MINI-COURSE: p-MODULAR REPRESENTATIONS OF p-ADIC GROUPS

FLORIAN HERZIG,
NOTES TRANSCRIBED BY KAROL KOZIOŁ

These notes are an introduction to the p-modular (or "mod-p") representation theory of p-adic reductive groups. We will focus on the group $\mathrm{GL}_2(\mathbb{Q}_p)$, but try to provide statements that generalize to an arbitrary p-adic reductive group G (for example, $\mathrm{GL}_n(\mathbb{Q}_p)$).[1]

1. Motivation

We fix two primes p and ℓ. The motivation for studying the representation theory of p-adic groups comes from Local Langlands Conjectures.

1.1. The case $\ell \neq p$. In this case, we are in the setting of the classical Local Langlands Correspondence, which can be stated (roughly) as follows: Let $n \geq 1$. We then have an injective map

(1)
$$\left\{\begin{array}{c}\text{continuous representations of}\\ \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\text{ on }n\text{-dimensional}\\ \overline{\mathbb{Q}}_\ell\text{-vector spaces, up to}\\ \text{isomorphism}\end{array}\right\} \hookrightarrow \left\{\begin{array}{c}\text{irreducible, admissible}\\ \text{representations of }\mathrm{GL}_n(\mathbb{Q}_p)\\ \text{on }\overline{\mathbb{Q}}_\ell\text{-vector spaces,}\\ \text{up to isomorphism}\end{array}\right\}$$

The statement above is a bit imprecise; one needs to impose that Frobenius acts semisimply on the left-hand side. Usually, the left-hand side is enlarged by replacing it by the set of Frobenius-semisimple Weil-Deligne representations, so that one obtains a bijection. This correspondence can be uniquely characterized by a list of properties (equivalence of L- and ε-factors of pairs, compatibility with contragredients, etc.). In particular, for $n = 1$, the correspondence reduces to local class field theory.

The correspondence was first established by Harris-Taylor ([13]) and Henniart ([14]) in 2000, and more recently by Scholze ([23]).

1.2. The case $\ell = p$. In this case, we would like to have a p-adic analog of (1), to be dubbed the "p-adic Local Langlands Correspondence." Additionally, we would like to have a "mod-p version;" that is, a "mod-p Local Langlands Correspondence," which would be compatible with the p-adic correspondence by reduction of lattices on both sides.

When $\ell = p$, we have "more" Galois representations, due to the fact that $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ have compatible topologies. Breuil proposed to

[1] We thank Vytas Paškūnas for his help with Section 10.

replace the right-hand side above with certain (not necessarily irreducible) Banach space representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\overline{\mathbb{Q}}_p$ (or, more precisely, a fixed finite extension of $\mathbb{Q}_p$). For the mod-p correspondence, one takes admissible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$.

When $n = 2$, such a correspondence has been made precise and proven by work of Breuil, Colmez, Paškūnas and others (see [7], [10], [12], [19], [22], and the references therein). However, for $n > 2$, there are fewer tools one has to attack this problem. For an overview of what is known, see [8].

The goal of these notes will be to describe the irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$, or more generally, an algebraically closed field E of characteristic p. The classification was first obtained by Barthel-Livné ([3], [4]), and completed by Breuil ([6]). One of the main differences between the mod-p theory and the classical theory (i.e., the theory of complex representations) is the absence of an $\overline{\mathbb{F}}_p$-valued Haar measure, which limits the techniques at ones disposal.

2. p-adic Groups

We begin with the basics. Let p be a prime number, and let $\mathbb{Z}_p$ denote the ring of p-adic integers; it is a discrete valuation ring with uniformizer p. We have

$$\mathbb{Z}_p/p^r\mathbb{Z}_p \cong \mathbb{Z}/p^r\mathbb{Z},$$

so in particular, the residue field of $\mathbb{Z}_p$ is $\mathbb{F}_p$. We let $\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p)$ be the field of p-adic numbers. The topology of $\mathbb{Q}_p$ is defined by a fundamental system of neighborhoods of 0:

$$\mathbb{Z}_p \supset p\mathbb{Z}_p \supset p^2\mathbb{Z}_p \supset \ldots \supset p^r\mathbb{Z}_p \supset \ldots$$

The basis for the topology is then given by the cosets of these neighborhoods. With this topology, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ are topological rings. We note that the sets $p^r\mathbb{Z}_p$ are all compact and open.

We take $n \geq 2$, and set $G = \mathrm{GL}_n(\mathbb{Q}_p)$. This is again a topological group, and a fundamental system of neighborhoods of 1 is given by

$$\mathrm{GL}_n(\mathbb{Z}_p) \supset 1 + p\mathrm{M}_n(\mathbb{Z}_p) \supset 1 + p^2\mathrm{M}_n(\mathbb{Z}_p) \supset \ldots \supset 1 + p^r\mathrm{M}_n(\mathbb{Z}_p) \supset \ldots.$$

We again note that the subgroups $1 + p^r\mathrm{M}_n(\mathbb{Z}_p)$ are compact and open. We define

$$\begin{aligned} K &:= \mathrm{GL}_n(\mathbb{Z}_p) \\ K(r) &:= 1 + p^r\mathrm{M}_n(\mathbb{Z}_p) \end{aligned}$$

The group K is a maximal compact subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$, and we have

$$K/K(r) \cong \mathrm{GL}_n(\mathbb{Z}_p/p^r\mathbb{Z}_p).$$

In particular, $K/K(1) \cong \mathrm{GL}_n(\mathbb{F}_p)$.

When $n = 2$, we define the following closed subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$:

$$B := \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \qquad T := \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \qquad U := \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix},$$
$$\overline{B} := \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, \qquad \overline{U} := \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix},$$

where a $*$ indicates an arbitrary entry. The group T is a maximal torus of $\mathrm{GL}_2(\mathbb{Q}_p)$, B is a Borel subgroup, and U its unipotent radical. We have factorizations as follows:

$$B = T \ltimes U, \quad \overline{B} = T \ltimes \overline{U}.$$

3. Smooth Representations

In what follows, we let Γ be a closed subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$ with the subspace topology, or a finite group with the discrete topology. These assumptions ensure that the identity element of Γ has a neighborhood basis of compact open subgroups. We take E to be an algebraically closed field of characteristic p (e.g., $\overline{\mathbb{F}}_p$). This will serve as the coefficient field for all representations we consider. Given a representation π of Γ, we will often identify π with its underlying vector space. Moreover, given a vector $v \in \pi$, we denote the action of $\gamma \in \Gamma$ on v by $\gamma.v$. If S is a subset of π and H is a submonoid of Γ, we let $\langle H.S \rangle_E$ be the smallest subspace of π containing S and stable by the action of H.

Definition 1. A representation π of Γ on an E-vector space is said to be *smooth* if

$$\pi = \bigcup_{\text{open subgroups } W} \pi^W,$$

where π^W denotes the subspace of vectors fixed by a subgroup W. Equivalently, π is smooth if the above equality holds with W running over compact open subgroups.

We remark that the smoothness condition is equivalent to saying that every vector is fixed by an open subgroup. This, in turn, is equivalent to saying that the action map

$$\begin{array}{ccc} \Gamma \times \pi & \longrightarrow & \pi \\ (\gamma, v) & \longmapsto & \gamma.v \end{array}$$

is continuous, where π is given the discrete topology.

Definition 2. If π is any representation of Γ (not necessarily smooth), we define

$$\pi^\infty := \bigcup_{\text{open subgroups } W} \pi^W \subset \pi$$

to be the largest smooth subrepresentation of π.

We now proceed to define two types of induction functors.

Assume first that H is a closed subgroup of Γ, and let σ be a smooth representation of H. We define

$$\mathrm{Ind}_H^\Gamma(\sigma) := \{f : \Gamma \longrightarrow \sigma : f(h\gamma) = h.f(\gamma) \text{ for all } h \in H, \gamma \in \Gamma\}^\infty, \tag{2}$$

where the action of Γ is given by

$$(g.f)(\gamma) = f(\gamma g)$$

for every $\gamma, g \in \Gamma$. This procedure gives a smooth representation of Γ, and we call this functor *induction.*

Assume now that W is an open subgroup of Γ. Since Γ is a topological group, this implies that W is also closed in Γ. Let τ be a smooth representation of W, and define

$$\mathrm{ind}_W^\Gamma(\tau) := \left\{ f : \Gamma \longrightarrow \tau : \begin{array}{l} \diamond\ f(w\gamma) = w.f(\gamma) \text{ for all } w \in W, \gamma \in \Gamma \\ \diamond\ W\backslash \mathrm{supp}(f) \text{ is compact} \end{array} \right\}. \tag{3}$$

We note that $\mathrm{supp}(f)$ will be a union of W-cosets, so the quotient $W\backslash\mathrm{supp}(f)$ makes sense. Additionally, since W is open, the condition of $W\backslash\mathrm{supp}(f)$ being compact is equivalent to $W\backslash\mathrm{supp}(f)$ being finite. We define an action of Γ on $\mathrm{ind}_W^\Gamma(\tau)$ as above, and again obtain a smooth representation (without having to take smooth vectors!). We call this functor *compact induction.*

For $\gamma \in \Gamma, x \in \tau$, we will denote by $[\gamma, x] \in \mathrm{ind}_W^\Gamma(\tau)$ the function satisfying $\mathrm{supp}([\gamma, x]) = W\gamma^{-1}$ and $[\gamma, x](\gamma^{-1}) = x$. We have

$$[\gamma w, x] = [\gamma, w.x]$$

for $w \in W$, and moreover, the action of $g \in \Gamma$ on $[\gamma, x]$ is given by

$$g.[\gamma, x] = [g\gamma, x].$$

Remark 3. If the index $(\Gamma : H)$ is finite, then the subgroup H is open if and only if it is closed. In this case, the functors $\mathrm{Ind}_H^\Gamma(-)$ and $\mathrm{ind}_H^\Gamma(-)$ coincide.

