ind}_K^G V$, by the commuting triangle on the bottom right. This proves that the second map is surjective.

Suppose $x \otimes 1 \in \text{c-Ind}_K^G V \otimes_{\mathcal{H},\chi} \bar{k}$ maps to zero. Let $\eta : \text{c-Ind}_K^G V \to \text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)}$ denote the natural map. Then $\eta(x) \in \mathfrak{m}(\text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)})$, so

$$\eta\varphi(x) = \varphi\eta(x) \in \mathfrak{m}(\varphi\,\text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)}) \subset \mathfrak{m}\,\eta(\text{c-Ind}_K^G V).$$

As η is injective, $\varphi(x) \in \mathfrak{m}(\text{c-Ind}_K^G V)$. Finally $x \otimes 1 = \chi(\varphi)^{-1}(\varphi(x) \otimes 1) = 0$. This proves that the second map is injective. $\qquad\square$

3. Parabolic inductions and compact inductions

The following theorem is inspired by work of Barthel–Livné for GL_2 (see [BL94, Thm. 25]). One aspect of the proof, namely the comparison of parahoric and parabolic inductions, crucially use ideas of Schneider–Stuhler [SS91] and Vignéras [Vig04]. See also the comment after Cor. 3.6.

Theorem 3.1. *Let $P = MN$ be a standard parabolic in G and suppose that V is an M-regular weight for G. Then for any algebra homomorphism $\chi : \mathcal{H}_M(V_{\overline{N}(k)}) \to \bar{k}$, there is a natural isomorphism of smooth G-representations,*

$$\text{c-Ind}_K^G V \otimes_{\mathcal{H}_G(V),\chi} \bar{k} \xrightarrow{\ \sim\ } \text{Ind}_{\overline{P}}^G \left\{ \text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)} \otimes_{\mathcal{H}_M(V_{\overline{N}(k)}),\chi} \bar{k} \right\}.$$

Note that χ becomes a character of $\mathcal{H}_G(V)$ by composing with the partial Satake homomorphism $'\mathcal{S}_G^M : \mathcal{H}_G(V) \to \mathcal{H}_M(V_{\overline{N}(k)})$.

Proof. We begin by defining maps

$$(3.2) \qquad \text{c-Ind}_K^G V \xrightarrow{\ \eta\ } \text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)} \xrightarrow{\ \zeta\ } \text{Ind}_{\overline{P}}^G(\text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)}).$$

The map η is the natural one that we already used in Prop. 2.23. Explicitly, η sends $f \in \text{c-Ind}_K^G V$ to $g \mapsto p_{\overline{N}}(f(g))$ in $\text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)}$.

We define ζ by the following formula, for $f \in \text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)}$:

$$\zeta(f)(g) = \sum_{\overline{P}(\mathcal{O}) \backslash \overline{P}} \overline{p}^{-1}[1, f(\overline{p}g)].$$

Note that $\overline{P}$ acts via M. By using that $\overline{P}(\mathcal{O}) = M(\mathcal{O})\overline{N}(\mathcal{O})$ we see that the term in the sum only depends on $\overline{p} \in \overline{P}(\mathcal{O})\backslash\overline{P}$. We check that the sum only involves finitely non-zero terms. Without loss of generality f is supported on a single coset $\overline{\mathcal{P}}\gamma$. Since $\overline{P} \cap \overline{\mathcal{P}} = \overline{P}(\mathcal{O})$, we see that the sum involves at most one term. Clearly $\zeta(f)$ is $\overline{P}$-equivariant and ζ is G-equivariant. It follows that $\zeta(f)$ is smooth, so $\zeta(f) \in \mathrm{Ind}_{\overline{P}}^{G}(\mathrm{c\text{-}Ind}_{M(\mathcal{O})}^{M} V_{\overline{N}(k)})$. Therefore ζ is well defined.

Let $\mathcal{H} = \mathcal{H}(V)$ as in §2.5. It acts on the first term of (3.2) via $\mathcal{H}_G(V)$, on the second term via $\mathcal{H}_{\overline{\mathcal{P}}}(V_{\overline{N}(k)})$ and on the third term via $\mathcal{H}_M(V_{\overline{N}(k)})$.

Step 1. Check that η and ζ are $\mathcal{H}$-equivariant. We already checked this for η in Prop. 2.23.

We introduce a useful shorthand for certain functions in $\mathrm{Ind}_{\overline{P}}^{G}\sigma$ for any smooth M-representation σ. For any compact open subset $\Omega \subset N$ and any $x \in \sigma$ we write $[\Omega, x]$ for the function that is supported on $\overline{P}\Omega^{-1}$ and sends all $\nu \in \Omega^{-1}$ to x. This function is locally constant, since $N \hookrightarrow \overline{P}\backslash G$ has open image and Ω is compact open. Note that for $m \in M$ and $n \in N$,

$$m[\Omega, x] = [{}^{m}\Omega, mx], \quad n[\Omega, x] = [n\Omega, x].$$

We claim that $\zeta([1, \overline{v}]) = [\overline{\mathcal{P}}^{+}, [1, \overline{v}]]$. Clearly the left-hand side is supported on $\overline{P\mathcal{P}} = \overline{P}\overline{\mathcal{P}}^{+}$. It is then an easy computation to check that both sides agree on $\overline{\mathcal{P}}^{+}$.

To check that ζ is $\mathcal{H}$-equivariant, we can immediately reduce to the case when $\varphi = T_\lambda$ for some $\lambda \in X_*(T)_-$ with $h := \lambda(\varpi) \in Z_M^-$. Then ${}'\mathcal{S}_G^M(T_\lambda) = T_h^M$ by Cor. 2.18. Suppose $\overline{v} \in V_{\overline{N}(k)}$. On the one hand, in $\mathrm{c\text{-}Ind}_{M(\mathcal{O})}^{M} V_{\overline{N}(k)}$ we have $T_h^M([1, \overline{v}]) = [h^{-1}, \overline{v}]$ by (2.10). Thus $({}'\mathcal{S}_G^M(\varphi) \circ \zeta)([1, \overline{v}]) = [\overline{\mathcal{P}}^{+}, [h^{-1}, \overline{v}]]$. On the other hand,

$$(\zeta \circ i_{\overline{\mathcal{P}}}(\varphi))([1, \overline{v}]) = \sum_{\overline{\mathcal{P}}\backslash\overline{\mathcal{P}}h\overline{\mathcal{P}}} g^{-1}[\overline{\mathcal{P}}^{+}, [1, \varphi(g)\overline{v}]]$$

$$= \sum_{(\overline{\mathcal{P}}\cap\overline{\mathcal{P}}^{h})\backslash\overline{\mathcal{P}}} \overline{p}^{-1}h^{-1}[\overline{\mathcal{P}}^{+}, [1, T_\lambda(h)\overline{p}\overline{v}]].$$

Since $(\overline{\mathcal{P}}^{+})^{h}\backslash\overline{\mathcal{P}}^{+} \to (\overline{\mathcal{P}}\cap\overline{\mathcal{P}}^{h})\backslash\overline{\mathcal{P}}$ is a bijection by Lemma 2.16 and taking into account that $\overline{\mathcal{P}}^{+}$ fixes $\overline{v}$ and that $T_\lambda(h)$ is trivial on $V_{\overline{N}(k)}$ (see the proof of Lemma 2.22), the sum above simplifies to

$$\sum_{(\overline{\mathcal{P}}^{+})^{h}\backslash\overline{\mathcal{P}}^{+}} [\overline{p}^{-1}(\overline{\mathcal{P}}^{+})^{h}, [h^{-1}, \overline{v}]] = [\overline{\mathcal{P}}^{+}, [h^{-1}, \overline{v}]].$$

Step 2. Check that $\zeta \circ \eta$ is $\mathcal{H}_G(V)$-equivariant (via partial Satake). Let $\theta : V \to \mathrm{Ind}_{\overline{P}}^{G}(\mathrm{c\text{-}Ind}_{M(\mathcal{O})}^{M} V_{\overline{N}(k)})$ be the composition of $V \to \mathrm{c\text{-}Ind}_{K}^{G} V$ with $\zeta \circ \eta$. It is K-linear. Note that by definition of ζ and since $\eta([1, v])$ is

supported on K,

$$\theta(v)(1) = \zeta(\eta([1,v]))(1) = [1, \eta([1,v])(1)] = [1, p_{\overline{N}}(v)].$$

Thus, in the notation of (2.13), $\overline{\theta} \in \operatorname{Hom}_{M(\mathcal{O})}(V_{\overline{N}(k)}, \text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)}) = \mathcal{H}_M(V_{\overline{N}(k)})$ is the natural map. By Lemma 2.14, $\overline{\theta * \varphi} = \overline{\theta} * \varphi_M$, where $\varphi_M = {}'\mathcal{S}_G^M \varphi$. Since $\mathcal{H}_M(V_{\overline{N}(k)})$ is commutative and since the map in (2.13) is natural in σ, we get that $\overline{\theta * \varphi} = \varphi_M \circ \overline{\theta} = \overline{\varphi_M \circ \theta}$. But the map in (2.13) is injective, so $\theta * \varphi = \varphi_M \circ \theta$, as required.

Step 3. Check that $\eta \otimes_{\mathcal{H},\chi} \overline{k}$ is an isomorphism. This is the content of Cor. 2.25.

Step 4. Check that $\zeta \otimes_{\mathcal{H},\chi} \overline{k}$ is surjective.

Fix any non-zero $\overline{v} \in V_{\overline{N}(k)}$. We claim that $\zeta([1,\overline{v}]) \otimes 1 = [\overline{\mathcal{P}}^+, [1,\overline{v}]] \otimes 1$ generates $\operatorname{Ind}_{\overline{P}}^G(\text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)}) \otimes_{\mathcal{H},\chi} \overline{k}$. Pick $h_0 \in Z_M^{--}$ and let $x_0 = [1,\overline{v}] \in \text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)}$. Since

$$[\overline{\mathcal{P}}^+, h_0^n x_0] \otimes 1 = [\overline{\mathcal{P}}^+, T_{h_0^{-n}}^M(x_0)] \otimes 1 = [\overline{\mathcal{P}}^+, x_0] \otimes \chi(T_{h_0^{-n}}^M),$$

it is enough to show that $[\overline{\mathcal{P}}^+, h_0^n x_0]$ ($n \in \mathbb{Z}$) generate $\operatorname{Ind}_{\overline{P}}^G(\text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)})$.

Note that the smooth M-representation $\sigma := \text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)}$ is generated by x_0. (This is the only property of σ that we will use.) We want to show that any $f \in \operatorname{Ind}_{\overline{P}}^G \sigma$ is contained in the G-representation generated by the $[\overline{\mathcal{P}}^+, h_0^n x_0]$ for $n \in \mathbb{Z}$. By writing $\overline{P} \backslash G$ as a (finite) disjoint union of compact open subsets, each of which is contained in a G-translate of $\overline{P} \backslash \overline{P} N$, we can reduce to the case that $\operatorname{supp}(f) \subset \overline{P} N$. Since $f|_N$ is locally constant and compactly supported, we can moreover assume that $f = [\Omega, x]$, for some compact open subset $\Omega \subset N$ and $x \in \sigma$. We can write $x = \sum_i \lambda_i m_i x_0$ as finite linear combination with $\lambda_i \in \overline{k}$, $m_i \in M$, so

$$f = \sum_i \lambda_i m_i [\Omega^{m_i}, x_0].$$

Thus we may assume that $f = [\Omega, x_0]$, for some compact open $\Omega \subset N$. By Lemma 2.16 there is an $n \gg 0$ such that $\Omega = \coprod_j \nu_j \cdot (\overline{\mathcal{P}}^+)^{h_0^n}$, a finite disjoint union with $\nu_j \in N$. Therefore

$$f = \sum_j \nu_j [(\overline{\mathcal{P}}^+)^{h_0^n}, x_0] = \sum_j \nu_j h_0^{-n} [\overline{\mathcal{P}}^+, h_0^n x_0],$$

which completes the argument.

Step 5. Check that $\zeta \otimes_{\mathcal{H},\chi} \overline{k}$ is injective.

Let $f' \in \text{c-Ind}_{\overline{\mathcal{P}}}^G V_{\overline{N}(k)}$ such that $\zeta(f') \otimes 1 = 0$. We need to show that $f' \otimes 1 = 0$. Note that for $h \in Z_M^-$ we have $\operatorname{supp}(E_h f') \subset \overline{\mathcal{P}} h \operatorname{supp}(f')$ since E_h is supported on $\overline{\mathcal{P}} h \overline{\mathcal{P}}$. (Recall that $E_h \in \mathcal{H}$ was defined in the proof of Lemma 2.21. It is identified with T_h^M.) Thus by Lemma 2.20 there is an

$h \in Z_M^-$ such that $\mathrm{supp}(E_h f') \subset \overline{\mathcal{P}} T^- K$. Since $E_h f' \otimes 1 = \chi(E_h)(f' \otimes 1)$ and $\chi(E_h) = \chi(T_h^M) \neq 0$, we may assume that $\mathrm{supp}(f') \subset \overline{\mathcal{P}} T^- K$.

Lemma 2.16 shows that for $t \in T^-$, ${}^t(\overline{\mathcal{P}}^-) \subset \overline{\mathcal{P}}^-$ and $(\overline{\mathcal{P}}^+)^t \subset \overline{\mathcal{P}}^+$. Thus for $h \in Z_M^-$ and $t \in T^-$, $\overline{\mathcal{P}} h \overline{\mathcal{P}} t K = \overline{\mathcal{P}}(h \overline{\mathcal{P}}^- \overline{\mathcal{P}}^0)(\overline{\mathcal{P}}^+ t) K \subset \overline{\mathcal{P}} h t K$. We can write $f' = \sum_{i=1}^r f_i'$ such that $\mathrm{supp}(f_i') = \overline{\mathcal{P}} t_i' k_i$ ($t_i' \in T^-$, $k_i \in K$). Since any element of Z_M can be written in the form $h' h^{-1}$ for $h', h \in Z_M^-$, we can find $h_i \in Z_M^-$ and $\tau_i \in T(\mathcal{O})$ such that $\tau_i h_i t_i' = \tau_j h_j t_j'$ whenever $T(\mathcal{O}) Z_M t_i' = T(\mathcal{O}) Z_M t_j'$. Let $f_i = \chi(E_{h_i})^{-1} E_{h_i} f_i'$, $f = \sum_i f_i$, and $t_i = \tau_i h_i t_i' \in T^-$. Then $f' \otimes 1 = f \otimes 1$, moreover $\mathrm{supp}(f_i) \subset \overline{\mathcal{P}} t_i K$ (as $T(\mathcal{O}) \subset \overline{\mathcal{P}}$) and

$$(3.3) \qquad t_i = t_j \text{ whenever } T(\mathcal{O}) Z_M t_i = T(\mathcal{O}) Z_M t_j.$$

We can now write $f_i = \sum f_{ij}$ (finite sum) such that $\mathrm{supp}(f_{ij}) = \overline{\mathcal{P}} t_i k_{ij}$ for some $k_{ij} \in K$. By combining those f_{ij} that have identical support, we may assume moreover:

$$(3.4) \qquad \text{the sets } \overline{\mathcal{P}} t_i k_{ij} \text{ are pairwise disjoint.}$$

We now show that $f = 0$. In fact we show that $f|_{\overline{\mathcal{P}}P} = 0$, but all our conditions on f are invariant under K, so $f = 0$ since $\overline{\mathcal{P}} P K = G$. (When we replace f by kf for some $k \in K$, then t_i is unchanged and k_{ij} is replaced by $k_{ij} k^{-1}$. Also note that $\zeta(kf) \otimes 1 = k \zeta(f) \otimes 1 = 0$.)

All we will use is that the image of $\zeta(f)(1)$ in $\mathrm{c\text{-}Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)} \otimes_{\mathcal{H}, \chi} \bar{k}$ is zero. Let us show that this latter space is naturally isomorphic to $\mathrm{c\text{-}Ind}_{M(\mathcal{O}) Z_M}^M V_{\overline{N}(k)}$. We let Z_M act on $V_{\overline{N}(k)}$ by declaring that $h \overline{v} = \chi(T_{h^{-1}}^M) \overline{v}$ for $h \in Z_M$ and $\overline{v} \in V_{\overline{N}(k)}$. This is compatible with the $M(\mathcal{O})$-action: for $h_0 \in Z_M(\mathcal{O})$, $T_{h_0^{-1}}^M = \omega(h_0) T_1^M = \omega(h_0)$ by (2.8) and (2.9), where $\omega : Z_M(k) \to k^\times$ is the central character of $V_{\overline{N}(k)}$, so $\chi(T_{h_0^{-1}}^M) = \omega(h_0)$. The map

$$\mathrm{c\text{-}Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)} \to \mathrm{c\text{-}Ind}_{M(\mathcal{O}) Z_M}^M V_{\overline{N}(k)}$$

is the obvious one induced by Frobenius reciprocity. It sends $[m, \overline{v}]$ to $[m, \overline{v}]_{Z_M}$, where the subscript is used to distinguish between the two induced representations. In particular it is surjective. For $h \in Z_M$ it sends $T_h^M [m, \overline{v}] = h^{-1}[m, \overline{v}]$ to $[h^{-1} m, \overline{v}]_{Z_M} = [m, h^{-1} \overline{v}]_{Z_M} = \chi(T_h^M)[m, \overline{v}]_{Z_M}$. (See (2.10) for the first identity.) The induced map

$$(3.5) \qquad \mathrm{c\text{-}Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)} \otimes_{\mathcal{H}, \chi} \bar{k} \twoheadrightarrow \mathrm{c\text{-}Ind}_{M(\mathcal{O}) Z_M}^M V_{\overline{N}(k)}$$

is injective: we can lift any element in the kernel to one of the form $\sum [m_i, \overline{v}_i] \in \mathrm{c\text{-}Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)}$, where the m_i lie in distinct $M(\mathcal{O}) Z_M$-cosets. By considering its image on the right-hand side we see that all $\overline{v}_i$ are zero.