Remark 4. Since the elements of the compact induction $\mathrm{ind}_W^\Gamma(\tau)$ have finite support modulo W, we have a Γ-equivariant isomorphism

$$\mathrm{ind}_W^\Gamma(\tau) \cong E[\Gamma] \otimes_{E[W]} \tau,$$

as in the representation theory of finite groups.

The following result is central in representation theory.

Theorem 5 (Frobenius Reciprocity)**.** *Let H be a closed subgroup of Γ and W an open subgroup of Γ. Assume that π is a smooth representations of Γ, σ a smooth representation of H, and τ a smooth representation of W. We then have natural isomorphisms:*

(i) $\mathrm{Hom}_\Gamma(\pi, \mathrm{Ind}_H^\Gamma(\sigma)) \cong \mathrm{Hom}_H(\pi|_H, \sigma)$
(ii) $\mathrm{Hom}_\Gamma(\mathrm{ind}_W^\Gamma(\tau), \pi) \cong \mathrm{Hom}_W(\tau, \pi|_W)$

Proof. (i) Given a homomorphism $\varphi \in \mathrm{Hom}_\Gamma(\pi, \mathrm{Ind}_H^\Gamma(\sigma))$, we obtain an element of $\mathrm{Hom}_H(\pi|_H, \sigma)$ by post-composing φ with the natural H-equivariant evaluation map

$$\begin{array}{rcl} \mathrm{Ind}_H^\Gamma(\sigma) & \longrightarrow & \sigma \\ f & \longmapsto & f(1). \end{array}$$

(ii) Again, given a homomorphism $\varphi \in \mathrm{Hom}_\Gamma(\mathrm{ind}_W^\Gamma(\tau), \pi)$, we obtain an element of $\mathrm{Hom}_W(\tau, \pi|_W)$ by pre-composing φ with the W-equivariant map θ, given by

$$\begin{array}{rcl} \tau & \stackrel{\theta}{\longrightarrow} & \mathrm{ind}_W^\Gamma(\tau) \\ x & \longmapsto & [1, x]. \end{array}$$

The remaining details are left as an exercise. $\square$

Remark 6. Note that any element of the compact induction $\mathrm{ind}_W^\Gamma(\tau)$ can be written as

$$\sum_{i=1}^r [\gamma_i, x_i] = \sum_{i=1}^r \gamma_i.[1, x_i] = \sum_{i=1}^r \gamma_i.\theta(x_i),$$

where $\gamma_i \in \Gamma, x_i \in \tau$. This shows that $\theta(\tau)$ generates $\mathrm{ind}_W^\Gamma(\tau)$ as a Γ-representation.

Example 7. Let $n = 2$, and $G = \mathrm{GL}_2(\mathbb{Q}_p)$. We let $\chi_1, \chi_2 : \mathbb{Q}_p^\times \longrightarrow E^\times$ be two smooth characters (i.e., smooth, one-dimensional representations) of $\mathbb{Q}_p^\times$, and denote by

$$\begin{array}{rccl} \chi_1 \otimes \chi_2 : & \overline{B} & \longrightarrow & E^\times \\ & \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} & \longmapsto & \chi_1(\alpha)\chi_2(\delta) \end{array}$$

the character of $\overline{B}$ obtained by inflation from T. Taking the induction of $\chi_1 \otimes \chi_2$ from $\overline{B}$ to G, we obtain a smooth G-representation $\mathrm{Ind}_{\overline{B}}^G(\chi_1 \otimes \chi_2)$, called a *principal series representation.* Note that this representation is infinite-dimensional: it is not hard to verify that we have a vector space isomorphism

$$\begin{array}{rcl} \{f \in \mathrm{Ind}_{\overline{B}}^G(\chi_1 \otimes \chi_2) : \mathrm{supp}(f) \subset \overline{B}U\} & \stackrel{\sim}{\longrightarrow} & C_c^\infty(\mathbb{Q}_p, E) \\ f & \longmapsto & \left(x \longmapsto f\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right), \end{array}$$

where $C_c^\infty(\mathbb{Q}_p, E)$ denotes the space of locally constant, compactly supported, E-valued functions on $\mathbb{Q}_p$. We also remark that this vector space isomorphism can actually be upgraded to an isomorphism of B-representations, as the left-hand side is B-stable.

We now proceed to discuss some distinguished features of representation theory in characteristic p.

Definition 8. A *pro-p group* is a topological group which is compact and Hausdorff, and possesses a fundamental system of neighborhoods of the identity consisting of normal subgroups of p-power index.

Examples 9.

(i) A finite p-group with the discrete topology is obviously a pro-p group.
(ii) The ring of p-adic integers $\mathbb{Z}_p$ is a pro-p group (using that $\mathbb{Z}_p/p^r\mathbb{Z}_p \cong \mathbb{Z}/p^r\mathbb{Z}$).
(iii) Using the fact that $K(r)/K(r+1) \cong \mathrm{M}_n(\mathbb{F}_p)$ for $r > 0$ (exercise), one sees that $K(1)$ is a pro-p group. This implies in particular that any E-valued Haar measure μ of G vanishes on $K(r)$ for all $r > 0$, hence $\mu = 0$.
(iv) Let $I(1)$ denote the preimage in $K = \mathrm{GL}_n(\mathbb{Z}_p)$ of the unipotent (p-Sylow) subgroup

$$\begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \subset \mathrm{GL}_n(\mathbb{F}_p).$$

This subgroup is pro-p by the same argument as in (iii), and is called the *pro-p-Iwahori subgroup* of K.

The following lemma is very useful in mod-p representation theory.

Lemma 10. *Any nonzero smooth representation τ of a pro-p group H has H-fixed vectors; that is, $\tau^H \neq 0$.*

Proof. Forgetting the E-vector-space structure on τ, we can reduce to the case where τ is an $\mathbb{F}_p$-linear representation of H. Fix a vector $x \in \tau - \{0\}$; since the action of H on τ is smooth, $\mathrm{Stab}_H(x)$ is open, and hence of finite index in H (by compactness). The orbit of x under H is then finite, and therefore $\langle H.x \rangle_{\mathbb{F}_p}$ is a finite dimensional $\mathbb{F}_p$-vector-space (isomorphic to $\mathbb{F}_p^d$, say). Now, since H is pro-p, the image of $H \longrightarrow \mathrm{GL}_d(\mathbb{F}_p)$ will be a p-group. As the p-Sylow subgroups of $\mathrm{GL}_d(\mathbb{F}_p)$ are all conjugate to

$$\begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix},$$

the image of H (after conjugation) will fix the first basis vector. □

We may now begin to analyze the mod-p representations of G.

Definition 11. We shall call the irreducible mod-p representations of $\mathrm{GL}_n(\mathbb{F}_p)$ *weights*.

Corollary 12.

(i) We have a canonical bijection

$$\{\text{irreducible smooth representations of } K \text{ over } E\}$$
$$\overset{1:1}{\longleftrightarrow} \{\text{irreducible representations of } \mathrm{GL}_n(\mathbb{F}_p) \text{ over } E\}.$$

(ii) If π is a nonzero smooth representation of $G = \mathrm{GL}_n(\mathbb{Q}_p)$, then $\pi|_K$ contains a weight.

Proof. (i) Since $K(1)$ is open and normal in K and $K/K(1) \cong \mathrm{GL}_n(\mathbb{F}_p)$, we obtain irreducible smooth representations of K by inflation from $K/K(1)$. Conversely, let V be a smooth irreducible representation of K. As $K(1)$ is pro-p, we obtain $V^{K(1)} \neq 0$ by Lemma 10, and $V^{K(1)}$ is K-stable (again using normality of $K(1)$ in K). Hence, $V = V^{K(1)}$, and V is a representation of $K/K(1)$.

(ii) Choosing $x \in \pi^{K(1)} - \{0\}$, we have (by abuse of notation)

$$\langle K.x\rangle_E = \langle \mathrm{GL}_n(\mathbb{F}_p).x\rangle_E.$$

As this space is finite dimensional, it contains an irreducible subrepresentation of K. $\square$

Example 13. Using Corollary 12 above and the decomposition $\mathbb{Q}_p^\times = \mathbb{Z}_p^\times \times p^{\mathbb{Z}}$, we obtain a bijection

$$\begin{array}{ccc} \{\text{smooth characters } \chi : \mathbb{Q}_p^\times \longrightarrow E^\times\} & \overset{1:1}{\longleftrightarrow} & \mathrm{Hom}(\mathbb{F}_p^\times, E^\times) \times E^\times \\ \chi & \longmapsto & (\chi|_{\mathbb{Z}_p^\times}, \chi(p)). \end{array}$$

4. Weights

From now on, we focus on the case $n = 2$. This will allow us to do computations very explicitly. We first describe the weights of $K = \mathrm{GL}_2(\mathbb{Z}_p)$.

For any group X defined over $\mathbb{Q}_p$, we denote by X_p denote the analogous group over $\mathbb{F}_p$; for example

$$G_p := \mathrm{GL}_2(\mathbb{F}_p), \quad B_p := \begin{pmatrix} \mathbb{F}_p^\times & \mathbb{F}_p \\ 0 & \mathbb{F}_p^\times \end{pmatrix},$$

$$T_p := \begin{pmatrix} \mathbb{F}_p^\times & 0 \\ 0 & \mathbb{F}_p^\times \end{pmatrix}, \quad U_p := \begin{pmatrix} 1 & \mathbb{F}_p \\ 0 & 1 \end{pmatrix}, \quad \text{etc.}$$

Proposition 14. *Up to isomorphism, the weights of $G_p = \mathrm{GL}_2(\mathbb{F}_p)$ are precisely the following:*

$$F(a,b) := \mathrm{Sym}^{a-b}(E^2) \otimes \det{}^b,$$

where $0 \leq a - b \leq p-1,\ 0 \leq b < p-1$.