Since $\zeta(f)(1) \otimes 1$ depends only on $f|_{\overline{\mathcal{P}}P}$, we will assume from now on that f is supported on $\overline{\mathcal{P}} P = \overline{\mathcal{P}}^+ P$. Thus $k_{ij} \in (\overline{\mathcal{P}}^+)^{t_i} \cdot \overline{P} \cap K = (\overline{\mathcal{P}}^+)^{t_i} \cdot \overline{P}(\mathcal{O})$ for all i. Thus we may assume that $k_{ij} \in \overline{P}(\mathcal{O}) = \overline{\mathcal{P}}^- \overline{\mathcal{P}}^0$ without changing $\overline{\mathcal{P}} t_i k_{ij}$. Since $t_i \in T^-$ shrinks $\overline{\mathcal{P}}^-$ we may even assume that $k_{ij} \in \overline{\mathcal{P}}^0 = $

$M(\mathcal{O})$. Let us write $f_{ij} = [k_{ij}^{-1}t_i^{-1}, \overline{v}_{ij}]$. As $k_{ij}^{-1}t_i^{-1} \in M$, we see that $\zeta(f_{ij})(1) \otimes 1 = [k_{ij}^{-1}t_i^{-1}, \overline{v}_{ij}]_{Z_M}$. By showing that $M(\mathcal{O})Z_M t_i k_{ij}$ are pairwise disjoint we will deduce from $\zeta(f)(1) \otimes 1 = 0$ that $\overline{v}_{ij} = 0$ for all i, which will complete the argument. Thus suppose that $M(\mathcal{O})h t_i k_{ij} = M(\mathcal{O})t_r k_{rs}$ for some $h \in Z_M$. Note that $Z_M T^-$ is the antidominant part of T with respect to M. Thus by the Cartan decomposition for M we have $T(\mathcal{O})h t_i = T(\mathcal{O})t_r$, so by (3.3) it follows that $t_i = t_r$ and hence that $h \in Z_M(\mathcal{O})$. Therefore $M(\mathcal{O})t_i k_{ij} = M(\mathcal{O})t_r k_{rs}$ which implies $\overline{\mathcal{P}}t_i k_{ij} = \overline{\mathcal{P}}t_r k_{rs}$, so by (3.4) we see that $(i,j) = (r,s)$ and we are done.

Step 6. We showed that $(\zeta \circ \eta) \otimes_{\mathcal{H},\chi} \overline{k}$ is an $\mathcal{H}_G(V)$-linear isomorphism (as $\mathcal{H}_G(V)$ commutative). By tensoring over $\mathcal{H}_G(V)$ with χ we see that $(\zeta \circ \eta) \otimes_{\mathcal{H}_G(V),\chi} \overline{k}$ is an isomorphism. The target of that isomorphism is isomorphic to $\mathrm{Ind}_P^G(\text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)}) \otimes_{\mathcal{H}_M(V_{\overline{N}(k)}),\chi} \overline{k}$, since $'\mathcal{S}_G^M : \mathcal{H}_G(V) \hookrightarrow \mathcal{H}_M(V_{\overline{N}(k)})$ is a localisation map. Since $\mathcal{H}_M(V_{\overline{N}(k)})$ is noetherian, we can pick a finite generating set $\varphi_1, \ldots, \varphi_n$ of the ideal $\ker \chi$. We have an exact sequence

$$\sigma^{\oplus n} \xrightarrow{\sum \varphi_i} \sigma \to \sigma \otimes_{\mathcal{H}_M(V_{\overline{N}(k)}),\chi} \overline{k} \to 0.$$

As Ind_P^G is exact, we deduce that $\mathrm{Ind}_P^G(\text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)}) \otimes_{\mathcal{H}_M(V_{\overline{N}(k)}),\chi} \overline{k}$ is isomorphic to $\mathrm{Ind}_P^G(\text{c-Ind}_{M(\mathcal{O})}^M V_{\overline{N}(k)} \otimes_{\mathcal{H}_M(V_{\overline{N}(k)}),\chi} \overline{k})$. This completes the proof. $\qquad\square$

We record the following corollary to the proof.

Corollary 3.6. *Suppose that $\overline{V}$ is a weight for M and that $\chi_M : Z_M \to \overline{k}^\times$ such that $\chi_M|_{Z_M(\mathcal{O})}$ is the central character of $\overline{V}$. Then there is an algebra homomorphism $\chi : \mathcal{H} \to \overline{k}$ such that $\chi(T_h^M) = \chi_M(h)^{-1}$ for all $h \in Z_M^-$ and we have*

$$\text{c-Ind}_{\overline{P}}^G \overline{V} \otimes_{\mathcal{H},\chi} \overline{k} \xrightarrow{\sim} \mathrm{Ind}_P^G(\text{c-Ind}_{M(\mathcal{O})Z_M}^M \overline{V}).$$

On the right-hand side we let $h \in Z_M$ act on $\overline{V}$ by $\chi_M(h)$.

Proof. This isomorphism was obtained in the above proof, in case $\overline{V}$ is of the form $V_{\overline{N}(k)}$, where V is M-regular, and χ is the restriction of an algebra homomorphism $\mathcal{H}_M(V_{\overline{N}(k)}) \to \overline{k}$. But it did not matter that $\overline{V}$ was of that form. (It is anyway, by Lemma 2.5.) Moreover, by Prop. 4.1 and Cor. 4.2 (or directly) the pair (M, χ_M) gives rise to $\chi : \mathcal{H}_M(\overline{V}) \to \overline{k}$ such that $\chi(T_h^M) = \chi_M(h)^{-1}$ for $h \in Z_M^-$. $\qquad\square$

If $G = \mathrm{GL}_n$ and $P = B$ is the Borel, this recovers the results of Schneider–Stuhler [SS91, Prop. 11] ($\chi = 1$) and Vignéras [Vig04, Thm. 4.10] (χ arbitrary). In those cases the right-hand side simplifies to the principal series $\mathrm{Ind}_B^G(\chi_T)$.

4. Hecke eigenvalues and supersingularity

The following proposition allows one to compare Hecke eigenvalues between different weights. It is analogous to the classical result parameterising unramified Hecke eigenvalues by unramified characters of the torus [Car79, Cor. 4.2]. It will be convenient to define a *standard Levi* to be the unique Levi subgroup containing T of a standard parabolic subgroup.

Proposition 4.1. *There is a natural bijection between $\chi \in \mathrm{Hom}_{\bar{k}\text{-}alg}(\mathcal{H}_G(V), \bar{k})$ and pairs (M, χ_M), where M is a standard Levi and $\chi_M : Z_M \to \bar{k}^\times$ a character such that $\chi_M|_{Z_M(\mathcal{O})}$ is the central character of $V_{\overline{N}(k)}$.*

Given such a pair (M, χ_M) the corresponding set of eigenvalues is the composite of algebra homomorphisms

$$\chi : \mathcal{H}_G(V) \xrightarrow{\,'\mathcal{S}_G} \mathcal{H}_T^-(V_{\overline{U}(k)}) \xrightarrow{\chi'} \bar{k},$$

where $\chi'(\varphi) = \sum_{Z_M(\mathcal{O}) \backslash Z_M} \varphi(z) \chi_M(z)^{-1}$.

We will say in the following that $\chi \in \mathrm{Hom}_{\bar{k}\text{-alg}}(\mathcal{H}_G(V), \bar{k})$ is *parameterised by the pair* (M, χ_M) if they correspond under the bijection in the proposition. Suppose now that (M, χ_M) consists of a standard Levi M and an arbitrary smooth character $\chi_M : Z_M \to \bar{k}^\times$. Then there may be more than one weight V such that the central character of $V_{\overline{N}(k)}$ equals $\chi_M|_{Z_M(\mathcal{O})}$; for each one we obtain a corresponding algebra homomorphism $\mathcal{H}_G(V) \to \bar{k}$. In §9 we will see that if $G = \mathrm{GL}_n$ and π is an irreducible admissible G-representation, then all Hecke eigenvalues in all weights of π are identified in this manner.

We will see below that M is in fact the smallest standard Levi such that χ factors through $'\mathcal{S}_G^M : \mathcal{H}_G(V) \to \mathcal{H}_M(V_{\overline{N}(k)})$ (as an algebra homomorphism). We have the following immediate consequence.

Corollary 4.2. *In the situation of Prop. 4.1, we have for $\lambda \in X_*(T)_-$,*

$$\chi'(\tau_\lambda) = \begin{cases} \chi_M(\lambda(\varpi))^{-1} & \text{if } \lambda(\varpi) \in Z_M, \\ 0 & \text{otherwise.} \end{cases}$$

Given $\chi \in \mathrm{Hom}_{\bar{k}\text{-alg}}(\mathcal{H}_G(V), \bar{k})$ we can consider its Satake transform $\chi' \in \mathrm{Hom}_{\bar{k}\text{-alg}}(\mathcal{H}_T^-(V_{\overline{U}(k)}), \bar{k})$. We will say that χ' *vanishes on an open subset* $X \subset T^-$ if χ' vanishes on all elements of $\mathcal{H}_T^-(V_{\overline{U}(k)})$ that are supported on X. Then χ' has a well-defined *support*, namely the complement in T^- of the biggest open subset on which it vanishes. It has the following alternative description: under the isomorphism $T^-/T(\mathcal{O}) \xrightarrow{\sim} X_*(T)_-$ it corresponds to a subset of $X_*(T)_-$. And this is precisely the subset of $X_*(T)_-$ on which $\chi' : \bar{k}[X_*(T)_-] \to \bar{k}$ is non-vanishing.

Lemma 4.3. *Suppose $P = MN$ is a standard parabolic. A $\bar{k}$-algebra homomorphism $\chi : \mathcal{H}_G(V) \to \bar{k}$ factors through $'\mathcal{S}_G^M$ if and only if $\mathrm{supp}\,\chi' \supset Z_M^- T(\mathcal{O})$.*

Proof. As we noted in the proof of Prop. 2.12, $\mathcal{H}_M(V_{\overline{N}(k)})$ is the localisation of $\mathcal{H}_G(V)$ at any element φ such that the support of ${}'\mathcal{S}_G(\varphi)$ is a single $T(\mathcal{O})$-coset in Z_M^{--}. If $\operatorname{supp}\chi' \supset Z_M^- T(\mathcal{O})$ then $\chi(\varphi) = \chi'({}'\mathcal{S}_G(\varphi)) \neq 0$ and so χ factors through $\mathcal{H}_M(V_{\overline{N}(k)})$.

Conversely, if χ factors through ${}'\mathcal{S}_G^M$, then χ' extends to the subalgebra of $\mathcal{H}_T(V_{\overline{U}(k)})$ consisting of those elements whose support in T is antidominant for M. Since $Z_M T(\mathcal{O})$ is a subgroup of T that is contained in the part of T that is antidominant for M, it follows that $\operatorname{supp}\chi' \supset Z_M^- T(\mathcal{O})$. □

Proof of Proposition 4.1. By the proof of Cor. 1.5 in [Her11], $\operatorname{supp}\chi'$ is of the form $Z_M^- T(\mathcal{O})$ for some standard parabolic $P = MN$. We will show that algebra homomorphisms χ with $\operatorname{supp}\chi' = Z_M^- T(\mathcal{O})$ biject with characters $\chi_M : Z_M \to \bar{k}^\times$ such that $\chi_M|_{Z_M(\mathcal{O})}$ is the central character of $V_{\overline{N}(k)}$. By Lemma 4.3, χ factors through an algebra homomorphism $\widetilde{\chi} : \mathcal{H}_M(V_{\overline{N}(k)}) \to \bar{k}$. It is not hard to verify that $\widetilde{\chi}'$ has support $Z_M T(\mathcal{O})$. Thus by replacing $(M, V_{\overline{N}(k)}, \widetilde{\chi})$ by (G, V, χ), we are reduced to the case $\operatorname{supp}\chi' = Z T(\mathcal{O})$.

Let us write $\mathcal{H}_T^-(V_{\overline{U}(k)}) = \mathcal{H} \oplus \mathcal{I}$, where $\mathcal{H}$ (resp., $\mathcal{I}$) consists of those elements φ whose support is contained in $Z T(\mathcal{O})$ (resp., disjoint from $Z T(\mathcal{O})$). Clearly $\mathcal{H}$ is a subalgebra and $\mathcal{I}$ is an ideal, so the χ' with support $Z T(\mathcal{O})$ biject with $\operatorname{Hom}_{\bar{k}\text{-alg}}(\mathcal{H}, \bar{k})$. By restricting functions to Z, $\mathcal{H}$ is isomorphic to

$$\{\varphi : Z \to \bar{k} : \varphi(z_0 z) = \omega(z_0)\varphi(z) \ \forall z_0 \in Z(\mathcal{O}), z \in Z; \ \operatorname{supp}\varphi \text{ cpt.}\}$$

(as algebra under convolution), where ω is the central character of V. The linear dual of this space consists of all functions $f : Z \to \bar{k}$ such that $f(z_0 z) = \omega(z_0)^{-1} f(z)$ under the pairing $\langle \varphi, f \rangle = \sum_{Z(\mathcal{O}) \backslash Z} \varphi(z) f(z)$. A simple argument shows that the linear map $\mathcal{H} \to \bar{k}$ induced by f is an algebra homomorphism if and only if f is a homomorphism $Z \to \bar{k}^\times$. Finally we take the inverse of this homomorphism. □

Lemma 4.4. *Suppose that π is a smooth G-representation that has a central character ω_π. Suppose that V is a weight and that $\chi : \mathcal{H}_G(V) \to \bar{k}$ is a set of Hecke eigenvalues on $\operatorname{Hom}_K(V, \pi)$. If χ is parameterised by the pair (M, χ_M), then $\chi_M|_Z = \omega_\pi$.*

Proof. It is enough to show that $\chi_M(z) = \omega_\pi(z)$ for all z of the form $\lambda(\varpi) \in Z$. First, note that ${}'\mathcal{S}_G(T_\lambda) = \tau_\lambda$: this follows, for example, directly from the definition of ${}'\mathcal{S}_G$ or from (2.7). By Cor. 4.2, $\chi(T_\lambda) = \chi_G(z)^{-1}$. Finally, consider the induced G-linear map $\text{c-Ind}_K^G V \to \pi$. It follows from (2.10) that $\chi(T_\lambda) = \omega_\pi(z)^{-1}$. □

Lemma 4.5. *Suppose that $P = MN$ is a standard parabolic and that σ is an admissible M-representation. Suppose that V is a weight for G. Then the Hecke eigenvalues of V in $\operatorname{Ind}_{\bar{P}}^G \sigma$ and of $V_{\overline{N}(k)}$ in σ are parameterised by the same set of pairs (L, χ_L). In particular, $L \subset M$ in each case.*

Proof. By Lemma 2.14 the possible $\mathcal{H}_G(V)$-eigenvalues in $\mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{P}}^G \sigma)$ are obtained from the possible $\mathcal{H}_M(V_{\overline{N}(k)})$-eigenvalues in $\mathrm{Hom}_{M(\mathcal{O})}(V_{\overline{N}(k)}, \sigma)$ by composing with $'\mathcal{S}_G^M$. By construction, the pair associated to $\chi : \mathcal{H}_M(V_{\overline{N}(k)}) \to \bar{k}$ is the same as the pair associated to $\chi \circ '\mathcal{S}_G^M$. $\square$

Lemma 4.6. *Suppose that π is a smooth G-representation and that $f \in \mathrm{Hom}_K(V, \pi)$ is an $\mathcal{H}_G(V)$-eigenvector. Suppose that $\eta : G \to \bar{k}^\times$ is a smooth character. If the Hecke eigenvalues of f are parameterised by (M, χ_M), then the Hecke eigenvalues of $f \otimes \eta$ are parameterised by $(M, \chi_M \eta|_{Z_M})$.*

Proof. Since $\mathrm{c\text{-}Ind}_K^G(V \otimes \eta) \cong \mathrm{c\text{-}Ind}_K^G V \otimes \eta$, we have a natural isomorphism $\mathcal{H}_G(V) \xrightarrow{\sim} \mathcal{H}_G(V \otimes \eta)$, $\varphi \mapsto \varphi_\eta$, under which the Hecke eigenvalues of f and $f \otimes \eta$ are identified. A calculation shows that $\varphi_\eta(g) = \eta(g)\varphi(g)$ when we canonically identify $\mathrm{End}\, V$ and $\mathrm{End}(V \otimes \eta)$, which implies that $'\mathcal{S}_G(\varphi_\eta)(t) = \eta(t)'\mathcal{S}_G(\varphi)(t)$. (Note that η is trivial on $\overline{U}$, as it kills all pro-p subgroups.) The claim now follows from the formula for χ' in Prop. 4.1. $\square$

Finally we define what it means for an irreducible admissible representation π to be supersingular. Roughly speaking, all Hecke eigenvalues occurring in π should be as trivial as possible. We fixed a hyperspecial maximal compact subgroup K of $G(F)$ at the beginning, but our definition should be independent of this choice and we should take into account all of them. (We thank M.-F. Vignéras for this observation.) When $G = \mathrm{GL}_n$ all hyperspecial maximal compact subgroups of $G(F)$ are conjugate and we can just work with our fixed choice of K.

Suppose the reductive group scheme $G'_{/\mathcal{O}}$ is another reductive integral structure of $G_{/F}$. Fix a maximal split torus T' of G' and a Borel subgroup B' containing T'. Let $K' = G'(\mathcal{O})$. We claim that if K and K' are $G(F)$-conjugate, then even (K, T, B) and (K', T', B') are $G(F)$-conjugate. Without loss of generality we may assume that $K = K'$. By [Tit79, 3.8.1], [BT84, II.5.1.40] reductive integral structures of $G_{/F}$ are naturally in bijection with hyperspecial points of the reduced building. In this bijection G corresponds to the unique fixed point of $G(\mathcal{O})$ in the reduced building. Thus $G = G'$. Now by the same argument that was used towards the end of the proof of Lemma 2.16 there is an element of $G(\mathcal{O})$ that conjugates (T, B) to (T', B').

A K'-*weight* is an irreducible representation V' of $G'(k)$, or equivalently of K'. (Hence a K-weight is a weight.) We denote the corresponding Hecke algebra by $\mathcal{H}_{G,K'}(V')$ to emphasise the dependence on K'.