The action of G_p on E^2 is the standard one, given by the inclusion $G_p \hookrightarrow \mathrm{GL}_2(E)$.

We may describe the action of G_p on $F(a,b)$ explicitly as follows. We have an isomorphism $F(a,b) \cong E[X,Y]_{(a-b)}$, where $E[X,Y]_{(a-b)}$ is the space of homogeneous polynomials of degree $a-b$, with the action of $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in G_p$ on $f \in E[X,Y]_{(a-b)}$ given by

$$\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} .f\right)(X,Y) = f(\alpha X + \gamma Y, \beta X + \delta Y)(\alpha\delta - \beta\gamma)^b.$$

We remark that this definition makes sense for $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2(E)$, so that $E[X,Y]_{(a-b)}$ becomes a representation of the algebraic group $\mathrm{GL}_2(E)$. For $\mathrm{GL}_n(\mathbb{F}_p)$ with $n > 2$, there is no simple known explicit way to describe the irreducible mod-p representations. However, they still arise by restricting certain irreducible representations of the algebraic group $\mathrm{GL}_n(E)$ to $\mathrm{GL}_n(\mathbb{F}_p)$.

Idea of proof (Prop. 14). The proof of irreducibility relies on several observations:

(i) The space $F(a,b)^{U_p}$ is one-dimensional, and

$$F(a,b)^{U_p} = EX^{a-b}. \tag{4}$$

The group T_p acts on this space by the character $\eta_a \otimes \eta_b$, where $\eta_a(x) = x^a$ for $x \in \mathbb{F}_p^\times$.

(ii) We have

$$\langle G_p.X^{a-b}\rangle_E = F(a,b).$$

To show this, it suffices to consider the action of $\overline{U}_p$, and then the claim follows from a computation with a Vandermonde determinant (that is, one checks that the elements $\left(\begin{smallmatrix} 1 & 0 \\ \gamma & 1 \end{smallmatrix}\right).X^{a-b}$ for $0 \leq \gamma \leq a-b$ span $F(a,b)$).

The first observation along with Lemma 10 implies that the U_p-invariants of any nonzero subrepresentation of $F(a,b)$ must contain X^{a-b}. Thus, this subrepresentation must be all of $F(a,b)$ by the second observation. This shows that the representations $F(a,b)$ are all irreducible.

By considering the action of T_p on $F(a,b)^{U_p}$, we see that the representations $F(a,b)$ are all pairwise inequivalent, except for possibly the representations $F(b,b)$ and $F(p-1+b,b)$. Since $\dim_E F(b,b) = 1$ and $\dim_E F(p-1+b,b) = p$, we conclude that all of the $F(a,b)$ are indeed distinct.

Now, the number of p-modular representations of the finite group G_p is equal to the number of p-regular conjugacy classes in G_p. Using rational canonical form, it follows that the latter number is exactly $p(p-1)$, which shows that the representations $F(a,b)$ with $0 \leq a-b \leq p-1$ and $0 \leq b < p-1$ form a full system of representatives for the p-modular representations of G_p. $\square$

Lemma 15. *We have*

$$F(a,b)_{\overline{U}_p} \cong \eta_a \otimes \eta_b$$

as T_p-representations, where $V_{\overline{U}_p}$ denotes the $\overline{U}_p$-coinvariants of a G_p- representation V.

Proof. This follows from observation (i) of the previous proof, and the identity

$$(V_{\overline{U}_p})^* \cong (V^*)^{\overline{U}_p}$$

(note that we have $F(a,b)^* \cong F(a,b) \otimes \det^{-a-b}$ and, by conjugation, $F(a,b)^{\overline{U}_p} = EY^{a-b}$). Concretely, we have

$$F(a,b)_{\overline{U}_p} = F(a,b)/\langle EX^iY^{a-b-i} : 0 \le i < a-b\rangle.$$

□

Corollary 16. *The natural map obtained by composing*

$$F(a,b)^{U_p} \hookrightarrow F(a,b) \twoheadrightarrow F(a,b)_{\overline{U}_p}$$

is a T_p-linear isomorphism.

Using the above results, we can classify the weights appearing in principal series representations.

Proposition 17. *Fix two smooth characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \longrightarrow E^\times$. Then*

$$\dim_E \operatorname{Hom}_K(V, \operatorname{Ind}_{\overline{B}}^{G}(\chi_1 \otimes \chi_2)|_K) \le 1$$

for all weights V.

If $\chi_1|_{\mathbb{Z}_p^\times} \ne \chi_2|_{\mathbb{Z}_p^\times}$, then there is precisely one V such that equality holds, and $\dim_E V > 1$. If $\chi_1|_{\mathbb{Z}_p^\times} = \chi_2|_{\mathbb{Z}_p^\times}$, then there are two choices of V such that equality holds, and either $\dim_E V = 1$ or $\dim_E V = p$.

Proof. The proof essentially follows from Frobenius Reciprocity. Note first that, by Corollary 12, the characters $\chi_i|_{\mathbb{Z}_p^\times} : \mathbb{Z}_p^\times \longrightarrow E^\times$ must factor as

$$\begin{tikzcd} \mathbb{Z}_p^\times \arrow[rr, "\chi_i|_{\mathbb{Z}_p^\times}"] \arrow[dr] & & E^\times \\ & \mathbb{F}_p^\times \arrow[ur, "\eta_i"'] & \end{tikzcd}$$

for some $\eta_i : \mathbb{F}_p^\times \longrightarrow E^\times$.

Now, the Iwasawa Decomposition says that we have a factorization of the form

$$G = \overline{B}K,$$

from which it follows that

$$\operatorname{Ind}_{\overline{B}}^{G}(\chi_1 \otimes \chi_2)|_K \cong \operatorname{Ind}_{\overline{B}\cap K}^{K}(\chi_1|_{\mathbb{Z}_p^\times} \otimes \chi_2|_{\mathbb{Z}_p^\times})$$

(this is a special case of the Mackey Decomposition). Both of these facts are left as an exercise for $\mathrm{GL}_2(\mathbb{Q}_p)$. Hence, we obtain

$$\begin{aligned}
\mathrm{Hom}_K(V, \mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)|_K) &\overset{\text{Mackey}}{\cong} \mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{B}\cap K}^K(\chi_1|_{\mathbb{Z}_p^\times} \otimes \chi_2|_{\mathbb{Z}_p^\times})) \\
&\overset{\text{Frobenius}}{\cong} \mathrm{Hom}_{\overline{B}\cap K}(V|_{\overline{B}\cap K},\ \chi_1|_{\mathbb{Z}_p^\times} \otimes \chi_2|_{\mathbb{Z}_p^\times}) \\
&\overset{\substack{\text{action factors}\\ \text{mod } p}}{\cong} \mathrm{Hom}_{\overline{B}_p}(V|_{\overline{B}_p},\ \eta_1 \otimes \eta_2) \\
&\overset{\substack{\overline{U}_p \text{ acts}\\ \text{trivially}}}{\cong} \mathrm{Hom}_{T_p}(V_{\overline{U}_p},\ \eta_1 \otimes \eta_2).
\end{aligned}$$

Lemma 15 shows that $\mathrm{Hom}_K(V, \mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)|_K)$ is one-dimensional if $V_{\overline{U}_p} \cong \eta_1 \otimes \eta_2$, and zero-dimensional otherwise. A quick computation verifies that if $\eta_1 \neq \eta_2$, there is one such weight V (satisfying $\dim_E V > 1$), and two otherwise (satisfying $\dim_E V = 1$ or p). □

5. Hecke Algebras

Throughout this discussion, fix a weight V, and π a smooth representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Since the representation $\pi|_K$ contains a weight (cf. Corollary 12), it is natural to consider the multiplicity space

$$\mathrm{Hom}_K(V, \pi|_K)$$

of the weight V in π. By Frobenius Reciprocity, we have

$$\mathrm{Hom}_K(V, \pi|_K) \cong \mathrm{Hom}_G(\mathrm{ind}_K^G(V), \pi),$$

and the latter space comes equipped with a right action of

$$\mathcal{H}(V) := \mathrm{End}_G(\mathrm{ind}_K^G(V))$$

by pre-composition. The following proposition gives a more concrete description of the algebra $\mathcal{H}(V)$.

Proposition 18. *We have*

$$\mathcal{H}(V) \cong \left\{ \varphi : G \longrightarrow \mathrm{End}_E(V) : \begin{array}{c} \diamond\ \varphi(k_1 g k_2) = k_1 \circ \varphi(g) \circ k_2 \\ \text{for all } k_1, k_2 \in K, g \in G \\ \diamond\ K\backslash \mathrm{supp}(\varphi)/K \text{ is finite} \end{array} \right\}.$$

The composition product on the left-hand side becomes the convolution product on the right-hand side: for $\varphi_1, \varphi_2 : G \longrightarrow \mathrm{End}_E(V)$, we have

$$(\varphi_1 * \varphi_2)(g) = \sum_{\gamma \in G/K} \varphi_1(g\gamma)\varphi_2(\gamma^{-1}).$$

Idea of proof. By Frobenius Reciprocity, we have

$$\mathcal{H}(V) \cong \mathrm{Hom}_K(V, \mathrm{ind}_K^G(V)|_K) \subset \mathrm{Map}(V, \mathrm{Map}(G, V)) \cong \mathrm{Map}(G, \mathrm{Map}(V, V)),$$

where the last isomorphism is the natural map. Since the right-hand side of Proposition 18 is naturally a subspace of

$$\mathrm{Map}(G, \mathrm{Map}(V, V)),$$

one simply needs to check that the conditions on $\mathcal{H}(V)$ cut out the same subspace (see, for example, [18], Proposition 12 for more details). It is then straightforward to verify the final statement. $\square$

Remark 19. Taking $V = \mathbf{1}_K$, the trivial representation of K, we obtain

$$\mathcal{H}(\mathbf{1}_K) \cong C_c(K\backslash G/K, E),$$

the usual double coset algebra of compactly supported, K-biinvariant, E-valued functions on G.