Definition 4.7. Let π be an irreducible admissible representation. We say that π is *supersingular* if for all triples (K', T', B') as above, for all K'-weights V' and for all Hecke eigenvalues χ' on $\mathrm{Hom}_{K'}(V', \pi)$ the following equivalent conditions hold.

    (i) χ' is parameterised by the pair (G, ω_π).
    (ii) χ' is parameterised by a pair (G, χ_G) for some $\chi_G : Z \to \bar{k}^\times$.

(iii) χ' does not factor through $'\mathcal{S}_G^{M'} : \mathcal{H}_{G,K'}(V') \to \mathcal{H}_{M',M'(\mathcal{O})}(V'_{\overline{N'}(k)})$
for any proper parabolic $P' = M'N'$ containing B'.

Note that the first two conditions are equivalent by Lemma 4.4 and the last two conditions are equivalent by Lemma 4.3 and the proof of Prop. 4.1.

It is easy to see that in this definition, the triple (K', T', B') only matters up to $G(F)$-conjugacy. Thus we only need to let K' run through a set of representatives for the finitely many conjugacy classes of hyperspecial maximal compact subgroups in $G(F)$ (and choose compatible T', B' for each).

Suppose now that $G = \mathrm{GL}_n$. By the above, the supersingularity condition has to be checked only for our fixed choice (K, T, B). Even better, as a corollary to our main results we will see that it has to be checked only for one weight V and for one χ. (See Cor. 9.10.) Moreover supersingularity can be characterised using parabolic inductions. (See Cor. 9.13.)

5. Computing the Satake transform

In this section we determine the inverse of the mod p Satake transform explicitly. We deduce this from the corresponding result over the complex numbers, the Lusztig–Kato formula. The final result has a very simple shape: since $q = 0$ in $\bar{k}$, only the constant terms of the intervening Kazhdan–Lusztig polynomials matter.

If M is a standard Levi, we will denote by $\geq_M$ the usual partial order on $X_*(T)$ with respect to M, i.e., $\lambda \geq_M \mu$ means that $\lambda - \mu$ is a non-negative integral linear combination of the simple coroots of M. We also write $\geq$ for $\geq_G$.

Proposition 5.1. *Suppose that the derived subgroup of G is simply connected. Let V be a weight. Let M be the standard Levi subgroup such that* $\mathrm{Stab}_W(V^{U(k)}) = W_M$. *Then for all $\mu \in X_*(T)_-$,*

$$\tau_\mu = \sum_{\substack{\lambda \in X_*(T)_- \\ \lambda \geq_M \mu}} \mathcal{S}_G(T_\lambda).$$

The assumption on G should be unnecessary.

Proof. We can easily reduce to the case when $M = G$ by Cor. 2.18(ii). Just note that if $\lambda \in X_*(T)$ is antidominant for M and $\lambda \geq_M \mu$, then $\lambda \in X_*(T)_-$. (The point is that $\langle \alpha^\vee, \beta \rangle \leq 0$ for all α, $\beta \in \Delta$.)

Next we will reduce to the case when V is the trivial weight, by showing that the coefficients of $\mathcal{S}_G(T_\lambda)$ in the basis $(\tau_\mu)_\mu$ do not depend on V. Since the derived subgroup of G is simply connected, we can write $V = F(\nu)$ for some q-restricted weight ν. We have $\nu \in X^0(T)$ since $\mathrm{Stab}_W(\nu) = W$. The dual Weyl module $H^0_{\mathcal{O}}(\nu)$ (see [Jan03, §II.8]) of $G_{/\mathcal{O}}$ of highest weight ν is free of rank one (for example by the Weyl character formula [Jan03, Prop. II.5.10]). In particular there is a character $\widetilde{\nu} : G \to F^\times$ that agrees

with ν on T and whose reduction modulo ϖ is the one-dimensional represen-
tation V. (Note that $\widetilde{\nu}(K) \subset \mathcal{O}^\times$, as $K = G(\mathcal{O})$ is compact. Alternatively
we could obtain $\widetilde{\nu}$ using the isogeny of F-tori $T \to G/G'$, where G' is the
derived subgroup of G.) Let $\lambda, \mu \in X_*(T)_-$ and put $t := \lambda(\varpi)$, $t' := \mu(\varpi)$.
For $g = k_1 t k_2 \in KtK$ we see that $T_\lambda(g) = k_1 k_2 \in \mathrm{End}_{\bar{k}}(V) = \bar{k}$, i.e.,
$T_\lambda(g) = \overline{\widetilde{\nu}(g)\widetilde{\nu}(t)^{-1}} \in \bar{k}^\times$. Now we follow a standard argument (e.g., [Gro98,
§3]). By the Iwasawa decomposition we can write $KtK = \coprod_{i=1}^r t_i u_i K$ with
$t_i = \lambda_i(\varpi)$, $\lambda_i \in X_*(T)$, $u_i \in U$. Then $t_i u_i K \cap t'U \neq \varnothing$ if and only if $\lambda_i = \mu$,
in which case the intersection equals $t'u_i U(\mathcal{O})$. Thus the coefficient of τ_μ in
$\mathcal{S}_G(T_\lambda)$ equals

$$\sum_{i:\lambda_i=\mu} T_\lambda(t'u_i) = \#\{i : \lambda_i = \mu\} \cdot \overline{\widetilde{\nu}(t')\widetilde{\nu}(t)^{-1}} = \#\{i : \lambda_i = \mu\},$$

where the last equality follows since $\mu \geq_\mathbb{R} \lambda$ if $KtK \cap t'U \neq \varnothing$ [Her11,
Lemma 3.6] and thus $\widetilde{\nu}(t't^{-1}) = \varpi^{\langle \nu, \mu-\lambda \rangle} = 1$. It is therefore indeed inde-
pendent of $\nu \in X^0(T)$.

We now prove the proposition in case V is trivial. Recall the Lusztig–
Kato formula [HKP10, Thm. 7.8.1] (see also [Gro98, §4]). This is an identity
in $\mathbb{Z}[q^{1/2}, q^{-1/2}][X_*(T)]$ which, when the variable $q^{1/2}$ is specialised to a
complex square root of $q = \#k$, gives the following in $\mathbb{C}[X_*(T)]$:

$$\mathrm{ch}\, V_\mu = \sum_{\lambda \leq \mu} q^{-\langle \mu, \rho \rangle} P_{w_\lambda, w_\mu}(q) 1^\vee_{K\lambda(\varpi)K}.$$

Here λ, μ denote any *dominant* coweights, V_μ is the irreducible complex
representation of highest weight μ of the dual group, ρ is the half-sum of all
positive roots, w_λ is the element $\lambda \cdot w_0$ in the extended affine Weyl group
$\widetilde{W} = X_*(T) \rtimes W$, where w_0 is the longest Weyl element, and $1^\vee_{K\lambda(\varpi)K}$ denotes
the classical (i.e., normalised) Satake transform of the characteristic function
of $K\lambda(\varpi)K$. Besides, for any elements $w \leq w'$ in $\widetilde{W}$, $P_{w,w'}(q) \in 1 + q\mathbb{Z}[q]$
denotes the corresponding Kazhdan–Lusztig polynomial. We remark that
$\lambda \leq \mu$ if and only if $w_\lambda \leq w_\mu$ in $\widetilde{W}$.

For any $\mu' \in X_*(T)$ take the coefficient of μ' in the above formula and
rescale:

$$q^{\langle \mu - w_0\mu', \rho \rangle} \dim V_\mu(\mu') = \sum_{\lambda \leq \mu} P_{w_\lambda, w_\mu}(q) \sum_{U/U(\mathcal{O})} 1_{K\lambda(\varpi)K}(\mu'(\varpi)u).$$

Here $V_\mu(\mu')$ denotes the μ'-weight space in V_μ. Also note that $\delta_B^{1/2}(\mu'(\varpi)) = q^{\langle w_0\mu', \rho \rangle}$ for the modulus character of B.

Consider the left-hand side. We have $V_\mu(\mu') = 0$ unless $w_0\mu \leq \mu' \leq \mu$. If
these inequalities hold, then $\langle \mu - w_0\mu', \rho \rangle \geq 0$ with equality if and only if
$\mu = w_0\mu'$. Therefore we may reduce both sides modulo p and obtain

$$\tau_{w_0\mu} = \sum_{\lambda \leq \mu} \mathcal{S}_G(T_{w_0\lambda}),$$

noting again that the Kazhdan–Lusztig polynomials have constant coeffi-cient 1. Finally we interchange λ with $w_0\lambda$ and μ with $w_0\mu$. $\square$

6. Maps between compact inductions

6.1. **Generalities.** Suppose that we are given compact open subgroups H_i of G and finite-dimensional smooth H_i-representations V_i ($i = 1, 2$). We define

$$\mathcal{H}_{H_1,H_2}(V_1, V_2) := \operatorname{Hom}_G(\text{c-Ind}_{H_1}^G V_1, \text{c-Ind}_{H_2}^G V_2).$$

If moreover V_3 is a finite-dimensional smooth representation of a compact open subgroup H_3, we have a natural bilinear map given by composition

$$(6.1) \qquad \mathcal{H}_{H_2,H_3}(V_2, V_3) \times \mathcal{H}_{H_1,H_2}(V_1, V_2) \to \mathcal{H}_{H_1,H_3}(V_1, V_3).$$

In particular $\mathcal{H}_{H_1,H_2}(V_1, V_2)$ is a Hecke bimodule, with $\mathcal{H}_{H_1}(V_1)$ acting on the right and $\mathcal{H}_{H_2}(V_2)$ acting on the left.

By Frobenius reciprocity, $\mathcal{H}_{H_1,H_2}(V_1, V_2)$ is isomorphic to $\operatorname{Hom}_{H_1}(V_1, \text{c-Ind}_{H_2}^G V_2)$ and thus, by thinking of it inside the space of functions on $G \times V_1 \to V_2$, to

$$\{\varphi : G \to \operatorname{Hom}_{\bar{k}}(V_1, V_2) : \operatorname{supp}\varphi \text{ compact},$$
$$\varphi(h_2 g h_1) = h_2 \circ \varphi(g) \circ h_1 \ \forall h_i \in H_i, g \in G\}.$$

In this language the composition (6.1) is given by convolution: $(\varphi' * \varphi)(g) = \sum_{G/H_2} \varphi'(gx)\varphi(x^{-1})$.

If $H_1 = H_2 = K$ and the V_i are weights, we just write $\mathcal{H}_G(V_1, V_2)$ for $\mathcal{H}_{K,K}(V_1, V_2)$.

Proposition 6.2. *Suppose that V_1, V_2 are two weights. Then $\mathcal{H}_G(V_1, V_2)$ is non-zero if and only if $V_1^{U(k)} \cong V_2^{U(k)}$ as $T(k)$-representations.*

If this is satisfied, the Hecke algebras $\mathcal{H}_G(V_1)$ and $\mathcal{H}_G(V_2)$ can naturally be identified via $\mathcal{S}_G$. Under this identification the actions of the two Hecke algebras on $\mathcal{H}_G(V_1, V_2)$ agree. If moreover the centre of G is connected and the derived subgroup of G is simply connected then $\mathcal{H}_G(V_1, V_2)$ is a free module of rank one under $\mathcal{H}_G(V_1) \cong \mathcal{H}_G(V_2)$.

This follows immediately from the following proposition and its proof.

Proposition 6.3. *Suppose that V_1, V_2 are two weights. The Satake transform*

$$\mathcal{S}_G : \mathcal{H}_G(V_1, V_2) \to \mathcal{H}_T(V_1^{U(k)}, V_2^{U(k)})$$

$$\varphi \mapsto \left(t \mapsto \sum_{u \in U/U(\mathcal{O})} \varphi(tu)\Big|_{V_1^{U(k)}}\right)$$

is injective. The transform is compatible with compositions:

(6.4)
$$
\begin{array}{ccccc}
\mathcal{H}_G(V_2, V_3) & \times & \mathcal{H}_G(V_1, V_2) & \longrightarrow & \mathcal{H}_G(V_1, V_3) \\
\downarrow{\scriptstyle \mathcal{S}_G} & & \downarrow{\scriptstyle \mathcal{S}_G} & & \downarrow{\scriptstyle \mathcal{S}_G} \\
\mathcal{H}_T(V_2^{U(k)}, V_3^{U(k)}) & \times & \mathcal{H}_T(V_1^{U(k)}, V_2^{U(k)}) & \longrightarrow & \mathcal{H}_T(V_1^{U(k)}, V_3^{U(k)})
\end{array}
$$

If $V_1^{U(k)} \cong V_2^{U(k)}$ as $T(k)$-representation, G has connected centre, and the derived subgroup of G is simply connected, the image of $\mathcal{S}_G$ is a free module of rank one under the compatible actions of $\mathcal{H}_T^-(V_1^{U(k)}) \cong \mathcal{H}_T^-(V_2^{U(k)})$.

Proof. That the transform is well defined and compatible with compositions is formal using the Iwasawa decomposition and follows exactly the same steps as in the case when $V_1 = V_2$ (see [Her11]).

Suppose $\lambda \in X_*(T)_-$. By Step 1 of the proof of [Her11, Thm. 1.2], we see that the vector space of $\varphi \in \mathcal{H}_G(V_1, V_2)$ that are supported on $K\lambda(\varpi)K$ is one-dimensional if $V_1^{N_{-\lambda}(k)} \cong V_2^{N_{-\lambda}(k)}$ as $M_\lambda(k)$-representations and zero otherwise. Since $N_{-\lambda}(k) \subset U(k)$ it follows that $\mathcal{H}_G(V_1, V_2) \neq 0$ implies $V_1^{U(k)} \cong V_2^{U(k)}$ as $T(k)$-representation. To see the converse, suppose $V_1^{U(k)} \cong V_2^{U(k)}$ and choose any $\lambda \in X_*(T)_-$ such that $\langle \lambda, \alpha \rangle < 0$ for all $\alpha \in \Delta$. Then $N_{-\lambda} = U$, so that there exists a non-zero Hecke operator supported on $K\lambda(\varpi)K$.

We pick $\varphi \neq 0$ in $\mathcal{H}_G(V_1, V_2)$ and show that $\mathcal{S}_G(\varphi)$ is non-zero. Suppose first that φ is supported on $K\lambda(\varpi)K$ for some $\lambda \in X_*(T)_-$. For $\mu \in X_*(T)$ the same argument as in Step 3 of [Her11, Thm. 1.2] shows that $\mathcal{S}_G(\varphi)(\mu(\varpi)) \neq 0$ implies $\mu \geq_{\mathbb{R}} \lambda$ and that $\mathcal{S}_G(\varphi)(\lambda(\varpi)) \neq 0$. For general φ, picking a minimal $\lambda \in X_*(T)_-$ for $\geq_{\mathbb{R}}$ such that φ is non-zero on $K\lambda(\varpi)K$, we see that $\mathcal{S}_G(\varphi)(\lambda(\varpi)) \neq 0$. Thus $\mathcal{S}_G$ is injective. (This gives an alternative proof that $\mathcal{H}_G(V_1, V_2) = 0$ unless $V_1^{U(k)} \cong V_2^{U(k)}$.)

Let us now assume that $V_1^{U(k)} \cong V_2^{U(k)}$ and that G has connected centre and simply connected derived subgroup. We first show that there is a $\lambda_0 \in X_*(T)_-$ such that there is a non-zero $\varphi \in \mathcal{H}_G(V_1, V_2)$ supported on $K\lambda(\varpi)K$ if and only if $\lambda - \lambda_0 \in X_*(T)_-$. We can write $V_i \cong F(\nu_i)$ with ν_i q-restricted. As $V_1^{U(k)} \cong V_2^{U(k)}$, we have $\nu_1 - \nu_2 \in (q-1)X^*(T)$. There is a non-zero $\varphi \in \mathcal{H}_G(V_1, V_2)$ supported on $K\lambda(\varpi)K$ if and only if $V_1^{N_{-\lambda}(k)} \cong V_2^{N_{-\lambda}(k)}$ if and only if $\langle \nu_1 - \nu_2, \alpha \rangle = 0$ for all roots α of M_λ, i.e., for all $\alpha \in \Phi$ such that $\langle \lambda, \alpha \rangle = 0$ (the last step uses [Her11, Lemma 2.5] and Prop. 1.3 in the appendix of [Her09]). Since the centre of G is connected we may choose $\lambda_0 \in X_*(T)_-$ such that for all simple roots α, $\langle \lambda_0, \alpha \rangle = 0$ if $\langle \nu_1 - \nu_2, \alpha \rangle = 0$ and $\langle \lambda_0, \alpha \rangle = -1$ otherwise. This clearly satisfies the desired property.

We can canonically identify $\mathcal{H}_T(V_1^{U(k)})$, $\mathcal{H}_T(V_2^{U(k)})$ and we denote them simply by $\mathcal{H}_T$. By Thm. 2.6 and (6.4) we see that $\mathcal{H}_T^-$ preserves the image of $\mathcal{S}_G$. Choose $\varphi_0 \in \mathcal{H}_G(V_1, V_2)$ that is non-zero and supported on $K\lambda_0(\varpi)K$.

We show below that $\mathcal{H}_T^- \mathcal{S}_G(\varphi_0) = \mathrm{im}(\mathcal{S}_G)$. Since $\mathcal{H}_T \cong \bar{k}[X_*(T)]$ is an integral domain, this completes the proof.

Claim: For all $\lambda \in \lambda_0 + X_*(T)_-$ there is a $\psi_\lambda \in \mathcal{H}_T^- \mathcal{S}_G(\varphi_0)$ such that for all $\mu \in \lambda_0 + X_*(T)_-$, $\psi_\lambda(\mu(\varpi)) \neq 0$ if and only if $\mu = \lambda$.

We prove the claim by induction with respect to $\geq_{\mathbb{R}}$, noting that $\{\mu \in X_*(T)_- : \mu \geq_{\mathbb{R}} \lambda\}$ is finite. Consider $\tau_{\lambda-\lambda_0} \mathcal{S}_G(\varphi_0) \in \mathcal{H}_T^- \mathcal{S}_G(\varphi_0)$. By the above this is zero on $\mu(\varpi)$ unless $\mu \geq_{\mathbb{R}} \lambda$ and non-zero on $\lambda(\varpi)$. If there is no $\mu >_{\mathbb{R}} \lambda$ in $\lambda_0 + X_*(T)_-$ we are done. Otherwise by induction we can subtract off multiples of ψ_μ for such μ to find a ψ_λ.

Finally given $\psi \in \mathrm{im}(\mathcal{S}_G)$, by the claim we can subtract off multiples of the $\psi_\lambda \in \mathcal{H}_T^- \mathcal{S}_G(\varphi_0)$ to assume without loss of generality that $\psi(\lambda(\varpi)) = 0$ for all $\lambda \in \lambda_0 + X_*(T)_-$. Then the above argument for the injectivity of $\mathcal{S}_G$ shows that $\psi = 0$. $\qquad\square$

Corollary 6.5. *Suppose V is a weight. Then c-$\mathrm{Ind}_K^G V$ is a torsion-free $\mathcal{H}_G(V)$-module.*

Proof. If c-$\mathrm{Ind}_K^G V$ has torsion, there is a non-zero $\varphi : \text{c-}\mathrm{Ind}_K^G V \to \text{c-}\mathrm{Ind}_K^G V$ in $\mathcal{H}_G(V)$ that has non-zero kernel. As a non-zero smooth G-representation, the kernel has to contain a weight V'. By Frobenius reciprocity we get a non-zero map c-$\mathrm{Ind}_K^G V' \to \text{c-}\mathrm{Ind}_K^G V$ whose composite with φ is zero. More generally suppose that a composite of non-zero maps $\varphi_1 : \text{c-}\mathrm{Ind}_K^G V_1 \to \text{c-}\mathrm{Ind}_K^G V_2$ and $\varphi_2 : \text{c-}\mathrm{Ind}_K^G V_2 \to \text{c-}\mathrm{Ind}_K^G V_3$ is zero. By Prop. 6.2, the $T(k)$-representations $V_1^{U(k)}$, $V_2^{U(k)}$, and $V_3^{U(k)}$ are isomorphic, and we identify them (non-canonically). Then for all i, j the bimodules $\mathcal{H}_T(V_i^{U(k)}, V_j^{U(k)})$ are all naturally identified with the integral domain $\mathcal{H}_T(V_1^{U(k)})$ such that moreover for any triple (i, j, k) the bimodule multiplication corresponds to the ring multiplication. Since $\mathcal{S}_G(\varphi_2) * \mathcal{S}_G(\varphi_1) = 0$ and $\mathcal{S}_G$ is injective, one of φ_1, φ_2 has to be zero. $\qquad\square$

6.2. The minuscule case.

Let V, V' denote distinct weights such that $V^{U(k)} \cong (V')^{U(k)}$. As we just saw, in this case we can identify $\mathcal{H}_G(V)$ and $\mathcal{H}_G(V')$, and we will denote them simply by $\mathcal{H}_G$. The goal of this subsection is to find an explicit criterion when

$$(6.6) \qquad \text{c-}\mathrm{Ind}_K^G V \otimes_{\mathcal{H}_G,\chi} \bar{k} \cong \text{c-}\mathrm{Ind}_K^G V' \otimes_{\mathcal{H}_G,\chi} \bar{k},$$

in case V and V' differ only minimally (in the sense that $V^{N(k)} \cong (V')^{N(k)}$ as $M(k)$-representations for some maximal parabolic $P = MN$) and provided there exists a corresponding minuscule fundamental coweight. We remark that when $G = \mathrm{GL}_n$, any simple root admits a minuscule fundamental coweight.

An isomorphism as in (6.6) is extremely useful because it allows us to "change the weight" in a smooth G-representation π. The point is that $\mathrm{Hom}_G(\text{c-}\mathrm{Ind}_K^G V \otimes_{\mathcal{H}_G,\chi} \bar{k}, \pi) \neq 0$ is equivalent to saying that V occurs in π with Hecke eigenvalues χ. So if (6.6) holds and V occurs in π with Hecke

eigenvalues χ, then V' occurs in π with Hecke eigenvalues χ. The results in this subsection will play a key role in the proofs of Theorems 8.1 and 9.8.

Suppose that G has simply connected derived subgroup. This implies that simple coroots possess fundamental weights. Fix a simple root α. We denote by ω_α a fundamental weight for $\alpha^\vee$, i.e., $\langle \beta^\vee, \omega_\alpha \rangle = \delta_{\alpha\beta}$ for all simple roots β. We make the following assumption.

There exists a minuscule fundamental coweight $-\lambda$ associated to α.

Recall that this means that $\langle -\lambda, \beta \rangle = \delta_{\alpha\beta}$ for $\beta \in \Delta$ and $\langle -\lambda, \gamma \rangle \in \{0, 1\}$ for all $\gamma \in \Phi^+$. In particular, $\lambda \in X_*(T)_-$.

Suppose that ν is a q-restricted weight satisfying $\langle \nu, \alpha^\vee \rangle = 0$. We let $\nu' := \nu + (q-1)\omega_\alpha$ (again q-restricted) and define weights $V := F(\nu)$, $V' := F(\nu')$. Note that $V^{U(k)} \cong (V')^{U(k)}$ as $T(k)$-representations. There exist Hecke operators $\varphi_\lambda^+ \in \mathcal{H}_G(V, V')$ and $\varphi_\lambda^- \in \mathcal{H}_G(V', V)$ whose support is $K\lambda(\varpi)K$. (In fact λ is precisely a possible λ_0 in the proof of Prop. 6.3.)

Proposition 6.7. *With the above notation, $\mathcal{S}_G(\varphi_\lambda^- * \varphi_\lambda^+) \in \mathcal{H}_T^-$ has support $\lambda(\varpi)^2 T(\mathcal{O}) \cup \lambda(\varpi)^2 \alpha^\vee(\varpi) T(\mathcal{O})$. The values at $\lambda(\varpi)^2$ and $\lambda(\varpi)^2 \alpha^\vee(\varpi)$ add to zero.*

Proof. We first compute $\varphi_\lambda^- * \varphi_\lambda^+ \in \mathcal{H}_G(V)$. Let $t = \lambda(\varpi)$.

Sublemma 6.8. *As $-\lambda$ is minuscule, we have $K \cap {}^t K = \mathrm{red}^{-1}(P_{-\lambda}(k))$.*

Note that this is elementary when $G = \mathrm{GL}_n$.

Proof of Sublemma 6.8. By Prop. 2.15 we know that the left-hand side is contained in the right-hand side. We can rewrite the right-hand side by Lemma 2.16 as

$$N_{-\lambda}(\mathcal{O}) M_\lambda(\mathcal{O}) \ker(N_\lambda(\mathcal{O}) \to N_\lambda(k)).$$

Since $t \in Z(M_\lambda)$, we see that $M_\lambda(\mathcal{O}) \subset K \cap {}^t K$. Lemma 2.16 shows that t^{-1} contracts $N_{-\lambda}(\mathcal{O})$ so that $N_{-\lambda}(\mathcal{O}) \subset K \cap {}^t K$.

Fix an order on the set of roots α such that $\langle \lambda, \alpha \rangle > 0$. Multiplication induces an isomorphism $\prod_{\langle \lambda, \alpha \rangle > 0} U_\alpha \xrightarrow{\sim} N_\lambda$ [Jan03, II.1.7(1)]. Moreover we choose root homomorphisms $x_\alpha : \mathbb{G}_a \xrightarrow{\sim} U_\alpha$ [Jan03, II.1.2]. An element of $N_\lambda(\mathcal{O})$ can then be expressed uniquely as $\prod_{\langle \lambda, \alpha \rangle > 0} x_\alpha(u_\alpha)$ with $u_\alpha \in \mathcal{O}$. If its reduction is trivial in $N_\lambda(k)$ then $u_\alpha \in \varpi\mathcal{O}$ for all α. Then $t^{-1} \prod x_\alpha(u_\alpha) t = \prod x_\alpha(\alpha(t^{-1})u_\alpha)$, and this is contained in K as $\langle \lambda, \alpha \rangle = 1$ whenever $\langle \lambda, \alpha \rangle > 0$ (as $-\lambda$ minuscule). It follows that $\ker(N_\lambda(\mathcal{O}) \to N_\lambda(k)) \subset K \cap {}^t K$. $\square$

For each $w \in W$ choose a representative $\dot{w} \in N(T)(\mathcal{O})$. It follows from the sublemma that

$$K/(K \cap {}^t K) \cong G(k)/P_{-\lambda}(k) \cong \coprod_{W/W_\lambda} U(k) \dot{w} P_{-\lambda}(k)/P_{-\lambda}(k),$$

where we used the rational Bruhat decomposition in the last step and where W_λ denotes $W_{M_\lambda} = \mathrm{Stab}_W(\lambda)$. Therefore $KtK = \bigcup_W U(\mathcal{O}) \dot{w} t K$.

We claim that the support of $\varphi_\lambda^- * \varphi_\lambda^+$ is Kt^2K. Let $\lambda' \in X_*(T)_-$ and let $t' = \lambda'(\varpi)$. When we compute $(\varphi_\lambda^- * \varphi_\lambda^+)(t')$, by the above all terms are of the form $\varphi_\lambda^-(u\dot{w}t)\varphi_\lambda^+(t^{-1}\dot{w}^{-1}u^{-1}t')$, where $u \in U(\mathcal{O})$ and $w \in W$. As $t' \in T^-$, $(t')^{-1}u^{-1}t' \in U(\mathcal{O})$, so if the term is non-zero then so is $\varphi_\lambda^-(^wt)\varphi_\lambda^+(^wt^{-1}t')$ (where we also pulled a $\dot{w}$ from the left to the right). By the refined Cartan decomposition we need that $-w\lambda + \lambda' = w'\lambda$ for some $w' \in W$. Moreover $\varphi_\lambda^-(^wt)\varphi_\lambda^+(^{w'}t) \neq 0$, which implies that $\varphi_\lambda^-(t)\dot{w}^{-1}\dot{w}'\varphi_\lambda^+(t) \neq 0$. This in turn implies that $p_{N_\lambda}(\dot{w}^{-1}\dot{w}'(V')^{N-\lambda(k)}) \neq 0$. The proof of Lemma 2.17 then shows that $w^{-1}w' \in W_\lambda$. (Note that $\mathrm{Stab}_W(\nu') \subset W_\lambda$ since the stabiliser is generated by simple reflections and since $\langle \nu', \alpha^\vee \rangle = q - 1 > 0$.) Thus $w'\lambda = w\lambda$. As $-w\lambda + \lambda' = w'\lambda$, we obtain $\lambda' = 2w\lambda$. Taking into account that λ' and λ are antidominant coweights, we see that $w\lambda = \lambda$ and that $\lambda' = 2\lambda$. The latter equation shows that the support of $\varphi_\lambda^- * \varphi_\lambda^+$ is contained in Kt^2K. But if we take $\lambda' = 2\lambda$, the former equation shows that $u\dot{w}t = t(t^{-1}ut)\dot{w} \in tK$, so that only the trivial term $\varphi_\lambda^-(t)\varphi_\lambda^+(t)$ contributes to $(\varphi_\lambda^- * \varphi_\lambda^+)(t^2)$, but that term is clearly non-zero.

To complete the proof, we will show that $\mathcal{S}_G(T_{2\lambda}) = \tau_{2\lambda} - \tau_{2\lambda+\alpha^\vee}$. First note that $2\lambda + \alpha^\vee \in X_*(T)_-$ since for $\beta \in \Delta$ we have

$$\langle 2\lambda + \alpha^\vee, \beta \rangle = \begin{cases} 0 & \text{if } \beta = \alpha, \\ \langle \alpha^\vee, \beta \rangle \leq 0 & \text{if } \beta \neq \alpha. \end{cases}$$

Let M be the standard Levi with $W_M = \mathrm{Stab}_W(\nu)$. Note that $\alpha \in \Delta_M$. By Prop. 5.1, it suffices to show the following:

Claim: Suppose $\mu \in X_*(T)_-$. Then $\mu \geq_M 2\lambda$ if and only if $\mu = 2\lambda$ or $\mu \geq_M 2\lambda + \alpha^\vee$.

It suffices to prove the claim with $\geq_M$ replaced by $\geq$. Suppose $\mu \geq 2\lambda$, so $\mu - 2\lambda = \sum_i \beta_i^\vee$, where the β_i are simple roots. Let $\alpha_0 \in X^*(T)$ be the sum of the longest roots of all irreducible components of the root system (it need not itself be a root). Then $\alpha_0 = \sum_{\beta \in \Delta} n_\beta \beta$ with $n_\beta \geq 1$. Moreover α_0 is dominant and $\langle \lambda, \alpha_0 \rangle = -n_\alpha = -1$ (as $-\lambda$ is minuscule). As μ is antidominant, we find

$$\langle \mu, \alpha \rangle \geq \sum_{\beta \in \Delta} n_\beta \langle \mu, \beta \rangle = \langle \mu, \alpha_0 \rangle \geq \langle 2\lambda, \alpha_0 \rangle = -2.$$

If $\langle \mu, \alpha \rangle > -2$ then $\sum_i \langle \beta_i^\vee, \alpha \rangle > 0$, so α has to occur among the simple roots β_i and thus $\mu \geq 2\lambda + \alpha^\vee$. If $\langle \mu, \alpha \rangle = -2$, then $\langle \mu, \beta \rangle = 0$ for all $\beta \in \Delta - \{\alpha\}$. Then $\mu - 2\lambda = 0$ since it is orthogonal to all roots and contained in $\mathbb{Z}\Phi^\vee$. $\square$

Question 6.9. Suppose that the centre of G is connected so that every simple root α possesses a fundamental coweight λ. Is the result of Prop. 6.7 true when λ is not assumed to be minuscule? It seems that this is the case when $G = \mathrm{GSp}_4$.

Corollary 6.10. *We keep the above notation. Suppose $\chi \in \mathrm{Hom}_{\bar{k}\text{-}alg}(\mathcal{H}_G, \bar{k})$ is parameterised by (M, χ_M). Assume that $\alpha \notin \Delta_M$. Then*

$$(6.11) \qquad \text{c-Ind}_K^G V \otimes_{\mathcal{H}_G, \chi} \bar{k} \cong \text{c-Ind}_K^G V' \otimes_{\mathcal{H}_G, \chi} \bar{k},$$

provided either $\alpha^\vee(\varpi) \notin Z_M$ or $\chi_M(\alpha^\vee(\varpi)) \neq 1$.

We can reformulate the conditions in a way that does not use ϖ. The first is equivalent to $\langle \alpha^\vee, \beta \rangle \neq 0$ for some $\beta \in \Delta_M$, the second is equivalent to $\chi_M \circ \alpha^\vee : F^\times \to k^\times$ being non-trivial.

Proof. By the above, φ_λ^+, φ_λ^- induce G-linear maps between the two representations in (6.11). If we can show that $\chi(\varphi_\lambda^- * \varphi_\lambda^+) = \chi(\varphi_\lambda^+ * \varphi_\lambda^-)$ is non-zero, we will be done. Let χ' be the Satake transform of χ. By Prop. 6.7 it suffices to show that $\chi'(\tau_{2\lambda}) \neq \chi'(\tau_{2\lambda+\alpha^\vee})$. Recall that $\text{supp}\, \chi' = Z_M^- T(\mathcal{O})$.

Since $\langle \lambda, \beta \rangle = 0$ for all simple roots $\beta \in \Delta_M$ (as $\alpha \notin \Delta_M$), we have $\lambda(\varpi)^2 \in Z_M^-$; moreover $\lambda(\varpi)^2 \alpha^\vee(\varpi) \in Z_M^-$ if and only if $\alpha^\vee(\varpi) \in Z_M$. We now use Cor. 4.2. If $\alpha^\vee(\varpi) \notin Z_M$ then $\chi'(\tau_{2\lambda}) \neq 0 = \chi'(\tau_{2\lambda+\alpha^\vee})$. Otherwise both are non-zero and $\chi'(\tau_{2\lambda})\chi'(\tau_{2\lambda+\alpha^\vee})^{-1} = \chi_M(\alpha^\vee(\varpi)) \neq 1$.

In the latter case let us verify that $\chi_M \circ \alpha^\vee$ is unramified, i.e., trivial on $k^\times$. We have that $\chi_M|_{Z_M(k)}$ is the central character of $F(\nu)_{\overline{N}(k)}$, hence it equals $\nu|_{Z_M(k)}$. Thus if $\langle \alpha^\vee, \beta \rangle = 0$ for all $\beta \in \Delta_M$ then $\alpha^\vee(x) \in Z_M(k)$ for $x \in k^\times$, so $\chi_M(\alpha^\vee(x)) = \nu(\alpha^\vee(x)) = 1$. (Recall that $\langle \nu, \alpha^\vee \rangle = 0$.) $\qquad\square$

Remark 6.12. Equation (6.11) also holds if $\alpha \in \Delta_M$ and $\langle \alpha^\vee, \beta \rangle = 0$ for all $\beta \in \Delta_M - \{\alpha\}$.

We showed in [Her11] (Step 4 of the proof of Thm. 1.2) that the image of $\mathcal{S}_G$ is supported on the "almost antidominant" part of the torus, namely the set of those $t \in T$ such that $\text{ord}_F(\alpha(t)) \leq 1$ for all $\alpha \in \Delta$. (The proof still goes through when $V_1 \not\cong V_2$.) We now show that the support of the image of $\mathcal{S}_G$ may extend outside T^- if $V_1 \not\cong V_2$.

Proposition 6.13. *With the above notation, $\mathcal{S}_G(\varphi_\lambda^+) \in \mathcal{H}_T$ has support $\lambda(\varpi)T(\mathcal{O}) \cup \lambda(\varpi)\alpha^\vee(\varpi)T(\mathcal{O})$.*

Note that $\langle \lambda + \alpha^\vee, \alpha \rangle = 1$.

Proof. By Prop. 6.7 it suffices to show that $\mathcal{S}_G(\varphi_\lambda^-)$ has support $\lambda(\varpi)T(\mathcal{O})$. Let $M = M_\lambda$. We first claim that $\mathcal{S}_G^M(\varphi_\lambda^-) \in \mathcal{H}_M((V')^{N(k)}, V^{N(k)})$ has support $M(\mathcal{O})\lambda(\varpi)M(\mathcal{O})$. Since $\langle \nu', \alpha^\vee \rangle > 0$ and $-\lambda$ is a fundamental coweight for α, it follows that V' is $M_\lambda = M$-regular. Then Cor. 2.18(ii) establishes the claim. (Note that the proof still goes through and that it only depends on the "source" weight V'.)