Remark 20. If $\varphi \in \mathcal{H}(V), f \in \mathrm{Hom}_K(V, \pi|_K)$, and $v \in V$, then the right action of $\mathcal{H}(V)$ on $\mathrm{Hom}_K(V, \pi|_K)$ is given by

$$(f * \varphi)(v) = \sum_{g \in K\backslash G} g^{-1}.f(\varphi(g).v). \tag{5}$$

To further analyze the structure of the algebra $\mathcal{H}(V)$, we will need to know about the double coset space $K\backslash G/K$:

Cartan Decomposition.

$$G = \bigsqcup_{\substack{r,s\in\mathbb{Z} \\ r\leq s}} K \begin{pmatrix} p^r & 0 \\ 0 & p^s \end{pmatrix} K$$

This decomposition holds more generally (cf. [25], Section 3.3.3); in the case $G = \mathrm{GL}_2(\mathbb{Q}_p)$, it can be proven using the theory of elementary divisors.

Theorem 21.

(i) For every $r, s \in \mathbb{Z}$ with $r \leq s$, there exists a unique function $\mathrm{T}_{r,s} \in \mathcal{H}(V)$ such that $\mathrm{supp}(\mathrm{T}_{r,s}) = K \left(\begin{smallmatrix} p^r & 0 \\ 0 & p^s \end{smallmatrix}\right) K$ and such that $\mathrm{T}_{r,s} \left(\begin{smallmatrix} p^r & 0 \\ 0 & p^s \end{smallmatrix}\right) \in \mathrm{End}_E(V)$ is a linear projection.

(ii) The set $\{\mathrm{T}_{r,s}\}_{r,s\in\mathbb{Z},r\leq s}$ forms a basis for $\mathcal{H}(V)$.

(iii) We have an algebra isomorphism

$$\mathcal{H}(V) \cong E[\mathrm{T}_1, \mathrm{T}_2^{\pm 1}],$$

where $\mathrm{T}_1 := \mathrm{T}_{0,1}$ and $\mathrm{T}_2 := \mathrm{T}_{1,1}$. In particular, $\mathcal{H}(V)$ is commutative.

Idea of proof. (i) As an example, consider the following identity in G:

$$\begin{pmatrix} \alpha & \beta \\ p\gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} \alpha & p\beta \\ \gamma & \delta \end{pmatrix},$$

where $\alpha, \delta \in \mathbb{Z}_p^\times, \beta, \gamma \in \mathbb{Z}_p$. Since the action of G on V factors through $\mathrm{GL}_2(\mathbb{F}_p)$, we have

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \circ \varphi \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \varphi \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \circ \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$$

for any $\varphi \in \mathcal{H}(V)$. Taking $\alpha = \delta = 1, \gamma = 0$ shows that the image of the operator $\varphi \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ is contained in V^{U_p}. Likewise, taking $\alpha = \delta = 1, \beta = 0$ shows that $\varphi \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ factors through $V_{\overline{U}_p}$. Hence, the map $\varphi \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) : V \longrightarrow V$ factors as

$$\begin{tikzcd} V \arrow[r, "\varphi\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)"] \arrow[d, two heads] & V \\ V_{\overline{U}_p} \arrow[r, dashed] & V^{U_p} \arrow[u, hook] \end{tikzcd}$$

Taking $\beta = \gamma = 0$, we see that the dotted arrow is T_p-linear. Thus, by Corollary 16, we see that there is a one-dimensional space of such maps. We take $\mathrm{T}_{0,1}$ to be the function supported on $K \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) K$, such that $\mathrm{T}_{0,1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ is (the map associated to) the inverse of the obvious map $V^{U_p} \xrightarrow{\sim} V_{\overline{U}_p}$ of Corollary 16. It follows that $\mathrm{T}_{0,1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ is a projection.

More generally, for $r < s$, the argument is the same as above. For $r = s$, it is more elementary, and left as an exercise.
(iii) It is clear that the elements $\{\mathrm{T}_{r,s}\}_{r,s \in \mathbb{Z}, r \leq s}$ form a basis, and one can show that this basis is related to $\{\mathrm{T}_1^i \mathrm{T}_2^j\}_{i \geq 0, j \in \mathbb{Z}}$ by a unitriangular change of coordinates (for a suitable ordering of both sides). □

To see this presentation of the Hecke algebra more naturally, we may use the mod-p Satake transform. We let

$$\mathcal{H}_T(V_{\overline{U}_p}) := \mathrm{End}_T(\mathrm{ind}_{T \cap K}^T(V_{\overline{U}_p})),$$

and note that Proposition 18 remains valid for $\mathcal{H}_T(V_{\overline{U}_p})$ if (G, K, V) is replaced with $(T, T \cap K, V_{\overline{U}_p})$. The Satake transform $\mathcal{S}_G : \mathcal{H}(V) \longrightarrow \mathcal{H}_T(V_{\overline{U}_p})$ is given explicitly by

$$\mathcal{S}_G(\varphi)(t) := \mathrm{pr}_{\overline{U}} \circ \left(\sum_{\overline{u} \in \overline{U} \cap K \backslash \overline{U}} \varphi(\overline{u} t) \right),$$

where $\varphi \in \mathcal{H}(V), t \in T$, and $\mathrm{pr}_{\overline{U}} : V \twoheadrightarrow V_{\overline{U}_p}$ is the natural map. It is straightforward to check that with this definition $\mathcal{S}_G$ is well-defined and an algebra homomorphism. With more work one shows that $\mathcal{S}_G$ is injective and that

$$\text{(6)} \qquad \mathrm{im}\ \mathcal{S}_G = \left\{ \psi \in \mathcal{H}_T(V_{\overline{U}_p}) : \psi \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = 0 \text{ if } yx^{-1} \notin \mathbb{Z}_p \right\}.$$

As T is abelian, one easily obtains the following analog of Theorem 21: $\mathcal{H}_T(V_{\overline{U}_p})$ has basis $\{\tau_{r,s} = \tau_1^{s-r}\tau_2^r\}_{r,s\in\mathbb{Z}}$, where $\tau_{r,s}$ is determined by

$$\mathrm{supp}(\tau_{r,s}) = (T\cap K)\begin{pmatrix} p^r & 0 \\ 0 & p^s \end{pmatrix}, \qquad \tau_{r,s}\begin{pmatrix} p^r & 0 \\ 0 & p^s \end{pmatrix} = 1,$$

and $\tau_1 := \tau_{0,1}, \tau_2 := \tau_{1,1}$. From (6), we get

$$\mathcal{H}(V) \cong \mathrm{im}\ \mathcal{S}_G = E[\tau_1, \tau_2^{\pm 1}].$$

In fact, it is not hard to verify that $\mathcal{S}_G(\mathrm{T}_1) = \tau_1$ and $\mathcal{S}_G(\mathrm{T}_2) = \tau_2$. We remark that the Satake transform exists more generally for any p-adic reductive group (see [16] and [15]).

Proposition 22. *Suppose $\chi_1, \chi_2 : \mathbb{Q}_p^\times \longrightarrow E^\times$ are two smooth characters, and let $f : V \hookrightarrow \mathrm{Ind}_B^G(\chi_1\otimes\chi_2)|_K$ be a nonzero K-linear map. Then*

$$\begin{aligned} f * \mathrm{T}_1 &= \chi_2(p)^{-1}f \\ f * \mathrm{T}_2 &= \chi_1(p)^{-1}\chi_2(p)^{-1}f. \end{aligned}$$

Proof. This is a calculation using equation (5). Alternatively, the map f induces a T-linear map

$$\overline{f} : \mathrm{ind}_{T\cap K}^T(V_{\overline{U}_p}) \twoheadrightarrow \chi_1\otimes\chi_2$$

(see the proof of Proposition 17). The map f then factors as $\mathrm{Ind}_B^G(\overline{f})\circ F_0$, where

$$F_0 : \mathrm{ind}_K^G(V) \longrightarrow \mathrm{Ind}_B^G(\mathrm{ind}_{T\cap K}^T(V_{\overline{U}_p}))$$

is the map introduced in the proof of Theorem 23. Since

$$F_0\circ\varphi = \mathrm{Ind}_B^G(\mathcal{S}_G(\varphi))\circ F_0,$$

one reduces the proof to the analogous, but much simpler, problem for T:

$$\begin{aligned} \overline{f}\circ\tau_1 &= \chi_2(p)^{-1}\overline{f} \\ \overline{f}\circ\tau_2 &= \chi_1(p)^{-1}\chi_2(p)^{-1}\overline{f}. \end{aligned}$$

□

6. Comparison Isomorphisms

Using the results of the previous section, we can now describe comparison isomorphisms between compact and parabolic induction.