Since $\nu' - \nu = (q-1)\omega_\alpha$ and ω_α is orthogonal to all simple coroots of M, we see that $(V')^{N(k)} \cong V^{N(k)}$. By fixing an isomorphism between them we are reduced to consider the image of T_λ^M under $\mathcal{S}_M : \mathcal{H}_M(V^{N(k)}) \to \mathcal{H}_T(V^{U(k)})$. But this equals τ_λ, as $\lambda(\varpi) \in Z_M$. (Just as in the proof of Lemma 4.4.) $\qquad\square$

Example 6.14. Consider $G = \mathrm{GL}_2$ with diagonal torus T and upper-triangular Borel B. A pair of integers (a,b) denotes the element of $X^*(T)$ that sends $\left(\begin{smallmatrix} t_1 & \\ & t_2 \end{smallmatrix}\right)$ to $t_1^a t_2^b$. Let $V = F(0,0)$, the trivial weight, and $V' = F(q-1,0) \cong \mathrm{Sym}^{q-1}\, \bar{k}^2$, the Steinberg weight. Then we are in the situation considered above, with $\alpha = (1,-1)$. In particular there are non-zero $\mathcal{H}_G[G]$-linear maps $\text{c-Ind}_K^G V \to \text{c-Ind}_K^G V'$ and $\text{c-Ind}_K^G V' \to \text{c-Ind}_K^G V$ whose support is $K\left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right)K$. A variant of the latter map was used in [Kis09, Lemma 1.5.5].

7. Generalised Steinberg representations

In this section we will determine the Jordan–Hölder factors of $\mathrm{Ind}_B^G 1$ (completing the work of Große-Klönne) as well as their weights and Hecke eigenvalues. As usual, $P = MN$ and Q denote standard parabolic subgroups of G throughout.

Recall that the *generalised Steinberg representations* are defined as follows:

$$(7.1) \qquad\qquad \mathrm{Sp}_P = \frac{\mathrm{Ind}_P^G 1}{\sum_{Q \supsetneq P} \mathrm{Ind}_Q^G 1}.$$

Theorem 7.2. *For any standard parabolic subgroup P, the generalised Steinberg representation Sp_P is irreducible and admissible.*

For the Steinberg representation Sp_B this was proved by Vignéras [Vig08, §4]. The general case was proved by Große-Klönne [GK, Cor. 4.3], under the assumption that G is of type A, B, C, or D. We recall Cor. 4.4 of [GK], which now holds without any restriction on the root system.

Corollary 7.3. *The generalised Steinberg representations Sp_P are pairwise non-isomorphic. They form the irreducible constituents of $\mathrm{Ind}_B^G 1$, each occurring with multiplicity one. In particular, $\mathrm{Ind}_B^G 1$ is of finite length $2^{\#\Delta}$.*

We deduce Thm. 7.2 from the work of Große-Klönne with the help of Thm. 3.1. More precisely we show that Sp_P contains a unique weight, that that weight lifts to $\mathrm{Ind}_P^G 1$, and finally that the lifted weight generates $\mathrm{Ind}_P^G 1$.

Proposition 7.4. *For any standard parabolic subgroup P, the representation Sp_P contains a unique weight V_P. It occurs with multiplicity one. The unique set of Hecke eigenvalues in weight V_P is parameterised by the pair $(T, 1)$.*

We will see in the proof that V_P is the unique M-regular weight such that $(V_P)^{N(k)} \cong (V_P)_{\overline{N}(k)}$ is trivial. It follows that $\mathrm{Stab}_W(V_P^{U(k)}) = W_M$, so that the weights V_P are pairwise distinct.

Proof of Theorem 7.2 and Proposition 7.4. Let $\overline{\mathrm{Sp}}_P$ denote the generalised Steinberg representation of $G(k)$, i.e.,

$$\overline{\mathrm{Sp}}_P = \frac{\mathrm{Ind}_{\overline{P}(k)}^{G(k)} 1}{\sum_{Q \supsetneq P} \mathrm{Ind}_{\overline{Q}(k)}^{G(k)} 1}.$$

Then by [GK, §3], there is a natural K-linear embedding $\overline{\mathrm{Sp}}_P \hookrightarrow \mathrm{Sp}_P$ such that

(7.5) $$(\overline{\mathrm{Sp}}_P)^{B(k)} = (\mathrm{Sp}_P)^I = (\mathrm{Sp}_P)^{I(1)}.$$

Since $I(1)$ is a pro-p group, we see in particular that Sp_P is admissible.

Step 1. We show that $\mathrm{soc}_K(\mathrm{Sp}_P)$ is irreducible. Let $V \subset \mathrm{Sp}_P$ be any weight. By (7.5) we have $V^{U(k)} \subset (\overline{\mathrm{Sp}}_P)^{B(k)}$, so $V^{U(k)} = V^{B(k)}$ is a one-dimensional subspace of $(\overline{\mathrm{Sp}}_P)^{B(k)}$, which is stable under the action of the Hecke algebra $\bar{k}[B(k)\backslash G(k)/B(k)]$. In [GK, Prop. 3.4] it is shown that any non-zero $\bar{k}[B(k)\backslash G(k)/B(k)]$-submodule of $(\overline{\mathrm{Sp}}_P)^{B(k)}$ contains the element g_P, which is by definition the image of $1_{\overline{P}(k)B(k)} \in \mathrm{Ind}_{\overline{P}(k)}^{G(k)} 1$ in $\overline{\mathrm{Sp}}_P$. This shows that V is generated by g_P as $G(k)$-module.

Step 2. By Lemma 2.5, there is a unique M-regular weight V_P such that $(V_P)_{\overline{N}(k)}$ is trivial. Let $\chi : \mathcal{H}_M(1) \to \bar{k}$ denote the Hecke eigenvalues of the trivial weight in the trivial M-representation. It is parameterised by the pair $(T, 1)$, for example by Lemmas 4.5 and 4.4 since the trivial M-representation is contained in the trivial principal series for M. From Thm. 3.1 we get a surjective G-linear map $\mathrm{c\text{-}Ind}_K^G V_P \otimes_{\mathcal{H}_G(V_P),\chi} \bar{k} \twoheadrightarrow \mathrm{Ind}_{\overline{P}}^G 1$. Thus the image of V_P generates $\mathrm{Ind}_{\overline{P}}^G 1$ as G-representation; a fortiori, its image in Sp_P is non-zero and generates. By Step 1, V_P is the unique weight of Sp_P.

Step 3. Suppose that $\pi \subset \mathrm{Sp}_P$ is a non-zero subrepresentation. Then π contains a weight. By the previous two steps, we know that this is V_P and that V_P generates Sp_P. Thus $\pi = \mathrm{Sp}_P$.

The final statement of Prop. 7.4 follows from the proof of Lemma 4.5. That is, $\chi \circ {}'\mathcal{S}_G^M$ is parameterised by $(T, 1)$ because χ is. $\qquad \square$

The following proposition will be useful later.

Proposition 7.6. *Suppose that $P = MN$ is a standard parabolic and that Q is a standard parabolic of M. Then the constituents of $\mathrm{Ind}_{\overline{P}}^G \mathrm{Sp}_Q$ are given by $\mathrm{Sp}_{P'}$, where P' runs through all standard parabolic subgroups of G such that $P' \cap M = Q$. They all occur with multiplicity one.*

Proof. The last statement follows from Cor. 7.3: $\mathrm{Ind}_{\overline{P}}^G \mathrm{Sp}_Q$ is a subquotient of $\mathrm{Ind}_{\overline{B}}^G 1$ since Sp_Q is a subquotient of $\mathrm{Ind}_{B \cap M}^M 1$.

For each standard parabolic subgroup P' of G, $P' \cap M$ is a standard parabolic subgroup of M. In the parameterisation of standard parabolic subgroups by subsets of Δ the map $P' \mapsto P' \cap M$ becomes $\Delta' \mapsto \Delta' \cap \Delta_M$. Thus there is a smallest standard parabolic subgroup Q_0' of G such that

$Q'_0 \cap M = Q$ and we have $P' \supset Q'_0$ if and only if $P' \cap M \supset Q$. In fact, $Q'_0 = QN$.

By exactness and transitivity of parabolic induction,

$$\mathrm{Ind}_P^G \mathrm{Sp}_Q \cong \frac{\mathrm{Ind}_P^G \mathrm{Ind}_Q^M 1}{\sum_{R \supsetneq Q} \mathrm{Ind}_P^G \mathrm{Ind}_R^M 1} \cong \frac{\mathrm{Ind}_{Q'_0}^G 1}{\sum_{R \supsetneq Q} \mathrm{Ind}_{R'_0}^G 1}.$$

Using (7.1) again it follows by induction that the constituents of $\mathrm{Ind}_P^G 1$ are given by $\mathrm{Sp}_{P'}$ for $P' \supset P$. The claim now follows, recalling that we have already established that all constituents occur with multiplicity one. $\square$

8. Irreducibility of parabolic inductions

The goal of this section is to construct many irreducible admissible G-representations by parabolic induction. The most precise results are obtained for GL_n.

We remark that parabolic induction preserves admissibility. Suppose that $P = MN$ is a standard parabolic and that σ is an admissible M-representation. It suffices to show that $(\mathrm{Ind}_P^G \sigma)^{K(1)}$ is finite-dimensional. This follows from the admissibility of σ since

$$(\mathrm{Ind}_P^G \sigma)^{K(1)} = (\mathrm{Ind}_{P(\mathcal{O})}^K \sigma)^{K(1)} = \mathrm{Ind}_{\overline{P}(k)}^{G(k)} (\sigma^{M(1)}),$$

where $M(1)$ is the kernel of $M(\mathcal{O}) \to M(k)$.

8.1. **The case of** GL_n**.** When $G = \mathrm{GL}_n$ we will always let T be the diagonal torus and B the upper-triangular Borel subgroup.

Theorem 8.1. *Suppose that $G = \mathrm{GL}_n$. Let P be the standard parabolic with Levi $\prod_i \mathrm{GL}_{n_i}$, where $\sum_i n_i = n$. Suppose that σ_i is an irreducible admissible supersingular representation of $\mathrm{GL}_{n_i}(F)$ for $i = 1, \ldots, r$. Then $\mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ is irreducible if and only if there is no i such that $n_i = n_{i+1} = 1$ and $\sigma_i \cong \sigma_{i+1}$.*

When $n_1 = \cdots = n_r = 1$, this was proved by Ollivier [Oll06]. Her method relied on a detailed knowledge of the $I(1)$-Hecke algebra and the study of intertwinings of certain induced $I(1)$-Hecke modules. Henniart (unpublished) found a different proof in that case, which utilises the structure of a principal series as B-representation and which works, under some conditions, for quasi-split groups. By contrast, our strategy is to show that every non-zero subrepresentation of an induced representation satisfying the criterion in Thm. 8.1 has to contain a sufficiently regular weight (by "changing the weight" as in §6.2), and that such a weight has to generate the whole representation (by using the surjectivity of the map in Thm. 3.1).

Lemma 8.2. *Suppose that $G = \prod_{i=1}^r G_i$ is a product of split reductive groups. If σ_i is an irreducible admissible G_i-representation for $i = 1, \ldots, r$, then $\sigma := \bigotimes_i \sigma_i$ is an irreducible admissible G-representation. Conversely, every irreducible admissible G-representation is of this form.*

The weights of σ are of the form $V := \bigotimes_i V_i$ where V_i is an arbitrary weight of σ_i. The Hecke eigenvalues of V in σ are parameterised by the pairs $(\prod_i M_i, \prod \chi_i)$, where each (M_i, χ_i) ranges over the pairs parameterising Hecke eigenvalues of V_i in σ_i.

Proof. The first part follows just as in the case of finite groups (using a proof that does not use character theory). Similarly, the weights for G are of the form $\bigotimes V_i$ where each V_i is a weight for G_i. Next note that $\bigotimes \mathrm{Hom}_{G_i(\mathcal{O})}(V_i, \sigma_i) \cong \mathrm{Hom}_{G(\mathcal{O})}(V, \sigma)$. Finally we have a natural isomorphism $\bigotimes \mathcal{H}_{G_i}(V_i) \cong \mathcal{H}_G(V)$ which is induced by the map $\otimes \varphi_i \mapsto \varphi$ with $\varphi(g_1, \ldots, g_r) = \bigotimes \varphi_i(g_i)$. (To see that it is a bijection, note that it sends $\bigotimes T_{\lambda_i}$ to $T_{\prod \lambda_i}$.) One also verifies that this isomorphism is compatible with the Satake transform (for G and the G_i). The lemma then follows easily from Prop. 4.1. $\square$

Proof of Theorem 8.1. Let $M_i = \mathrm{GL}_{n_i}$. We have $P = MN$ with $M = \prod_i \mathrm{GL}_{n_i}$. We let $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r$; it is an irreducible admissible representation of M. Note first that the criterion is clearly necessary since otherwise σ extends to a representation of a larger parabolic subgroup.

To show that the criterion is sufficient, assume that $\pi \subset \mathrm{Ind}_{\overline{P}}^{G} \sigma$ is a non-zero subrepresentation. Let V be a weight of π. We claim that the action of $\mathcal{H}_G(V)$ on $\mathrm{Hom}_K(V, \pi)$ factors through $'\mathcal{S}_G^M : \mathcal{H}_G(V) \to \mathcal{H}_M(V_{\overline{N}(k)})$, which is a localisation map by Prop. 2.12. Consider the natural maps of finite-dimensional $\mathcal{H}_G(V)$-modules,

$$(8.3) \qquad \mathrm{Hom}_K(V, \pi) \hookrightarrow \mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{P}}^{G} \sigma) \xrightarrow{\sim} \mathrm{Hom}_{M(\mathcal{O})}(V_{\overline{N}(k)}, \sigma),$$

where $\mathcal{H}_G(V)$ acts on the final term via $'\mathcal{S}_G^M$ (see Lemma 2.14). Any vector space automorphism of the right-hand side that preserves the left-hand side obviously acts invertibly on it. The claim follows since $'\mathcal{S}_G^M$ is a localisation map. The maps in (8.3) are now $\mathcal{H}_M(V_{\overline{N}(k)})$-linear with the right-hand side carrying its natural $\mathcal{H}_M(V_{\overline{N}(k)})$-action.

By the above we can pick Hecke eigenvalues $\chi : \mathcal{H}_M(V_{\overline{N}(k)}) \to \bar{k}$ and a corresponding Hecke eigenvector $f : V \to \pi$ that occur in the finite-dimensional $\mathcal{H}_M(V_{\overline{N}(k)})$-module $\mathrm{Hom}_K(V, \pi)$. By (8.3) we can think of f also as $\mathcal{H}_M(V_{\overline{N}(k)})$-eigenvector with eigenvalues χ in $\mathrm{Hom}_{M(\mathcal{O})}(V_{\overline{N}(k)}, \sigma)$ and we get a surjection

$$\text{c-Ind}_{M(\mathcal{O})}^{M} V_{\overline{N}(k)} \otimes_{\mathcal{H}_M(V_{\overline{N}(k)}), \chi} \bar{k} \twoheadrightarrow \sigma.$$

Case 1: V is M-regular. Then Thm. 3.1 applies. Since $\mathrm{Ind}_{\overline{P}}^{G}$ is an exact functor, we obtain

$$(8.4) \qquad \text{c-Ind}_K^G V \otimes_{\mathcal{H}_G(V), \chi} \bar{k} \twoheadrightarrow \mathrm{Ind}_{\overline{P}}^{G} \sigma.$$

This map is naturally induced from $V \xrightarrow{f} \pi \subset \mathrm{Ind}_{\overline{P}}^{G} \sigma$. (See the computation of $\bar{\theta}$ in Step 2 of the proof of Thm. 3.1.) As V generates the left-hand side
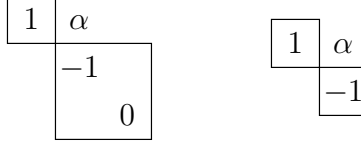
FIGURE 8.1. Final part of the proof of Thm. 8.1: α and $\alpha^\vee$.

of (8.4), it follows that $\mathrm{Ind}_{\bar{P}}^G \sigma$ is generated by $f(V)$ as G-representation. In other words, $\pi = \mathrm{Ind}_{\bar{P}}^G \sigma$.

Case 2: V is not M-regular. We show by induction that π has to contain an M-regular weight (in fact the one corresponding to $V^{N(k)}$ by Lemma 2.5). Then Case 1 implies that $\pi = \mathrm{Ind}_{\bar{P}}^G \sigma$, which concludes the proof that $\mathrm{Ind}_{\bar{P}}^G \sigma$ is irreducible.

Since the derived subgroup of G is simply connected, we can write $V \cong F(\nu)$. As $\mathrm{Stab}_W(\nu) \not\subset W_M$ and since the left-hand side is generated by simple reflections, there is a simple root $\alpha \in \Delta - \Delta_M$ such that $s_\alpha(\nu) = \nu$. Since the centre of G is connected, there is a fundamental coweight $-\lambda$ associated to α, which is minuscule as $G = \mathrm{GL}_n$. Just as in §6.2 we now put $\nu' = \nu + (q-1)\omega_\alpha$ and $V' = F(\nu')$. The map f above gives rise to a non-zero map c-$\mathrm{Ind}_K^G V \otimes_{\mathcal{H}_{G,\chi}} \bar{k} \to \pi$ (we also write χ for $\chi \circ' \mathcal{S}_G^M$). If we can show that Cor. 6.10 applies, there is a non-zero map c-$\mathrm{Ind}_K^G V' \otimes_{\mathcal{H}_{G,\chi}} \bar{k} \to \pi$, in particular V' is a weight of π. Since $\mathrm{Stab}_W(\nu')$ is strictly smaller than $\mathrm{Stab}_W(\nu)$, by induction we eventually find that π has to contain an M-regular weight.

Finally we show that the criterion in Thm. 8.1 implies that the assumptions in Cor. 6.10 are satisfied. By Lemma 8.2 and Def. 4.7, the Hecke eigenvalues χ are parameterised by $(M, \prod \chi_i)$, where χ_i is the central character of σ_i. We have $\alpha \notin \Delta_M$. Suppose first that there exists $\beta \in \Delta_M$ that is adjacent to α in the Dynkin diagram. In this case $\langle \alpha^\vee, \beta \rangle \neq 0$, so $\alpha^\vee(\varpi) \notin Z_M$ and we are done. See the left part of Fig. 8.1. Otherwise, we have $n_i = n_{i+1} = 1$ for the two Levi blocks (M_i, σ_i) and (M_{i+1}, σ_{i+1}) that α "separates". (We work with the diagonal torus and the upper triangular Borel.) See the right part of Fig. 8.1. In particular $\sigma_i = \chi_i$, $\sigma_{i+1} = \chi_{i+1}$. Therefore on $F^\times$ we have $\chi_M \circ \alpha^\vee = \chi_i \chi_{i+1}^{-1} \neq 1$ by assumption, so we are done. $\qquad\square$

Theorem 8.5. *Suppose that $G = \mathrm{GL}_n$. Let P be the standard parabolic with Levi $\prod_i \mathrm{GL}_{n_i}$, where $\sum_i n_i = n$. Suppose that σ_i is an irreducible admissible supersingular representation of $\mathrm{GL}_{n_i}(F)$ for $i = 1, \ldots, r$. Then $\mathrm{Ind}_{\bar{P}}^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ has finite length.*

By the transitivity of parabolic induction, we can rewrite such a representation in the form $\mathrm{Ind}_{\bar{Q}}^G(\tau_1 \otimes \cdots \otimes \tau_s)$ with $Q \supset P$ the standard parabolic with Levi $\prod_i \mathrm{GL}_{m_i}$, such that for all i either

○ τ_i is supersingular and $m_i > 1$, or

○ $\tau_i \cong \mathrm{Ind}_{\overline{B_i}}^{\mathrm{GL}_{m_i}} \eta_i$ for some $\eta_i : F^\times \to \bar{k}^\times$

and such that $\eta_i \neq \eta_{i+1}$ whenever both τ_i and τ_{i+1} fall into the second case. (Here and in the following, we often write η when we really mean $\eta \circ \det$. Moreover B_i denotes the Borel in GL_{m_i}.)

By Cor. 7.3, the irreducible constituents of $\mathrm{Ind}_{\overline{B_i}}^{\mathrm{GL}_{m_i}} \eta_i$ are of the form $\mathrm{Sp}_{Q_i} \otimes \eta_i$, where Q_i ranges over the standard parabolics of GL_{m_i}.

We have thus reduced the proof of Thm. 8.5 to the following generalisation of Thm. 8.1. It shows that $\mathrm{Ind}_{\overline{Q}}^{G}(\tau_1 \otimes \cdots \otimes \tau_s)$ has length $2^{\sum_i d_i}$, where $d_i = 0$ or $m_i - 1$ depending on whether τ_i falls into the first or into the second case.

Theorem 8.6. *Suppose that $G = \mathrm{GL}_n$. Let P be the standard parabolic with Levi $\prod_i \mathrm{GL}_{n_i}$, where $\sum_i n_i = n$. Suppose that σ_i is an irreducible admissible representation of $\mathrm{GL}_{n_i}(F)$ for $i = 1, \ldots, r$ such that for all i either*

○ *σ_i is supersingular and $n_i > 1$, or*

○ *$\sigma_i \cong \mathrm{Sp}_{Q_i} \otimes \eta_i$ for some η_i and some standard parabolic $Q_i \subset \mathrm{GL}_{n_i}$.*

Assume that $\eta_i \neq \eta_{i+1}$ whenever both σ_i and σ_{i+1} fall into the second case. Then $\mathrm{Ind}_{\overline{P}}^{G}(\sigma_1 \otimes \cdots \otimes \sigma_r)$ is irreducible.

Proof. The proof differs from the one of Thm. 8.1 only in the final part. This time the Hecke eigenvalues are parameterised by the pair $(M' = \prod M_i', \prod \chi_i)$, where $M_i' = M_i$ and χ_i is the central character of σ_i if σ_i falls into the first case, whereas M_i' is the torus of M_i and $\chi_i = \eta_i \circ \det$ if σ_i falls into the second case. It follows that $\alpha^\vee(\varpi) \notin Z_{M'}$ unless the blocks adjacent to α, (M_i, σ_i) and (M_{i+1}, σ_{i+1}), both fall into the second case and the argument goes through since $\eta_i \eta_{i+1}^{-1} \neq 1$. $\qquad\square$

Theorem 8.7. *Suppose that the final condition on the η_i is dropped in Thm. 8.6. Then $\mathrm{Ind}_{\overline{P}}^{G}(\sigma_1 \otimes \cdots \otimes \sigma_r)$ is of finite length with explicit irreducible constituents, each occurring with multiplicity one.*

In the proof we describe the irreducible constituents; in particular we show that the length of $\mathrm{Ind}_{\overline{P}}^{G}(\sigma_1 \otimes \cdots \otimes \sigma_r)$ equals 2^δ, where δ is the number of times the condition $\eta_i \neq \eta_{i+1}$ fails.

Proof. Note that $\bigotimes_{i=1}^s \mathrm{Sp}_{Q_i} \cong \mathrm{Sp}_{Q_1 \times \cdots \times Q_s}$ as representation of $\prod_{i=1}^s \mathrm{GL}_{n_i}$. (For example, directly from (7.1).) We may thus collect consecutive σ_i with the same twisting character η_i and use transitivity of parabolic induction to rewrite our representation as $\mathrm{Ind}_{\overline{Q}}^{G}(\tau_1 \otimes \cdots \otimes \tau_s)$, where $Q \supset P$ is the standard parabolic with Levi $\prod_i \mathrm{GL}_{m_i}$, such that for all i either

○ τ_i is supersingular and $m_i > 1$, or

○ $\tau_i \cong \mathrm{Ind}_{\overline{R_i}}^{\mathrm{GL}_{m_i}} \mathrm{Sp}_{S_i} \otimes \eta_i'$ for some $\eta_i' : F^\times \to \bar{k}^\times$

and such that $\eta_i' \neq \eta_{i+1}'$ whenever both τ_i and τ_{i+1} fall into the second case. Here $R_i = L_i N_i'$ is a standard parabolic of GL_{m_i} and S_i a standard parabolic of L_i. (Note that L_i is a product of consecutive GL_{n_j} and S_i a product

of consecutive Q_j.) By exactness of parabolic induction, by Prop. 7.6 and Thm. 8.6 it follows that the irreducible constituents are obtained by replacing each τ_i that falls into the second case by $\mathrm{Sp}_{S_i'} \otimes \eta_i'$ for any standard parabolic S_i' of GL_{m_i} such that $S_i' \cap L_i = S_i$.

The multiplicity one assertion is true because these irreducible representations do not share any common weight. (This holds for the generalised Steinberg representations by the results of Section 7. Now use (2.13).) Alternatively this follows using ordinary parts; see Cor. 9.10. $\square$

8.2. **General results.** We show that parabolic inductions of irreducible admissible representations are usually irreducible, even for general G. As we do not have the results of §6.2 available in general, we need to put stronger hypotheses on the weights that are allowed to occur. On the other hand, the representation of the Levi does not have to be supersingular.

Theorem 8.8. *Let $P = MN$ be a standard parabolic and suppose that σ is an irreducible admissible M-representation satisfying:*

> (∗) *For all simple roots $\alpha \in \Delta - \Delta_M$, the restriction to $k^\times$ via the coroot $\alpha^\vee$ of the $T(k)$-representation $(\mathrm{soc}_{M(\mathcal{O})}\, \sigma)_{(\overline{U} \cap M)(k)}$ does not contain the trivial representation.*

Then $\mathrm{Ind}_P^G \sigma$ is irreducible.

Proof. This follows by the proof of Thm. 8.1, since we now show that (∗) implies that all weights V of $\mathrm{Ind}_P^G \sigma$ are M-regular. First assume that the derived subgroup of G is simply connected. In this case, $V \cong F(\nu)$ for some q-restricted weight ν and $V_{\overline{N}(k)} \cong F^M(\nu)$. Assumption (∗) shows that for all simple roots $\alpha \in \Delta - \Delta_M$, $\langle \nu, \alpha^\vee \rangle \not\equiv 0 \pmod{q-1}$ (as $V_{\overline{N}(k)}$ is a direct summand of $\mathrm{soc}_{M(\mathcal{O})}\, \sigma$ and $(V_{\overline{N}(k)})_{(\overline{U} \cap M)(k)}$ is isomorphic to $\nu|_{T(k)}$). In particular, $\langle \nu, \alpha^\vee \rangle > 0$ for such α. Since $\mathrm{Stab}_W(\nu)$ is generated by simple reflections, we see that $\mathrm{Stab}_W(\nu) \subset W_M$, so V is indeed M-regular.

If G is general, pick a weight V of $\mathrm{Ind}_P^G \sigma$. The restriction of the $T(k)$-representation $V_{\overline{U}(k)}$ to $k^\times$ via $\alpha^\vee$ is non-trivial for $\alpha \in \Delta - \Delta_M$ by (∗). Now take a z-extension $\widetilde{G} \twoheadrightarrow G$ of the special fibre, just as in [Her11, Lemma 2.5]. Since coroots are compatible in z-extensions, we get the analogous statement for V as $\widetilde{G}(k)$-representation. By the previous paragraph, V is $\widetilde{M}$-regular and therefore M-regular. $\square$

Remark 8.9. Condition (∗) is best possible in the sense that it is equivalent to the condition that all weights of $\mathrm{Ind}_P^G \sigma$ are M-regular.

Here is a simple application to principal series representations.

Corollary 8.10. *Suppose that $\chi : T \to \overline{k}^\times$ is a smooth character. Then the principal series representation $\mathrm{Ind}_B^G \chi$ is irreducible provided that $\chi \circ \alpha^\vee|_{k^\times} \neq 1$ for all simple roots α. (Note that $\chi|_{T(\mathcal{O})}$ factors through $T(k)$.)*

Proof. This follows from the theorem using (2.13). $\square$

9. CLASSIFICATION RESULTS

9.1. **The case of** GL_n**.** In this subsection, $G = \mathrm{GL}_n$. The goal is to show that all irreducible admissible representations are of the form as in Thm. 8.6. Our strategy is roughly speaking as follows. If π is irreducible admissible, it contains a weight V and Hecke eigenvalues χ in that weight, so c-$\mathrm{Ind}_K^G V \otimes_{\mathcal{H}_G(V),\chi} \bar{k} \twoheadrightarrow \pi$. If there is a standard parabolic $P = MN \neq G$ such that V is M-regular and χ factors through $'\mathcal{S}_G^M$, then we can apply Thm. 3.1 to express π as a quotient of a (non-trivial) parabolic induction. We can then use Emerton's ordinary parts functor (a right adjoint to parabolic induction) to write π as a quotient of $\mathrm{Ind}_P^G \sigma$ with σ irreducible admissible, and we can induct. If the process gets stuck, we try to "change the weight" (Cor. 6.10). The one crucial ingredient that is still needed is that we can detect the trivial representation by its weight and Hecke eigenvalues; see Prop. 9.1 below.

We recall that Emerton's functor Ord_P [Eme10a] sends smooth G-representations to smooth M-representations. It is left exact and preserves admissibility. It is a right adjoint to Ind_P^G between the full subcategories of admissible representations.

Proposition 9.1. *Suppose π is a smooth G-representation such that π contains the trivial weight with Hecke eigenvalues parameterised by $(T, 1)$. Then either π contains the trivial G-representation or there exists a proper standard parabolic P such that $\mathrm{Ord}_P \pi \neq 0$.*

In fact we will see in Cor. 9.11 below that if π is moreover irreducible and admissible, then π is the trivial G-representation. (Note also that $\mathrm{Ord}_P 1 = 0$ for any proper parabolic P.)

Proof. The first assumption means that $\pi^K \neq 0$. We use the more classical notation $[KgK]$ for a Hecke operator in the unramified Hecke algebra $\mathcal{H}_G(1)$. The action on π^K is the natural left action: if $KgK = \coprod g_\alpha K$, then $[KgK]v := \sum g_\alpha v \in \pi^K$. Note that $[KgK] = 1_{Kg^{-1}K} \in \mathcal{H}_G(1)$. Let $T_i := [Kt_iK]$, where $t_i = \mathrm{diag}(\varpi, \ldots, \varpi, 1, \ldots, 1)$ (i copies of ϖ).

Pick $v \in \pi^K$, a Hecke eigenvector with Hecke eigenvalues parameterised by $(T, 1)$. This means that $T_i v = v$ for all $1 \leq i \leq n$. (This follows from Cor. 4.2. Note that $\geq_{\mathbb{R}}$ is the same as $\geq$ on $X_*(T)$ for GL_n and so by (2.7), $\mathcal{S}_G(T_\lambda) = \tau_\lambda$ whenever $-\lambda$ is minuscule.) From $i = n$ or from Lemma 4.4 we see that π has trivial central character.

Suppose that $\mathrm{Ord}_P \pi = 0$ for all proper standard parabolics P.

Step 1. We show that $U_i := [It_iI] \in \mathcal{H}_I(1)$ acts nilpotently on v for $i = 1$, $\ldots$, $n-1$. (Recall that I denotes the Iwahori subgroup.)

Let P be the standard parabolic subgroup with Levi $\mathrm{GL}_i \times \mathrm{GL}_{n-i}$ and let $\mathcal{P}$ be the corresponding parahoric subgroup. Suppose that the Hecke operator $[\mathcal{P}t_i\mathcal{P}]$ has a non-zero eigenvalue on $\pi^{\mathcal{P}}$. Then a corresponding eigenvector is even stable under $\mathcal{H}(1) \subset \mathcal{H}_{\mathcal{P}}(1)$ because $\mathcal{H}(1)$ is generated by $E_h = [\mathcal{P}h^{-1}\mathcal{P}]$ for $h = t_i^{-1}$ and $h = t_n^{-1}$ (in the notation of Lemma 2.21).

Then there is a unique unramified character $\chi_M : Z_M \to \bar{k}^\times$ that sends h^{-1} to the (non-zero!) eigenvalue of E_h for $h = t_i^{-1}$ and $h = t_n^{-1}$. Since E_h is identified with $T_h^M \in \mathcal{H}_M(1)$, Prop. 9.14 applies with $\overline{V} = 1$ and we see that $\mathrm{Ord}_P \pi \neq 0$. Contradiction.

Thus $[\mathcal{P}t_i\mathcal{P}]$ is nilpotent on $\pi^\mathcal{P}$. But $[\mathcal{P}t_i\mathcal{P}] = [It_iI]$ on $\pi^\mathcal{P} \subset \pi^I$, so U_i acts nilpotently on v. (To see this, note that $\mathcal{P}t_i\mathcal{P}/\mathcal{P}$ naturally bijects with $N(\mathcal{O})/^{t_i}N(\mathcal{O})$ by Lemma 2.16. Similarly It_iI/I naturally bijects with $U(\mathcal{O})/^{t_i}U(\mathcal{O})$. Now use that $U = N \rtimes (M \cap U)$.)

Step 2. Setup and some lemmas.

Define the following elements in the Iwahori Hecke algebra $\mathcal{H}_I(1)$:

$$S_1 = \left[I \begin{pmatrix} 1 & 1 \\ & \ddots \\ & & 1 \end{pmatrix} I\right], \ldots, S_{n-1} = \left[I \begin{pmatrix} 1 \\ & \ddots \\ & & 1 & 1 \end{pmatrix} I\right], \Pi = \left[I \begin{pmatrix} & 1 \\ & & \ddots \\ \varpi & & & 1 \end{pmatrix} I\right],$$

where S_i is defined by the i-th simple reflection. It is straightforward to verify the following relations.

$$\begin{aligned}
S_i^2 &= -S_i & &\text{for all } i, \\
S_i S_j &= S_j S_i & &\text{whenever } |i - j| > 1, \\
S_k S_{k+1} S_k &= S_{k+1} S_k S_{k+1} & &\text{for all } k < n - 1, \\
S_k \Pi &= \Pi S_{k+1} & &\text{for all } k < n - 1.
\end{aligned}$$

We also have

$$\Pi^n = 1 \text{ on } \pi^I.$$

The quadratic relations take a simple form, since we work in characteristic p. For the last property note that the matrix defining Π normalises I. (We remark that the above is a presentation of $\mathcal{H}_I(1)$, but we will not need that fact. It can be deduced from Cor. 4 in [Vig05] for $R = \overline{\mathbb{F}}_p$ and γ consisting of the trivial character.)

Note that

$$T_1 v = \sum \begin{pmatrix} \varpi & a_2 & a_3 & \cdots \\ & 1 & \\ & & 1 \\ & & & \ddots \end{pmatrix} v + \sum \begin{pmatrix} 1 & & & \\ & \varpi & a_3 & \cdots \\ & & 1 \\ & & & \ddots \end{pmatrix} v + \cdots,$$

where in each sum, the a_i run over representatives of $\mathcal{O}$ modulo ϖ.

It is not hard to express this in terms of the Iwahori Hecke action. First, in each of these sums we can interchange the first column with the column containing ϖ (since $v \in \pi^K$). For $1 \leq i \leq n$ let $x_i \in G$ be the product of the matrices defining $S_{i(i+1)\cdots(n-1)}\Pi$. It will suffice to show that $S_{i(i+1)\cdots(n-1)}\Pi = [Ix_iI]$ and that Ix_iI is the disjoint union of the cosets $u(a_{i+1}, \ldots, a_n)x_iI$, where $u(a_{i+1}, \ldots, a_n) \in U(\mathcal{O})$ is zero above the diagonal except for the i-th row which ends in $(a_{i+1}, \ldots, a_n)$, and the a_j run through representatives of $\mathcal{O}$ modulo ϖ. The second claim is a simple calculation. The first claim follows since both sides contain q^{n-i} one-sided cosets. Since

$T_1 v = v$, we get:

$$(9.2) \qquad v = \sum_{i=1}^{n} S_{i(i+1)\cdots(n-1)}\Pi v,$$

where we abbreviate $S_{i(i+1)\cdots(j-1)} := S_i S_{i+1} \cdots S_{j-1}$ for any $1 \le i \le j \le n$. (This equals 1 when $i = j$.)