We fix two smooth characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \longrightarrow E^\times$, let V be a weight, and suppose

$$f : V \hookrightarrow \mathrm{Ind}_B^G(\chi_1\otimes\chi_2)|_K$$

is a nonzero K-linear homomorphism. Frobenius Reciprocity provides us with a nonzero G-linear map

$$\widetilde{f} : \mathrm{ind}_K^G(V) \longrightarrow \mathrm{Ind}_B^G(\chi_1\otimes\chi_2).$$

By Proposition 22, if $\varphi \in \mathcal{H}(V)$, we have

$$\widetilde{f} \circ \varphi = \chi'(\varphi)\widetilde{f},$$

where $\chi' : \mathcal{H}(V) \longrightarrow E$ is the algebra homomorphism defined by

$$\begin{aligned} \chi'(\mathrm{T}_1) &= \chi_2(p)^{-1} \\ \chi'(\mathrm{T}_2) &= \chi_1(p)^{-1}\chi_2(p)^{-1}. \end{aligned}$$

The universal property of tensor products implies that we have a G-linear map

$$\overline{f} : \mathrm{ind}_K^G(V) \otimes_{\mathcal{H}(V)} \chi' \longrightarrow \mathrm{Ind}_{\overline{B}}^G(\chi_1 \otimes \chi_2).$$

Theorem 23. *The map* $\overline{f}$ *is an isomorphism if* $\dim_E V > 1$.

Idea of proof. One can even prove a "universal" version of this theorem: consider the G-linear map

$$F_0 : \mathrm{ind}_K^G(V) \longrightarrow \mathrm{Ind}_{\overline{B}}^G(\mathrm{ind}_{T\cap K}^T(V_{\overline{U}_p}))$$

obtained as in the proof of Proposition 17 from the $T\cap K$-linear map $V_{\overline{U}_p} \longrightarrow \mathrm{ind}_{T\cap K}^T(V_{\overline{U}_p})$, which corresponds to the identity map in $\mathrm{End}_T(\mathrm{ind}_{T\cap K}^T(V_{\overline{U}_p}))$ under Frobenius Reciprocity. A calculation shows that F_0 is $\mathcal{H}(V)$-linear with respect to $\mathcal{S}_G$, i.e., that $F_0 \circ \varphi = \mathrm{Ind}_{\overline{B}}^G(\mathcal{S}_G(\varphi)) \circ F_0$ for all $\varphi \in \mathcal{H}(V)$. As $\mathcal{S}_G(\mathrm{T}_1) = \tau_1$ is invertible in $\mathcal{H}_T(V_{\overline{U}_p})$ and the actions of G and $\mathcal{H}(V)$ commute, F_0 induces a G-linear and $\mathcal{H}(V)[\mathrm{T}_1^{-1}]$-linear map

$$F : \mathrm{ind}_K^G(V)[\mathrm{T}_1^{-1}] \longrightarrow \mathrm{Ind}_{\overline{B}}^G(\mathrm{ind}_{T\cap K}^T(V_{\overline{U}_p})). \tag{7}$$

It now suffices to show F is an isomorphism, and then specialize T_1 and T_2 (that is, we apply the functor $- \otimes_{\mathcal{H}(V)[\mathrm{T}_1^{-1}]} \chi'$ to both sides of the isomorphism (7) above to recover the map $\overline{f}$).

To show F is surjective, one shows that the elements $F(\mathrm{T}_1^{-n}([1,x]))$, where $n \geq 0$ and $x \in V^{U_p} - \{0\}$, generate the right-hand side of (7) as a G-representation (this argument uses the Bruhat Decomposition and the fact that $\dim_E V > 1$). For the injectivity of F, one can use an elegant argument of Abe which uses the Satake transform (and which works even if $\dim_E V = 1$). See, for example, [18], Theorems 16 and 32 for more details. □

Remark 24. Notice that any pair (V, χ') subject to $\dim_E V > 1$ and $\chi'(\mathrm{T}_1) \neq 0$ arises in this theorem. Hence, for any such pair, $\mathrm{ind}_K^G(V) \otimes_{\mathcal{H}(V)} \chi'$ is a principal series representation.

Corollary 25. *If* $\dim_E V > 1$, *the weight* $f(V)$ *generates* $\mathrm{Ind}_{\overline{B}}^G(\chi_1 \otimes \chi_2)$ *as a* G*-representation.*

Proof. We have already seen that the image of V inside $\mathrm{ind}_K^G(V)$ generates the compact induction as a G-representation. By Theorem 23, we have

$$\mathrm{ind}_K^G(V) \twoheadrightarrow \mathrm{ind}_K^G(V) \otimes_{\mathcal{H}(V)} \chi' \cong \mathrm{Ind}_{\overline{B}}^G(\chi_1 \otimes \chi_2),$$

and the claim follows. □

Corollary 26. *Let $\chi_1, \chi_2 : \mathbb{Q}_p^\times \longrightarrow E^\times$ be two smooth characters, and suppose $\chi_1|_{\mathbb{Z}_p^\times} \neq \chi_2|_{\mathbb{Z}_p^\times}$. Then*

$$\pi = \mathrm{Ind}_{\overline{B}}^{G}(\chi_1 \otimes \chi_2)$$

is an irreducible G-representation.

Proof. By Proposition 17, π contains a unique weight V and it satisfies $\dim_E V > 1$. Moreover, V generates π as a G-representation by Corollary 25. Suppose σ is a nonzero subrepresentation of π; then σ contains the unique weight V, and thus has to be the whole of π. $\square$

7. The Steinberg Representation

In this section we discuss the Steinberg representation, which arises when one takes $\chi_1 = \chi_2 = \mathbf{1}$ (the trivial character) in the definition of principal series.

Definition 27. We define the *Steinberg representation* St by the following short exact sequence:

$$0 \longrightarrow \mathbf{1}_G \longrightarrow \mathrm{Ind}_{\overline{B}}^{G}(\mathbf{1} \otimes \mathbf{1}) \longrightarrow \mathrm{St} \longrightarrow 0,$$

where the first map arises from Frobenius Reciprocity. More concretely, using (2), we can identify St as the quotient

$$\{\text{locally constant functions on } \overline{B}\backslash G \cong \mathbb{P}^1(\mathbb{Q}_p)\}/\{\text{constant functions}\}.$$

Theorem 28. *The representation* St *is irreducible.*

Idea of proof. Recall that the subgroup $I(1)$, defined in Example 9(iv), is a pro-p subgroup of G. One can prove in a straightforward way that $\dim_E \mathrm{St}^{I(1)} = 1$ (one needs to show that any $I(1)$-invariant function in the quotient is the image of an $I(1)$-invariant function in $\mathrm{Ind}_{\overline{B}}^{G}(\mathbf{1} \otimes \mathbf{1})$; see [4], Lemma 27 for more details). This implies that St contains a unique weight, which is of multiplicity 1.

Let $V = F(p-1, 0) = \mathrm{Sym}^{p-1}(E^2)$, and note that $\dim_E V = p$. We thus obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_K(V, \mathbf{1}_G|_K) \longrightarrow \mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{B}}^{G}(\mathbf{1} \otimes \mathbf{1})|_K) \longrightarrow \mathrm{Hom}_K(V, \mathrm{St}|_K).$$

Since $\mathrm{Hom}_K(V, \mathbf{1}_G|_K) = 0$ and $\dim_E \mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{B}}^{G}(\mathbf{1} \otimes \mathbf{1})|_K) = 1$ (by Proposition 17), we obtain $\dim_E \mathrm{Hom}_K(V, \mathrm{St}|_K) \geq 1$, which shows that V is the aforementioned unique weight of St. By Corollary 25, V generates $\mathrm{Ind}_{\overline{B}}^{G}(\mathbf{1} \otimes \mathbf{1})$ as a G-representation, and therefore generates St. The result now follows as in the proof of Corollary 26 above. $\square$

Remark 29. The sequence defining St does not split. To see this, just note that the unique nontrivial weight of $\mathrm{Ind}_{\overline{B}}^{G}(\mathbf{1} \otimes \mathbf{1})$ generates $\mathrm{Ind}_{\overline{B}}^{G}(\mathbf{1} \otimes \mathbf{1})$ as a G-representation, by Corollary 25.

Remark 30. If $\chi : \mathbb{Q}_p^\times \longrightarrow E^\times$ is a smooth character, we can twist the exact sequence defining St to obtain

$$0 \longrightarrow \chi \circ \det \longrightarrow \mathrm{Ind}_B^G(\chi \otimes \chi) \longrightarrow \mathrm{St} \otimes (\chi \circ \det) \longrightarrow 0.$$

By what we've proved for the representation St, we know that the representations $\chi \circ \det$ and $\mathrm{St} \otimes (\chi \circ \det)$ are irreducible, and contain unique weights (of dimensions 1 and p, respectively). Moreover, the Hecke eigenvalues on these representations are the same as those of $\mathrm{Ind}_B^G(\chi \otimes \chi)$, and are given by

$$\mathrm{T}_1 \longmapsto \chi(p)^{-1}, \qquad \mathrm{T}_2 \longmapsto \chi(p)^{-2}.$$

8. Change of Weight

We now fix two weights V, V', and consider the module of intertwiners given by

$$\mathcal{H}(V, V') := \mathrm{Hom}_G(\mathrm{ind}_K^G(V), \mathrm{ind}_K^G(V')).$$

It naturally has the structure of an $(\mathcal{H}(V'), \mathcal{H}(V))$-bimodule by pre- and post-composition.

Proposition 31.

(i) We have

$$\mathcal{H}(V, V') \cong \left\{ \varphi : G \longrightarrow \mathrm{Hom}_E(V, V') : \begin{array}{c} \diamond\ \varphi(k_1 g k_2) = k_1 \circ \varphi(g) \circ k_2 \\ \text{for all } k_1, k_2 \in K, g \in G \\ \diamond\ K\backslash \mathrm{supp}(\varphi)/K \text{ is finite} \end{array} \right\}.$$

The bimodule structure on the right-hand side is given by convolution (as in Proposition 18).

(ii) We have $\mathcal{H}(V, V') \neq 0$ if and only if $V_{\overline{U}_p} \cong V'_{\overline{U}_p}$ as T_p-representations.

(iii) If $V \ncong V'$ and $V_{\overline{U}_p} \cong V'_{\overline{U}_p}$, then there exists $\varphi \in \mathcal{H}(V, V')$ satisfying $\mathrm{supp}(\varphi) = K \begin{pmatrix} p^r & 0 \\ 0 & p^s \end{pmatrix} K$ if and only if $r < s$.

Proof. The proof is exactly the same as for Proposition 18. □

In case (iii) of the above proposition, the only possible choices for V and V' are $V = F(b, b)$ and $V' = F(p-1+b, b)$ for $0 \leq b < p-1$ (or vice versa). Therefore, there exist G-linear maps

$$\mathrm{ind}_K^G(V) \underset{\varphi^-}{\overset{\varphi^+}{\rightleftarrows}} \mathrm{ind}_K^G(V')$$

satisfying $\mathrm{supp}(\varphi^-) = \mathrm{supp}(\varphi^+) = K \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} K$. These maps are unique up to a scalar.

For the next proposition, it will be convenient to make the identifications

$$\mathcal{H}(V) \cong E[\mathrm{T}_1, \mathrm{T}_2^{\pm 1}] \cong \mathcal{H}(V'),$$

and call both algebras $\mathcal{H}$. (More naturally, both algebras are identified with the same subalgebra of $\mathcal{H}_T(V_{\overline{U}_p}) = \mathcal{H}_T(V'_{\overline{U}_p})$ by the Satake transform.)

Proposition 32.
 (i) The maps φ^-, φ^+ are $\mathcal{H}$-linear and commute with each other.
 (ii) We have (up to a scalar)

$$\varphi^+ \circ \varphi^- = \mathrm{T}_1^2 - \mathrm{T}_2.$$

Sketch of proof. The mod-p Satake transform gives an injective map $\mathcal{H}(V, V') \hookrightarrow \mathcal{H}_T(V_{\overline{U}_p}, V'_{\overline{U}_p})$ with the same formula as before. This map is compatible with convolution. Then (i) follows from the fact that $\mathcal{H}_T(V_{\overline{U}_p}) = \mathcal{H}_T(V'_{\overline{U}_p})$ is commutative, while (ii) is a calculation. □

Corollary 33. *If $\chi' : \mathcal{H} \longrightarrow E$ is an algebra homomorphism such that $\chi'(\mathrm{T}_1^2 - \mathrm{T}_2) \neq 0$, then we obtain a G-linear isomorphism*

$$\mathrm{ind}_K^G(V) \otimes_{\mathcal{H}} \chi' \cong \mathrm{ind}_K^G(V') \otimes_{\mathcal{H}} \chi'.$$

Proof. By Proposition 32, part (i), the maps φ^-, φ^+ induce G-linear maps between the two representations. Their composite is $\chi'(\mathrm{T}_1^2 - \mathrm{T}_2) \neq 0$, so we obtain an isomorphism. □

Proposition 34. *Let $\chi_1, \chi_2 : \mathbb{Q}_p^\times \longrightarrow E^\times$ be two smooth characters, and suppose $\chi_1 \neq \chi_2$. Then*

$$\pi = \mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$$

is an irreducible G-representation.

Proof. If $\chi_1|_{\mathbb{Z}_p^\times} \neq \chi_2|_{\mathbb{Z}_p^\times}$, then the claim follows from Corollary 26. We therefore may assume $\chi_1|_{\mathbb{Z}_p^\times} = \chi_2|_{\mathbb{Z}_p^\times}$ and

$$\chi_1(p) \neq \chi_2(p). \tag{8}$$

By Proposition 17, π contains two weights V, V' of the form $F(b, b)$ and $F(p-1+b, b)$ with $0 \le b < p-1$ (or vice versa). Let us label these weights so that $V = F(b, b)$ and $V' = F(p-1+b, b)$. Corollary 25 then implies that V' generates π as a G-representation, since $\dim_E V' = p > 1$.

Now let σ be a nonzero G-subrepresentation of π. We claim that σ must contain V'. Indeed, if this were not the case, we would necessarily have a K-linear injection $V \hookrightarrow \sigma|_K$ (as $\sigma|_K$ has to contain a weight). This implies the existence of a nonzero G-linear map

$$\mathrm{ind}_K^G(V) \longrightarrow \sigma,$$

which descends to

$$\mathrm{ind}_K^G(V) \otimes_{\mathcal{H}} \chi' \longrightarrow \sigma$$

for some $\chi' : \mathcal{H} \longrightarrow E$, as $\mathrm{Hom}_K(V, \pi|_K)$ is one-dimensional. By Proposition 22, $\chi' : \mathcal{H} \longrightarrow E$ is the character given by

$$\chi'(\mathrm{T}_1) = \chi_2(p)^{-1}, \qquad \chi'(\mathrm{T}_2) = \chi_1(p)^{-1}\chi_2(p)^{-1}.$$

Hence, we have

$$\chi'(\mathrm{T}_1^2 - \mathrm{T}_2) = \chi_2(p)^{-1}(\chi_2(p)^{-1} - \chi_1(p)^{-1}) \neq 0,$$

by equation (8). Therefore, Corollary 33 shows that we have a nonzero G-linear map

$$\mathrm{ind}_K^G(V') \otimes_{\mathcal{H}} \chi' \longrightarrow \sigma,$$

which, by Frobenius Reciprocity, gives a nonzero map $V' \hookrightarrow \sigma|_K$.

Since σ contains V' and V' generates π as a G-representation, we obtain $\sigma = \pi$, and therefore π is irreducible. $\square$

9. Classification of Representations

We are now in a position to give a classification of smooth irreducible representations of $G = \mathrm{GL}_2(\mathbb{Q}_p)$, at least under an admissibility assumption. We remark that this assumption is in fact unnecessary (see [5], [3], [6]).

For the following definition and proposition, suppose that Γ is a closed subgroup of G.

Definition 35. A smooth Γ-representation π is called *admissible* if $\dim_E \pi^W < \infty$ for all open subgroups W of Γ.

Proposition 36. *Let π be a smooth Γ-representation. Then π is admissible if and only if $\dim_E \pi^W < \infty$ for* one *open* pro-p *subgroup W of Γ.*

Proof. Let W be a fixed pro-p subgroup such that $\dim_E \pi^W < \infty$, and let W' be an arbitrary open subgroup. Since $\dim_E \pi^{W'} \leq \dim_E \pi^{W\cap W'}$, it is enough to show $\dim_E \pi^{W\cap W'} < \infty$. We may therefore assume that W' is an open subgroup of W (and hence of finite index). This gives

$$\pi^{W'} = \mathrm{Hom}_{W'}(\mathbf{1}_{W'}, \pi|_{W'}) \cong \mathrm{Hom}_W(\mathrm{ind}_{W'}^W(\mathbf{1}_{W'}), \pi|_W).$$

It thus suffices to show that $\mathrm{Hom}_W(M, \pi|_W)$ is finite-dimensional for any finite-dimensional smooth W-representation M. We argue by induction on $\dim_E M$. By Lemma 10, we have a short exact sequence

$$0 \longrightarrow \mathbf{1}_W \longrightarrow M \longrightarrow M/\mathbf{1}_W \longrightarrow 0.$$

Applying $\mathrm{Hom}_W(-, \pi|_W)$ yields

$$0 \longrightarrow \mathrm{Hom}_W(M/\mathbf{1}_W, \pi|_W) \longrightarrow \mathrm{Hom}_W(M, \pi|_W) \longrightarrow \mathrm{Hom}_W(\mathbf{1}_W, \pi|_W) \cong \pi^W.$$

The last term is finite-dimensional by assumption and the first term is finite-dimensional by induction. Hence the middle term is finite-dimensional. $\square$

The proof of the following proposition is left as an exercise (see also Lemmas 23 and 24 in [18]).

Proposition 37. *Let π be a smooth representation of G.*

(i) The representation π is admissible if and only if $\dim_E \mathrm{Hom}_K(V, \pi|_K) < \infty$ for any weight V.

(ii) If π is admissible, then π possesses a central character.

Corollary 38. *All principal series representations and all representations of the form* $\mathrm{St} \otimes (\chi \circ \det)$*, with* $\chi : \mathbb{Q}_p^\times \longrightarrow E^\times$ *a smooth character, are admissible.*

Proof. This follows from part (ii) of the previous proposition, Proposition 17, and (the proof of) Theorem 28. ☐

Definition 39. Let π be an irreducible, admissible G-representation. We say π is *supersingular* if for any weight V the action of T_1 on $\mathrm{Hom}_K(V, \pi|_K)$ is nilpotent (or, equivalently, if all eigenvalues of T_1 are zero).

The following gives a coarse classification of representations of G.

Theorem 40 (Barthel-Livné)**.** *Every irreducible, admissible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ falls into one of the following four families:*

(i) the principal series $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$ *with* $\chi_1 \neq \chi_2$*,*
(ii) the characters $\chi \circ \det$*,*
(iii) twists of the Steinberg representation $\mathrm{St} \otimes (\chi \circ \det)$*,*
(iv) the supersingular representations.

The four families are disjoint, and the characters appearing in cases (i) to (iii) are uniquely determined.