Sublemma 9.3. *Suppose $i \le k \le \ell \le j$. Then*

$$S_{i(i+1)\cdots j} S_{k(k+1)\cdots(\ell-1)} = S_{(k+1)(k+2)\cdots\ell} S_{i(i+1)\cdots j}$$

Proof. It suffices to consider the case $\ell - 1 = k$. In that case, $i \le k \le j - 1$. By the braid relations,

$$\begin{aligned}
S_{i(i+1)\cdots j} S_k &= S_{i(i+1)\cdots(k-1)k(k+1)} S_k S_{(k+2)\cdots j} \\
&= S_{i(i+1)\cdots(k-1)} S_{(k+1)} S_{k(k+1)(k+2)\cdots j} \\
&= S_{(k+1)} S_{i(i+1)\cdots j}.
\end{aligned}$$

$\square$

Sublemma 9.4. *We have $S_{n-1}\Pi^2 v = 0$.*

Proof. We first note that for all i we have $S_i v = 0$. This is immediate from the fact that $v \in \pi^K$ and that the double coset defining S_i is a disjoint union of $q = \#k$ one-sided cosets.

Then we get $S_{n-1}\Pi^2 v = \Pi^2 S_1 v = 0$ (since $\Pi^2 v = \Pi^{-(n-2)} v$). $\square$

Sublemma 9.5. *For all i, we have $U_i = (S_{i\cdots(n-1)}\Pi)^i$.*

Proof. If we multiply out the double cosets on the right-hand side, we certainly find the matrix t_i. (We just multiply the defining matrices of all these double cosets.) To see that the product of the double cosets yields only the double coset of this diagonal matrix, it's enough to count the number of one-sided cosets. On the left-hand side the number of cosets is $q^{i(n-i)}$. Since each S_i contains q cosets and Π contains just one, the result follows. $\square$

Step 3. We show that $U_1 v = S_{12\cdots(n-1)}\Pi v = 0$.

Sublemma 9.5 tells us that $U_1 = S_{12\cdots(n-1)}\Pi$. We now see that

$$(S_{12\cdots(n-1)}\Pi)^2 v = S_{12\cdots(n-1)}\Pi \left(v - \sum_{i=2}^{n} S_{i\cdots(n-1)}\Pi v \right)$$

by (9.2),

$$= S_{12\cdots(n-1)}\Pi v - \sum_{i=2}^{n} S_{12\cdots(n-1)} S_{(i-1)\cdots(n-2)}\Pi^2 v$$

by pushing Π to the right,

$$= S_{12\cdots(n-1)}\Pi v - \sum_{i=2}^{n} S_{i\cdots(n-1)} S_{12\cdots(n-1)}\Pi^2 v$$

by Sublemma 9.3,

$$= S_{12\cdots(n-1)}\Pi v,$$

by Sublemma 9.4. In other words, $U_1^2 v = U_1 v$. So by Step 1, we have $U_1 v = 0$.

Step 4. We show that $U_2 v = S_{23\cdots(n-1)}\Pi v = 0$.

By Step 3, the first term in (9.2) vanishes and we get

$$(9.6) \qquad\qquad v = \sum_{i=2}^{n} S_{i(i+1)\cdots(n-1)}\Pi v.$$

We repeat the calculation we did in the previous step, using (9.6) instead of (9.2); we find:

$$(S_{23\cdots(n-1)}\Pi)^2 v = S_{23\cdots(n-1)}\Pi v.$$

It follows from Sublemma 9.5 that

$$(9.7) \qquad\qquad U_2 v = (S_{23\cdots(n-1)}\Pi)^2 v = S_{23\cdots(n-1)}\Pi v$$

and moreover that $U_2^2 v = U_2 v$. By Step 1 we see that $U_2 v = 0$.

$$\vdots$$

Step $n+2$. We show that $v \in \pi^G$.

From Step $i+2$, we know that $U_i v = S_{i(i+1)\cdots(n-1)}\Pi v = 0$ ($1 \le i \le n-1$). Equation (9.2) thus simplifies to $v = \Pi v$. Since G is generated by K and
$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ \varpi & & \end{pmatrix} \in N_G(I)$ (for example by the Cartan decomposition, generating t_i first as in the proof of Sublemma 9.5), the claim follows. This completes the proof. $\qquad\square$

Theorem 9.8. *Let π be any irreducible admissible G-representation. Then there exists a standard parabolic P with Levi $\prod_{i=1}^{r} \mathrm{GL}_{n_i}$ and irreducible admissible representations σ_i of $\mathrm{GL}_{n_i}(F)$ such that $\pi \cong \mathrm{Ind}_{\bar{P}}^{G}(\sigma_1 \otimes \cdots \otimes \sigma_r)$ and such that for all i either*

- *σ_i is supersingular and $n_i > 1$, or*
- *$\sigma_i \cong \mathrm{Sp}_{Q_i} \otimes \eta_i$ for some η_i and some standard parabolic $Q_i \subset \mathrm{GL}_{n_i}$.*

Moreover $\eta_i \ne \eta_{i+1}$ whenever both σ_i and σ_{i+1} fall into the second case.

Together with Thm. 8.6, this achieves the classification of irreducible admissible $\mathrm{GL}_n(F)$-representations.

We first prove a useful lemma using ordinary parts.

Lemma 9.9. *Suppose that $Q = LN'$ is a standard parabolic subgroup. Suppose there is a G-invariant surjection $\mathrm{Ind}_{\bar{Q}}^{G}\tau \twoheadrightarrow \pi$ for some smooth L-representation τ having a central character and some irreducible admissible G-representation π. Then there exists an irreducible admissible L-representation σ and a G-invariant surjection $\mathrm{Ind}_{\bar{Q}}^{G}\sigma \twoheadrightarrow \pi$.*

Proof. As τ is locally Z_L-finite and π is smooth, the natural map

$$\mathrm{Hom}_G(\mathrm{Ind}_Q^G \tau, \pi) \to \mathrm{Hom}_L(\tau, \mathrm{Ord}_Q \pi)$$

is injective: this follows from the injectivity of the first map in [Eme10a, (4.4.7)] (a geometric fact) and the isomorphism in [Eme10a, Cor. 4.2.8]. In particular, $\mathrm{Ord}_Q \pi \neq 0$. As $\mathrm{Ord}_Q \pi$ is admissible ([Eme10a, Thm. 3.3.3]), it has an irreducible admissible subrepresentation $\sigma \hookrightarrow \mathrm{Ord}_Q \pi$ (the dual of an admissible L-representation is a finitely-generated module over the noetherian ring $\bar{k}[[L(\mathcal{O})]]$, see [Eme10a, Lemma 2.2.11]). By the adjunction [Eme10a, Thm. 4.4.6] we get a G-linear map $\mathrm{Ind}_Q^G \sigma \twoheadrightarrow \pi$. $\qquad\square$

Proof of Theorem 9.8. We argue by induction on n. For $n = 1$, there is nothing to show. For $n > 1$, we may assume that there is at least one weight V that occurs in π with non-supersingular Hecke eigenvalues χ parameterised by (M, χ_M) (otherwise π is supersingular and we are done). This means that $M \neq G$.

Let $\Delta_V = \{\alpha \in \Delta : s_\alpha \in \mathrm{Stab}_W(V^{U(k)})\}$, where $s_\alpha \in W$ denotes the simple reflection corresponding to the root α. Thus V is M-regular if and only if $\Delta_V \subset \Delta_M$.

Case 1: $\Delta_V \cup \Delta_M \neq \Delta$. Pick $\alpha \in \Delta - (\Delta_V \cup \Delta_M)$. Let $Q = LN'$ be the standard, maximal parabolic defined by α, so $\Delta_L = \Delta - \{\alpha\} \supset \Delta_M$. Then V is L-regular and χ factors through $\mathcal{H}_L(V_{\overline{N'}(k)})$. Thm. 3.1 gives rises to

$$\mathrm{Ind}_Q^G(\text{c-Ind}_{L(\mathcal{O})}^L V_{\overline{N'}(k)} \otimes_{\mathcal{H}_L(V_{\overline{N'}(k)}),\chi} \bar{k}) \twoheadrightarrow \pi.$$

Note that the L-representation that is being induced on the left-hand side has a central character (namely $h \mapsto \chi(T_{h^{-1}}^L)$). Thus by Lemma 9.9 there is an irreducible admissible L-representation σ and a G-linear surjection $\mathrm{Ind}_Q^G \sigma \twoheadrightarrow \pi$. We can now decompose σ as a tensor product of irreducible admissible representations of the Levi blocks (Lemma 8.2). By induction, each of them is of the desired form. By transitivity of parabolic induction we see that π is a quotient of a parabolic induction of the desired form, except that consecutive η_i might be equal. But by Thm. 8.7 we know that all Jordan–Hölder factors of this parabolic induction are of the desired form.

Case 2: $\Delta_V \cup \Delta_M = \Delta$. Suppose first that there is an $\alpha \in \Delta - \Delta_M$ such that $\alpha^\vee(\varpi) \notin Z_M$ or $\chi_M(\alpha^\vee(\varpi)) \neq 1$. Then by Cor. 6.10, there is a weight V' that occurs in π with the same Hecke eigenvalues χ and such that $\Delta_{V'} = \Delta_V - \{\alpha\}$. We are thus reduced to Case 1.

Otherwise, for all $\alpha \in \Delta - \Delta_M$ we have $\alpha^\vee(\varpi) \in Z_M$ and $\chi_M(\alpha^\vee(\varpi)) = 1$. The first condition implies that $M = T$ and then the second condition implies that $\chi_M = \eta \circ \det$, for some smooth character η. By twisting we may assume that $\chi_M = 1$ (see Lemma 4.6). Thus $V_{\overline{U}(k)}$ is trivial. Since we are in Case 2, we also have $\Delta_V = \Delta$. Putting together these two facts and using Lemma 2.3, we see that V is the trivial weight. By Prop. 9.1, either π is trivial (and we are done) or $\mathrm{Ord}_Q \pi \neq 0$ for some proper standard parabolic Q (and we induct as in Case 1). $\qquad\square$

Corollary 9.10. *Suppose that π is an irreducible admissible G-representation.*

(i) *All Hecke eigenvalues of π are parameterised by the same pair $(M', \chi_{M'})$.*
(ii) *There is a unique datum $(P, (\sigma_i)_{i=1}^r)$ as in Thm. 9.8 such that $\pi \cong \operatorname{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$. In other words, there are no non-trivial intertwinings between the representations in Thm. 9.8.*

Proof. By Thm. 9.8 we know that $\pi \cong \operatorname{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ with $(P = MN, (\sigma_i)_{i=1}^r)$ satisfying the conditions in that theorem.

As in Thm. 8.6 we write $M = \prod M_i$ as product of Levi blocks. By Lemmas 4.5 and 8.2 we see that the Hecke eigenvalues of π are parameterised by $(\prod M_i', \prod \chi_i)$, where (M_i', χ_i) runs over the Hecke eigenvalues of weights of σ_i. If σ_i is supersingular, we have $M_i' = M_i$ and χ_i is the central character of σ_i (see Def. 4.7). If $\sigma_i \cong \operatorname{Sp}_{Q_i} \otimes \eta_i$, then M_i' is the torus in M_i and $\chi_i = \eta_i \circ \det$ (use Prop. 7.4 and Lemma 4.6). Part (i) follows.

Since consecutive η_i are distinct, these common Hecke eigenvalues determine M. Part (ii) follows since $\operatorname{Ord}_P(\operatorname{Ind}_P^G \sigma) \cong \sigma$ [Eme10a, Prop. 4.3.4]. $\square$

We can now prove a converse to Prop. 7.4.

Corollary 9.11. *Suppose that π is an irreducible admissible G-representation. Suppose that Q is a standard parabolic and that the weight V_Q occurs in π with Hecke eigenvalues parameterised by $(T, 1)$. Then $\pi \cong \operatorname{Sp}_Q$.*

Proof. If $\pi \cong \operatorname{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ as in Thm. 9.8, then the analysis of Hecke eigenvalues in the proof of Cor. 9.10 shows that $P = G$, and that σ_1 is a generalised Steinberg representation (note that $r = 1$). Then Prop. 7.4 show that $\pi \cong \operatorname{Sp}_Q$. $\square$

Definition 9.12. Suppose that π is an irreducible admissible G-representation. We say that π is *supercuspidal* if it does not occur as subquotient in $\operatorname{Ind}_P^G \sigma$, where $P = MN$ is any proper standard parabolic and σ any irreducible admissible M-representation.

Corollary 9.13. *Suppose π is an irreducible admissible G-representation. Suppose $P = MN$ is a standard parabolic and σ an irreducible admissible M-representation.*

(i) *$\operatorname{Ind}_P^G \sigma$ is of finite length, and all constituents occur with multiplicity one. All Hecke eigenvalues of all constituents are parameterised by the same pair $(M', \chi_{M'})$.*
(ii) *π is supersingular if and only if π is supercuspidal.*

Proof. Part (i) follows from Thm. 8.7 and the analysis of Hecke eigenvalues in the proof of Cor. 9.10. (In terms of the classification in Thm. 9.8, only the parabolics Q_i differ among the constituents. But Q_i plays no role for the Hecke eigenvalues.) It is clear that the Hecke eigenvalues of σ are parameterised by the same pair $(M', \chi_{M'})$. In particular, $M' \subset M$.

If π is supercuspidal, it follows from Thm. 9.8 and the definition of the generalised Steinberg representations that π is supersingular. Conversely,

suppose that π occurs in $\mathrm{Ind}_P^G \sigma$ for some proper parabolic P. By the above, the Hecke eigenvalues of π are parameterised by $(M', \chi_{M'})$ with $M' \subset M \neq G$. Thus π cannot be supersingular. $\qquad\square$

9.2. **Some general results.** We show for general G that any irreducible admissible G-representation is parabolically induced from a supersingular representation, provided that it does not contain certain weights at the boundary. First we need the following result on the weights of ordinary parts.

Proposition 9.14. *Suppose that $\overline{V}$ is a weight for M. Suppose $\chi_M : Z_M \to \bar{k}^\times$ is a homomorphism such that $\chi_M|_{Z_M(\mathcal{O})}$ is the central character of $\overline{V}$. Then there is an algebra homomorphism $\chi : \mathcal{H}_M(\overline{V}) \to \bar{k}$ such that $\chi(T_h^M) = \chi_M(h)^{-1}$ for all $h \in Z_M$ and for any smooth G-representation π we have a natural isomorphism*

$$(9.15) \qquad \mathrm{Hom}_{M(\mathcal{O})}(\overline{V}, (\mathrm{Ord}_P \pi)^{Z_M = \chi_M}) \cong \mathrm{Hom}_G(\text{c-Ind}_{\mathcal{P}}^G \overline{V} \otimes_{\mathcal{H}(\overline{V}),\chi} \bar{k}, \pi).$$

Proof. We let $h \in Z_M$ act on $\overline{V}$ by $\chi_M(h)$. As this agrees with the usual action on $Z_M(\mathcal{O})$, the weight $\overline{V}$ becomes an $M(\mathcal{O})Z_M$-representation. Thus by the definition of $\mathrm{Ord}_P \pi$ [Eme10a, Def. 3.1.9], the left-hand side of (9.15) is naturally isomorphic to

$$\mathrm{Hom}_{M(\mathcal{O})Z_M}(\overline{V}, \mathrm{Hom}_{Z_M^+}(Z_M, \pi^{N(\mathcal{O})})) \cong \mathrm{Hom}_{M(\mathcal{O})Z_M^+}(\overline{V}, \pi^{N(\mathcal{O})}).$$

(Even though we let Z_M denote the *connected* centre, as opposed to [Eme10a], it is straightforward to verify that this is irrelevant for the definition of Ord_P.) We verify that any $M(\mathcal{O})Z_M^+$-linear map $f : \overline{V} \to \pi^{N(\mathcal{O})}$ factors through $\pi^{\mathcal{P}(1)}$, where $\mathcal{P}(1) = \ker(\mathcal{P} \to M(k))$. By the smoothness of π and by Lemma 2.16, there is an $h \in Z_M^{--}$ such that $f(\overline{V})$ is fixed by $^h(\mathcal{P}^-)$. Then $h^{-1}f(\overline{V})$ is fixed by $\mathcal{P}^-(M \cap K(1))(\mathcal{P}^+)^h = \mathcal{P}(1) \cap \mathcal{P}(1)^h$. Note that the natural map $N(\mathcal{O})/N(\mathcal{O})^h \to \mathcal{P}(1)/(\mathcal{P}(1) \cap \mathcal{P}(1)^h)$ is a bijection. By the definition of the Hecke Z_M^+-action on $\pi^{N(\mathcal{O})}$, we have for all $\overline{v} \in \overline{V}$:

$$\chi_M(h^{-1})f(\overline{v}) = \sum_{N(\mathcal{O})/N(\mathcal{O})^h} nh^{-1}f(\overline{v}) = \sum_{\mathcal{P}(1)/(\mathcal{P}(1)\cap\mathcal{P}(1)^h)} ph^{-1}f(\overline{v}).$$

This implies that $f(\overline{v}) \in \pi^{\mathcal{P}(1)}$. In particular f is $\mathcal{P}$-linear.

By Prop. 4.1 and Cor. 4.2 (or directly) the pair (M, χ_M) gives rise to $\chi : \mathcal{H}_M(\overline{V}) \to \bar{k}$ such that $\chi(T_h^M) = \chi_M(h)^{-1}$ for $h \in Z_M$. We verify that the $\mathcal{P}$-linear map $f : \overline{V} \to \pi^{\mathcal{P}(1)} \hookrightarrow \pi$ is an $\mathcal{H}(\overline{V})$-eigenvector with eigenvalues χ. For $h \in Z_M^-$,

$$(f * E_h)(\overline{v}) = \sum_{\mathcal{P}/(\mathcal{P}\cap\mathcal{P}^h)} ph^{-1}f(E_h(hp^{-1})\overline{v}) = \sum_{N(\mathcal{O})/N(\mathcal{O})^h} nh^{-1}f(\overline{v}) = \chi_M(h^{-1})f(\overline{v}),$$

since $N(\mathcal{O})/N(\mathcal{O})^h \to \mathcal{P}/(\mathcal{P} \cap \mathcal{P}^h)$ is a bijection and by definition of the Z_M^+-Hecke action on $\pi^{N(\mathcal{O})}$. (Note that E_h was defined in the proof of Lemma 2.21.) Now note that $\chi(E_h) = \chi(T_h^M) = \chi_M(h^{-1})$.