Proof. Let π be an irreducible, admissible representation, and let V be a weight of π. As π is admissible and $\mathcal{H}(V)$ is commutative, the (finite-dimensional) weight space $\mathrm{Hom}_K(V, \pi|_K)$ contains a common eigenvector $f : V \hookrightarrow \pi|_K$ for $\mathcal{H}(V)$, with eigenvalues given by an algebra homomorphism $\chi' : \mathcal{H}(V) \longrightarrow E$. If $\chi'(\mathrm{T}_1) = 0$ for all such pairs (V, χ'), then π is supersingular. We may therefore assume without loss of generality that $\chi'(\mathrm{T}_1) \neq 0$.

In this case, we get a nonzero (and in fact surjective) G-linear map

$$\mathrm{ind}_K^G(V) \otimes_{\mathcal{H}(V)} \chi' \twoheadrightarrow \pi.$$

We consider several possibilities:

- If $\dim_E V > 1$, then $\mathrm{ind}_K^G(V) \otimes_{\mathcal{H}(V)} \chi' \cong \mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$ for some choice of χ_1, χ_2 (cf. Remark 24), and we obtain that π is either an irreducible principal series representation or a twist of a Steinberg representation.
- If $\dim_E V = 1$ and $\chi'(\mathrm{T}_1^2 - \mathrm{T}_2) \neq 0$, then by Corollary 33, we have $\mathrm{ind}_K^G(V) \otimes_{\mathcal{H}(V)} \chi' \cong \mathrm{ind}_K^G(V') \otimes_{\mathcal{H}(V)} \chi'$, for some p-dimensional weight V'. We may now proceed as in the previous case.
- If $\dim_E V = 1$ and $\chi'(\mathrm{T}_1^2 - \mathrm{T}_2) = 0$, then (after twisting π by a character of the form $\chi \circ \det$) we may assume $V = \mathbf{1}_K$ and $\chi'(\mathrm{T}_1) = \chi'(\mathrm{T}_2) = 1$. The claim now follows from the following fact (cf. [3], Theorem 30 or [1], Proposition 4.7):

$$(\mathrm{ind}_K^G(\mathbf{1}_K) \otimes_{\mathcal{H}(\mathbf{1}_K)} \chi')^{\mathrm{ss}} \cong (\mathrm{Ind}_B^G(\mathbf{1} \otimes \mathbf{1}))^{\mathrm{ss}} \cong \mathbf{1}_G \oplus \mathrm{St}.$$

To show that the four families are disjoint, we analyze their weights and Hecke eigenvalues. Note firstly that if π is any subquotient of $\mathrm{Ind}_{\overline{B}}^{G}(\chi_1 \otimes \chi_2)$ and V is any weight of π, we have

$$V_{\overline{U}_p} \cong \chi_1|_{\mathbb{Z}_p^\times} \otimes \chi_2|_{\mathbb{Z}_p^\times}$$

as T_p-representations. This implies that the eigenvalues of $\mathcal{H}(V)$ on the space $\mathrm{Hom}_K(V, \pi|_K)$ are given by

$$\chi'(\mathrm{T}_1) = \chi_2(p)^{-1}, \tag{9}$$

$$\chi'(\mathrm{T}_2) = \chi_1(p)^{-1}\chi_2(p)^{-1}. \tag{10}$$

The condition $\chi'(\mathrm{T}_1) = 0$ distinguishes the supersingular representations (family (iv)). The irreducible principal series representations (family (i)) are distinguished by the condition $1 < \dim_E V < p$ or $\chi'(\mathrm{T}_1^2 - \mathrm{T}_2) \neq 0$. Finally, the characters of G (family (ii)) are determined by $\dim_E V = 1$ and $\chi'(\mathrm{T}_1^2 - \mathrm{T}_2) = 0$, while the twists of the Steinberg representation (family (iii)) are determined by $\dim_E V = p$ and $\chi'(\mathrm{T}_1^2 - \mathrm{T}_2) = 0$. Moreover, equations (9) and (10) above (along with knowledge of $V_{\overline{U}_p}$) imply how to uniquely recover the characters χ_1, χ_2 (respectively, χ) from χ'. □

Definition 41. An irreducible, admissible G-representation is called *supercuspidal* if it is not a subquotient of a principal series representation.

Corollary 42. *If π is an irreducible, admissible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$, then π is supercuspidal if and only if π is supersingular.*

Remark 43. The above description of irreducible, admissible representations in terms of parabolic induction, supersingular representations, and (generalized) Steinberg representations generalizes to arbitrary p-adic reductive groups. Also, Corollary 42 still holds. See [17] for the case of GL_n, [1] for the case of a general split, connected, reductive group, and [2] for the general case.

10. Supersingular Representations

Almost all of the arguments we have considered up to this point apply with little change to the group $\mathrm{GL}_2(F)$, where F is a finite extension of $\mathbb{Q}_p$. In this section, however, we will crucially use the fact that $F = \mathbb{Q}_p$ to give a more precise description of the classification of Theorem 40.

Our main goal will be to prove the following theorem, due to Breuil. We follow an argument due to Paškūnas [21] and Emerton [11].

Theorem 44 (Breuil)**.** *The irreducible, admissible, supersingular representations of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ are exactly*

$$\mathrm{ind}_K^G(V) \otimes_{\mathcal{H}(V)} \chi',$$

where V is a weight and $\chi'(\mathrm{T}_1) = 0$.

By Theorem 40 above, any irreducible, admissible, supersingular representation of G is a quotient of $\mathrm{ind}_K^G(V) \otimes_{\mathcal{H}(V)} \chi'$ with $\chi'(\mathrm{T}_1) = 0$, so it is enough to show that this quotient is irreducible and admissible. In order to do this, we shall use a slightly different model for $\mathrm{ind}_K^G(V) \otimes_{\mathcal{H}(V)} \chi'$, given as follows.

Let Z denote the center of G. We inflate V to a representation of KZ by decreeing that the element $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ acts trivially.

Proposition 45. *We have an isomorphism*

$$\mathrm{End}_G(\mathrm{ind}_{KZ}^G(V)) \cong E[\mathrm{T}],$$

where (via an isomorphism analogous to that of Proposition 18) T *is the function supported on the double coset* $KZ \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) KZ$*, such that* $\mathrm{T}\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \in \mathrm{End}_E(V)$ *is a linear projection.*

Proof. The proof is analogous to the proof of Theorem 21. $\square$

Now let $\eta : \mathbb{Q}_p^\times \longrightarrow E^\times$ be a (smooth) character which is trivial on $\mathbb{Z}_p^\times$, and which satisfies $\eta(p)^{-2} = \chi'(\mathrm{T}_2)$. Then one easily checks that we have a surjective G-linear map

$$\begin{array}{rcl} \mathrm{ind}_K^G(V) & \twoheadrightarrow & \mathrm{ind}_{KZ}^G(V) \otimes \eta \circ \det \\ {[g, v]_K} & \longmapsto & [g, v]_{KZ} \otimes \eta(\det g), \end{array}$$

where the subscript denotes the group with respect to which the element $[g, v]$ is equivariant. Note that the action of T_1 on the left is compatible with the action of $\mathrm{T} \cdot \eta(p)^{-1}$ on the right, while the action of T_2 is compatible with the action of the scalar $\eta(p)^{-2} = \chi'(\mathrm{T}_2)$. It is not hard to check that this induces an isomorphism

$$\mathrm{ind}_K^G(V) \otimes_{\mathcal{H}(V)} \chi' \xrightarrow{\sim} \frac{\mathrm{ind}_{KZ}^G(V)}{(\mathrm{T} - \chi'(\mathrm{T}_1)\eta(p))} \otimes (\eta \circ \det).$$

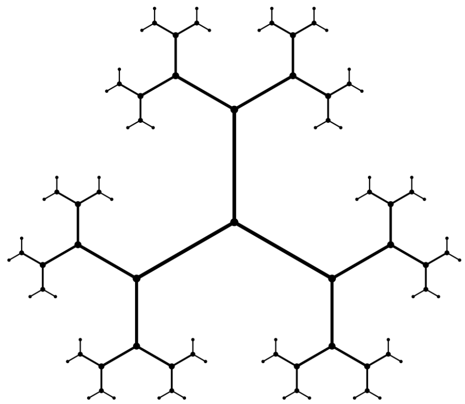
Therefore, to prove Theorem 44 it is enough to show $\mathrm{ind}_{KZ}^G(V)/(\mathrm{T})$ is irreducible and admissible as a G-representation.

Proposition 46. *The representation* $\mathrm{ind}_{KZ}^G(V)/(\mathrm{T})$ *is nonzero.*

Idea of proof. As the operator T is not invertible (by Proposition 45), it suffices to show that T is injective on $\mathrm{ind}_{KZ}^G(V)$. Consider the special case where $V = \mathbf{1}_K$. We then have

$$\mathrm{ind}_{KZ}^G(\mathbf{1}_K) = C_c(KZ\backslash G, E),$$

so that the compactly induced representation is equal to the space of finitely supported E-valued functions on $KZ\backslash G$. In this context, it is convenient to use the Bruhat-Tits tree $\mathfrak{X}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ (pictured below for $p = 2$; see [24] for an overview).



The vertices of $\mathfrak{X}$ correspond to homothety classes of $\mathbb{Z}_p$-lattices contained in $\mathbb{Q}_p^{\oplus 2}$. The group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts transitively on the set of such lattices, and the stabilizer of the class of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is exactly KZ. Hence, the representation $\mathrm{ind}_{KZ}^G(\mathbf{1}_K)$ is exactly the set of compactly supported functions on the vertices of $\mathfrak{X}$. In this setup, the operator T is simply the "sum over neighbors" map; that is, $(\mathrm{T}f)(x) = \sum f(y)$, where y runs through the neighbors of x.