Thus we have a natural map

$$\mathrm{Hom}_{M(\mathcal{O})Z_M^+}(\overline{V}, \pi^{N(\mathcal{O})}) \to \mathrm{Hom}_G(\text{c-Ind}_{\mathcal{P}}^G \overline{V} \otimes_{\mathcal{H}(\overline{V}),\chi} \bar{k}, \pi).$$

Reversing the above argument we obtain an inverse map. This completes the proof. □

Lemma 9.16. *Suppose that π is an irreducible admissible G-representation. Then there exists a standard parabolic $P = MN$ and an irreducible admissible M-representation σ such that $\mathrm{Ind}_{\overline{P}}^G \sigma \twoheadrightarrow \pi$ and such that for all triples (K_M', T_M', B_M') as in Def. 4.7, all T_M'-regular K_M'-weights of σ are supersingular.*

Proof. Suppose that π contains a T'-regular K'-weight that is non-supersingular, for some triple (K', T', B'). Relabel (K', T', B') as (K, T, B) for a moment. Pick a T-regular weight V that occurs in π with Hecke eigenvalues parameterised by the pair (M, χ_M), $M \neq G$. In the notation of the proof of Thm. 9.8 we see that $\Delta_V = \varnothing$ and $\Delta_M \neq \Delta$. By Case 1 of the proof of Thm. 9.8, there is a *proper* standard parabolic $Q = LN'$, an irreducible admissible L-representation σ, and a G-linear surjection $\mathrm{Ind}_{\overline{Q}}^G \sigma \twoheadrightarrow \pi$. When we change back to the original notation, $\overline{Q}$ gets replaced by some parabolic subgroup, but we can easily rewrite any parabolic induction as an induction from a standard parabolic subgroup. Now induct. □

Theorem 9.17. *Suppose that π is an irreducible admissible G-representation such that the following condition is satisfied:*

$(*')$ *For all triples (K', T', B') as in Def. 4.7 and for all simple roots $\alpha \in X^*(T')$, the restriction to $k^\times$ via the coroot $\alpha^\vee$ of the $T'(k)$-representation $(\mathrm{soc}_{K'} \pi)^{U'(k)}$ does not contain the trivial representation.*

Then there exists a standard parabolic $P = MN$ and an irreducible admissible supersingular M-representation σ such that $\pi \cong \mathrm{Ind}_{\overline{P}}^G \sigma$.

In the situation of the theorem all Hecke eigenvalues of π can be naturally identified. We can compare Hecke eigenvalues in the following way. Suppose we work with a triple (K', T', B'). The Hecke eigenvalues of a K'-weight are then parameterised by a pair $(L', \chi_{L'})$, where L' is a standard Levi with respect to (T', B'). The pair (T', B') can be conjugated to (T, B) by an element of G, which is unique up to T. After conjugating $(L', \chi_{L'})$ by such an element we get a well-defined pair (L, χ_L) with L standard and $\chi_L : Z_L \to \bar{k}^\times$ a smooth character. (To be precise we mean $L_{/F}$: the integral structure depends on the triple we started with.) It is now clear from Def. 4.7 and Lemma 4.5 that in this way all Hecke eigenvalues of π give rise to the pair (M, ω_σ), where ω_σ is the central character of σ (just as in Cor. 9.10).

Proof. By Thm. 9.16 there exists an irreducible admissible M-representation σ such that $\mathrm{Ind}_{\overline{P}}^G \sigma \twoheadrightarrow \pi$ and such that all T_M'-regular K_M'-weights of σ are

supersingular. It will suffice to show that condition $(*')$ implies that $\mathrm{Ind}_P^G \sigma$ is irreducible and that all K'_M-weights of σ are T'_M-regular (for any triple (K'_M, T'_M, B'_M)).

We first show that for any weight $\overline{V}$ of σ the unique M-regular weight V of G such that $V^{N(k)} \cong \overline{V}$ (see Lemma 2.5) is a weight of π. Let $\chi_M : Z_M \to \bar{k}^\times$ be the central character of σ, so we have an $M(\mathcal{O})$-linear map $\overline{V} \hookrightarrow \sigma \hookrightarrow (\mathrm{Ord}_P \pi)^{Z_M = \chi_M}$. By Prop. 9.14 we obtain an algebra homomorphism $\chi : \mathcal{H}_M(\overline{V}) \to \bar{k}$ and a G-linear surjection c-$\mathrm{Ind}_P^G \overline{V} \otimes_{\mathcal{H}(\overline{V}),\chi} \bar{k} \twoheadrightarrow \pi$. As V is M-regular, we may apply Cor. 2.25 and get a G-linear surjection c-$\mathrm{Ind}_K^G V \otimes_{\mathcal{H}(\overline{V}),\chi} \bar{k} \twoheadrightarrow \pi$. Thus V is a weight of π.

To show that $\mathrm{Ind}_P^G \sigma$ is irreducible it suffices to show that σ satisfies condition $(*)$ of Thm. 8.8. Suppose that $\overline{V}$ is a weight of σ. By what we just showed, $\overline{V} \cong V^{N(k)}$ for some weight V of π. So $\overline{V}_{(\overline{U} \cap M)(k)} \cong \overline{V}^{(U \cap M)(k)} \cong V^{U(k)}$. Thus the condition follows from $(*')$.

To complete the proof, it will be useful to have an alternative description of a triple (K, T, B). We claim that K is a hyperspecial subgroup corresponding to a point in the apartment of $T_{/F}$. Conversely we claim that if we start with a hyperspecial subgroup K' corresponding to a point in the apartment of a torus $T'_{/F}$ and a Borel subgroup $B'_{/F}$ containing $T'_{/F}$, then $K' = G'(\mathcal{O})$ for a unique reductive integral structure G' of $G_{/F}$. Moreover, $T'_{/F}$ extends (uniquely) to a split maximal torus T' of G' and $B'_{/F}$ extends (uniquely) to a Borel subgroup of G' containing T'.

To see the first claim, we already recalled in Section 4 that the reductive integral structure G of $G_{/F}$ corresponds to a unique hyperspecial point x in the building of $G_{/F}$ (the unique fixed point of K). Choose a torus $T_{x/F}$ in $G_{/F}$ whose apartment contains x. By Bruhat–Tits theory, the torus $T_{x/F}$ extends (uniquely) to a split maximal torus T_x of G. But then T and T_x are conjugate by an element of $G(\mathcal{O}) = K$ [G$^+$70, Exp. XXVI, Prop. 6.16], so x also lies in the apartment of $T_{/F}$. For the second claim, the corresponding hyperspecial point gives rise to the required reductive integral structure G' of $G_{/F}$. The remaining assertions are established by Bruhat–Tits ([BT84, §II.4.6]; see [Her11, §3] for more details).

Finally we deduce that all K'_M-weights of σ are T'_M-regular. By conjugating (K'_M, T'_M, B'_M) by an element of M, we may assume that (T'_M, B'_M) agrees with $(T, B \cap M)$ on the generic fibre. We claim that there is a reductive integral structure G' of $G_{/F}$ with a torus T' and a Borel subgroup B' containing it such that (T', B') agrees with (T, B) on the generic fibre and such that $G'(\mathcal{O}) \cap M = K'_M$. By the above claims it suffices to find a hyperspecial subgroup K' corresponding to a point in the apartment of $T_{/F}$ such that $K' \cap M = K'_M$. By [BT72, §7.6] and [BT84, II.4.2.18] the extended building of $M_{/F}$ can be embedded M-equivariantly in the extended building of $G_{/F}$ such that the apartments $A_M(T_{/F})$ and $A_G(T_{/F})$ of $T_{/F}$ coincide. Moreover any wall in $A_M(T_{/F})$ is a wall in $A_G(T_{/F})$. Let y be a

(hyperspecial) point in $A_M(T_{/F})$ whose stabiliser in M is K'_M. (Note that y need not be unique, since we work with the extended building.) As y is hyperspecial, for each simple root $\alpha \in \Delta_M$ there is an α-wall H_α that passes through y. The stabiliser in M of any point in $\bigcap_{\alpha \in \Delta_M} H_\alpha$ is K'_M. (It projects to a single point in the reduced building of $M_{/F}$.) For each simple root $\beta \in \Delta - \Delta_M$ pick an arbitrary β-wall H_β in $A_G(T_{/F})$. As simple roots are linearly independent, we see that $\bigcap_{\beta \in \Delta} H_\beta \neq \varnothing$. Let x be any point in this intersection. It is hyperspecial and its stabiliser K' in G is as required.

After relabelling (K', T', B') as (K, T, B) we can now assume that $(K'_M, T'_M, B'_M) = (M(\mathcal{O}), T, B \cap M)$. Then notice that we established above that condition $(*)$ holds for σ, but even for *all* simple roots α. We show that the condition for simple roots $\alpha \in \Delta_M$ implies the T-regularity of all weights of σ. If the derived subgroup of M is simply connected, we can write $\overline{V} \cong F^M(\nu)$ for some $\nu \in X^*(T)$. If $s_\alpha(\nu) = \nu$ for some $\alpha \in \Delta_M$ it follows that $\nu \circ \alpha^\vee$ is trivial on $k^\times$, contradicting the condition. In the general case one uses a z-extension of $M_{/k}$. $\square$

10. Submodule structure

In this section we determine the submodule structure of parabolically induced representations in two situations: for the trivial principal series (G general) and for the parabolic induction of an irreducible admissible representation (when $G = \mathrm{GL}_n$).

We recall some elementary facts about the submodule structure of finite length modules whose constituents occur with multiplicity one. Let M be such a module over some, not necessarily commutative, ring. Let J denote the (finite) set of irreducible constituents of M. For each $j \in J$ there is a unique submodule M_j whose cosocle is j. There is a partial order $\leq$ on J such that $i \leq j$ if and only if M_j contains i as constituent. Then the lattice of submodules of M (with respect to inclusion) is naturally isomorphic to the lattice of lower sets in $(J, \leq)$ (with respect to inclusion). (By a *lower set* in a partially ordered set $(J, \leq)$ we mean a subset $J' \subset J$ such that $i \in J'$ whenever $i \leq j$ and $j \in J'$.) A submodule N is sent to the lower set $\mathrm{JH}(N)$ of constituents of N, whereas a lower set $J' \subset J$ is sent to the submodule $\sum_{j \in J'} M_j$.

Recall from Cor. 7.3 that all constituents of $\mathrm{Ind}_B^G 1$ occur with multiplicity one.

Proposition 10.1. *Let Π denote the set of standard parabolic subgroups. The lattice of submodules of $\mathrm{Ind}_B^G 1$ is naturally isomorphic to the lattice of lower sets in $(\Pi, \supset)$.*

Note that we can identify the set of constituents of $\mathrm{Ind}_B^G 1$ with the set of standard parabolic subgroups via $\mathrm{Sp}_P \leftrightarrow P$. With this identification a submodule N corresponds to the lower set $\mathrm{JH}(N)$.

We have an analogous result for $\mathrm{Ind}_P^G \mathrm{Sp}_Q$ as in Prop. 7.6: the submodule lattice is isomorphic to the lattice of lower sets in $(\Pi_Q, \supset)$, where Π_Q consists of all standard parabolic subgroups P' such that $P' \cap M = Q$. (Just note that $\mathrm{Ind}_P^G \mathrm{Sp}_Q$ is a subquotient of $\mathrm{Ind}_B^G 1$.)

Proof. Let P be a standard parabolic subgroup. We will show that $\mathrm{Ind}_P^G 1$ is the unique submodule with cosocle Sp_P. This implies the result: we have $\mathrm{JH}(\mathrm{Ind}_P^G 1) = \{\mathrm{Sp}_Q : Q \supset P\}$ by using (7.1) and induction. Recall from Section 7 that each Sp_Q contains a unique weight V_Q and that these weights are pairwise distinct. Moreover we saw in the proof of Thm. 7.2 that V_P is a weight of $\mathrm{Ind}_P^G 1$ and that it generates $\mathrm{Ind}_P^G 1$ as G-representation. Thus the image of V_P generates any quotient of $\mathrm{Ind}_P^G 1$, so Sp_P is the unique irreducible quotient of $\mathrm{Ind}_P^G 1$. $\qquad\square$

Now let $G = \mathrm{GL}_n$. Let $P = MN$ be a standard parabolic subgroup and σ an irreducible admissible M-representation. We will determine the submodule structure of $\mathrm{Ind}_P^G \sigma$. Recall from Cor. 9.13 that $\mathrm{Ind}_P^G \sigma$ has finite length and that the constituents occur with multiplicity one.

We can use Thm. 9.8 to express $\pi := \mathrm{Ind}_P^G \sigma$ in the same form as in Thm. 8.7. As in the proof of Thm. 8.7 we can further rewrite π as $\mathrm{Ind}_Q^G(\tau_1 \otimes \cdots \otimes \tau_s)$, where Q is the standard parabolic with Levi $\prod_i \mathrm{GL}_{m_i}$, such that for all i either

- τ_i is supersingular and $m_i > 1$, or
- $\tau_i \cong \mathrm{Ind}_{R_i}^{\mathrm{GL}_{m_i}} \mathrm{Sp}_{S_i} \otimes \eta_i'$ for some $\eta_i' : F^\times \to \bar{k}^\times$

and such that $\eta_i' \neq \eta_{i+1}'$ whenever both τ_i and τ_{i+1} fall into the second case. Here $R_i = L_i N_i'$ is a standard parabolic of GL_{m_i} and S_i a standard parabolic of L_i. Let $I \subset \{1, \dots, s\}$ be the subset of those i such that τ_i falls into the second case. Let X be the set consisting of tuples $(S_i')_{i \in I}$ such that each S_i' is a standard parabolic subgroup of GL_{m_i} satisfying $S_i' \cap L_i = S_i$. We define a partial order $\leq_X$ on X by declaring that $(S_i')_{i \in I} \leq_X (S_i'')_{i \in I}$ if and only if $S_i' \supset S_i''$ for all $i \in I$.

Proposition 10.2. *With the above notation, the lattice of submodules of $\mathrm{Ind}_P^G \sigma$ is naturally isomorphic to the lattice of all lower sets in $(X, \leq_X)$.*

Proof. Whenever we are given GL_{m_i}-representations τ_i' for $i \in I$, let $\varsigma(\tau_i' : i \in I)$ denote the $\prod_i \mathrm{GL}_{m_i}$-representation $\bigotimes_{i=1}^s \tau_i'$, where $\tau_i' = \tau_i$ for $i \notin I$. Also let $\pi(\tau_i' : i \in I)$ denote $\mathrm{Ind}_Q^G(\varsigma(\tau_i' : i \in I))$. Then the constituents of π are the $\pi(\mathrm{Sp}_{S_i'} : i \in I)$ for $(S_i')_{i \in I} \in X$.

Note that π is a subquotient of $\pi' := \pi(\mathrm{Ind}_{B_i}^{\mathrm{GL}_{m_i}} \eta_i' : i \in I)$, where B_i is the Borel in GL_{m_i}. The corresponding partially ordered sets $(X, \leq_X)$, $(X', \leq_{X'})$ are compatible in the sense that $\leq_X$ is the restriction of $\leq_{X'}$ to X. Without loss of generality we can therefore assume that $\pi = \pi'$. (I.e., $R_i = B_i$ and $S_i = L_i = T$.) Then π has constituents $\pi(\mathrm{Sp}_{R_i} \otimes \eta_i' : i \in I)$,

where the R_i run over all standard parabolic subgroups of GL_{m_i} (for $i \in I$). Fix now such R_i. We claim that the submodule $\pi(\mathrm{Ind}_{\overline{R_i}}^{\mathrm{GL}_{m_i}} \eta_i' : i \in I)$ has irreducible cosocle $\pi(\mathrm{Sp}_{R_i} \otimes \eta_i' : i \in I)$. To see this, let $\overline{V_i}$ be any weight of τ_i for $i \notin I$ and the unique weight of $\mathrm{Sp}_{R_i} \otimes \eta_i'$ for $i \in I$. Then $\overline{V} := \bigotimes \overline{V_i}$ is a weight of $\varsigma(\mathrm{Ind}_{\overline{R_i}}^{\mathrm{GL}_{m_i}} \eta_i' : i \in I)$ and generates it as $\prod_i \mathrm{GL}_{m_i}$-representation (by the proof of Thm. 7.2). Let V be the $\prod_i \mathrm{GL}_{m_i}$-regular weight for G that corresponds to $\overline{V}$ under the bijection of Lemma 2.5. By Thm. 3.1 it follows that V generates $\pi(\mathrm{Ind}_{\overline{R_i}}^{\mathrm{GL}_{m_i}} \eta_i' : i \in I)$ as G-representation. (Just pick any set of Hecke eigenvalues χ for the weight $\overline{V}$.) But (2.13) and the fact that generalised Steinberg representations can be distinguished by their weights show that $\pi(\mathrm{Ind}_{\overline{R_i}}^{\mathrm{GL}_{m_i}} \eta_i' : i \in I)$ indeed has irreducible cosocle $\pi(\mathrm{Sp}_{R_i} \otimes \eta_i' : i \in I)$.

Finally notice that the constituents of $\pi(\mathrm{Ind}_{\overline{R_i}}^{\mathrm{GL}_{m_i}} \eta_i' : i \in I)$ are all $\pi(\mathrm{Sp}_{R_i'} \otimes \eta_i' : i \in I)$ with $R_i' \supset R_i$ for all $i \in I$. (Decompose each $\mathrm{Ind}_{\overline{R_i}}^{\mathrm{GL}_{m_i}} \eta_i'$ and apply Thm. 8.6.) $\qquad\square$

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Department of Mathematics, 2033 Sheridan Road, Evanston IL 60208-2730, USA

E-mail address: `herzig@math.northwestern.edu`