Now let $f \in \mathrm{ind}_{KZ}^G(\mathbf{1}_K)$ be an arbitrary nonzero function, and let $\mathfrak{T}$ be the convex hull of $\mathrm{supp}(f)$; it is a finite subtree of $\mathfrak{X}$. If we let x denote an extremal vertex of $\mathfrak{T}$ and $y \notin \mathfrak{T}$ a neighbor of x, then we have $(\mathrm{T}f)(y) \neq 0$, which shows T is injective. $\square$

We introduce some additional notation for the proof of Theorem 44. Let
$$\mathrm{red} : K = \mathrm{GL}_2(\mathbb{Z}_p) \twoheadrightarrow G_p = \mathrm{GL}_2(\mathbb{F}_p)$$
denote the "reduction-modulo-p" map, and define
$$I := \mathrm{red}^{-1}(B_p), \qquad I(1) := \mathrm{red}^{-1}(U_p).$$
We let
$$\Pi := \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix};$$
we then have
$$\Pi I(1) \Pi^{-1} = I(1) \tag{11}$$
$$\Pi^2 = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}. \tag{12}$$
We also set
$$t := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \qquad s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{X} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1 \in E[I \cap U].$$
Note that $\Pi = st$.

We will also need the technique of weight cycling mentioned above. Let π denote the quotient $\mathrm{ind}_{KZ}^G(V)/(\mathrm{T})$, and write $V = F(a,b)$. We denote by f the composition of natural maps

$$V \longrightarrow \mathrm{ind}_{KZ}^G(V) \twoheadrightarrow \pi.$$

Note that f is injective, as V generates $\mathrm{ind}_{KZ}^G(V)$, and $f * \mathrm{T} = 0$, by definition of π.

The map f allows us to identify V with a K-subrepresentation of π. Let us fix a nonzero element $v \in V^{U_p} \subset \pi^{I(1)}$. By equation (4), we see that I acts on v by the character $\eta_a \otimes \eta_b$ (using the identification $I/I(1) \cong T_p$). By (11) above, the element $v' := \Pi.v$ also lies in $\pi^{I(1)}$, and I acts on v' by the character $\eta_b \otimes \eta_a$. Hence, by Frobenius Reciprocity, we get a nonzero K-linear map

$$\begin{array}{rcl} \mathrm{ind}_I^K(\eta_b \otimes \eta_a) & \xrightarrow{j} & \pi \\ {[1,1]} & \longmapsto & v'. \end{array}$$

Additionally, using Lemma 15 and Frobenius Reciprocity, we have a complex

$$0 \longrightarrow V = F(a,b) \xrightarrow{i} \mathrm{ind}_I^K(\eta_b \otimes \eta_a) \longrightarrow V' := F(b+p-1,a) \longrightarrow 0.$$

See also [9], Theorem 7.1 for an alternate derivation of this complex. Since $\dim_E F(a,b) = a-b+1$ and $\dim_E F(b+p-1,a) = p+b-a$, the complex above is exact.

Lemma 47. *We have $j \circ i = f * \mathrm{T}$ (up to a scalar).*

Proof. Calculation. $\square$

The lemma holds equally well for any smooth representation π. In our situation we therefore have $j \circ i = 0$, so we see that the map j factors through the projection to V':

$$\begin{tikzcd} \mathrm{ind}_I^K(\eta_b \otimes \eta_a) \arrow[r, two heads] \arrow[d, "j"'] & V' \arrow[dl, dashed] \\ \pi & \end{tikzcd}$$

This implies that the K-subrepresentation of π generated by v' is isomorphic to V'. The upshot of this discussion is that Π acts on $\pi^{I(1)}$, exchanging v and v' (by (12) above), and we have

$$\langle K.v \rangle_E \cong F(a,b), \qquad \langle K.v' \rangle_E \cong F(b+p-1,a).$$

Note that these two weights are nonisomorphic.

Lemma 48 (Iwahori Decomposition)**.** *We have a factorization*

$$I = \underbrace{(I \cap U)}_{I^+}\underbrace{(I \cap T)}_{I^0}\underbrace{(I \cap \overline{U})}_{I^-} = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & \mathbb{Z}_p^\times \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p\mathbb{Z}_p & 1 \end{pmatrix},$$

where the factors may be taken in any order. In addition, we have

$$tI^+t^{-1} \subset I^+, \qquad tI^0t^{-1} = I^0, \qquad tI^-t^{-1} \supset I^-.$$

Proof. This decomposition is a general fact about reductive groups over local fields. For the case of $\mathrm{GL}_2(\mathbb{Q}_p)$, one can prove this directly: for example, one has (for $\alpha, \delta \in \mathbb{Z}_p^\times, \beta, \gamma \in \mathbb{Z}_p$)

$$\begin{pmatrix} \alpha & \beta \\ p\gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & \beta\delta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha - p\beta\gamma\delta^{-1} & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p\gamma\delta^{-1} & 1 \end{pmatrix}.$$

□

Proposition 49.

(i) Let

$$M := \langle I^+ t^{\mathbb{N}}.(Ev \oplus Ev')\rangle_E \subset \pi$$

be the subspace of π generated by the orbit of $I^+t^{\mathbb{N}}$ on the vectors v and v'. Then M is stable by and irreducible for the action of the monoid

$$I^0 I^+ t^{\mathbb{N}} = \begin{pmatrix} \mathbb{Z}_p - \{0\} & \mathbb{Z}_p \\ 0 & \mathbb{Z}_p^\times \end{pmatrix}.$$

(ii) We have $V \subset M$.

Given this proposition, we may prove irreducibility of the representation $\pi = \mathrm{ind}_{KZ}^G(V)/(\mathrm{T})$.

Proof of Theorem 44 (irreducibility), assuming Proposition 49. By Proposition 49, we have

$$M = \langle I^0 I^+ t^{\mathbb{N}}.V\rangle_E = \left\langle \begin{pmatrix} \mathbb{Z}_p - \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}.V \right\rangle_E,$$

since I^0 normalizes I^+. Moreover, since $t.M \subset M$, we have an increasing chain of subspaces

$$M \subset t^{-1}.M \subset t^{-2}.M \subset \dots.$$

Using the Iwasawa decomposition (cf. proof of Proposition 17) and the fact that the center Z acts by scalars on π, we obtain

$$\begin{aligned} \bigcup_{n \geq 0} t^{-n}.M &= \left\langle \begin{pmatrix} \mathbb{Q}_p - \{0\} & \mathbb{Q}_p \\ 0 & 1 \end{pmatrix}.V \right\rangle_E \\ &\overset{Z \text{ acts by scalars}}{=} \langle B.V\rangle_E \\ &\overset{V \text{ is a weight}}{=} \langle BK.V\rangle_E \\ &\overset{\text{Iwasawa}}{=} \langle G.V\rangle_E \\ &= \pi. \end{aligned}$$

Now, let $\sigma \subset \pi|_B$ be a nonzero B-subrepresentation. Then, by the above computation, there exists an integer $n \geq 0$ and $m \in M - \{0\}$ such that

$$t^{-n}.m \in \sigma.$$

As σ is B-stable, we have

$$m \in t^n.\sigma = \sigma,$$

which implies $M \subset \sigma$ by Proposition 49. Hence, we obtain

$$\pi = \bigcup_{n \geq 0} t^{-n}.M \subset \sigma,$$

which shows that π is irreducible, *even as a B-representation.* □

Thus, we have shown that it suffices to prove Proposition 49 to show the irreducibility of π. We let

$$N := \langle I^+ t^{2\mathbb{N}}.v \rangle_E \subset M, \qquad N' := \langle I^+ t^{2\mathbb{N}}.v' \rangle_E \subset M,$$

and set $r := a - b$. The proof of Proposition 49 will follow from the following lemmas.

Lemma 50.

(i) We have $t\mathcal{X}t^{-1} = \mathcal{X}^p$.

(ii) We have

$$\begin{aligned} v &= (*)\mathcal{X}^r t.v' \\ v' &= (*)\mathcal{X}^{p-1-r} t.v, \end{aligned}$$

where $(*)$ is some element of $E^\times$.

Proof. (i) Since E has characteristic p, we have

$$t\mathcal{X}t^{-1} = t\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1\right)t^{-1} = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} - 1 = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1\right)^p = \mathcal{X}^p.$$

(ii) Note that, if $X^{r-i}Y^i$ is an element of $V = F(a,b) = E[X,Y]_{(r)}$ with $0 \leq i \leq r$, we have

$$\mathcal{X}.X^{r-i}Y^i = X^{r-i}(Y+X)^i - X^{r-i}Y^i = iX^{r-i+1}Y^{i-1} + Q(X,Y),$$

where $Q(X,Y)$ is a homogeneous polynomial of degree r, such that the degree of $Q(X,Y)$ as a polynomial in Y is strictly less than $i-1$. Applying this r times, we obtain

$$\mathcal{X}^r.X^{r-i}Y^i = \begin{cases} 0 & \text{if } i \neq r, \\ r!X^r & \text{if } i = r. \end{cases}$$

Hence, as $\det s = -1$, we have

$$\mathcal{X}^r t.v' = \mathcal{X}^r s\Pi.v' = \mathcal{X}^r s.v = (-1)^b r! v,$$

and by symmetry, we obtain

$$\mathcal{X}^{p-1-r} t.v = (-1)^a (p-1-r)! v'.$$

Note that the condition $0 \leq r \leq p-1$ implies $r!$ and $(p-1-r)!$ are nonzero in E. □

Lemma 51. *We have*

$$N^{I^+} = Ev, \qquad (N')^{I^+} = Ev'.$$

Remark 52. Note that N^{I^+} and $(N')^{I^+}$ are both nonzero, since I^+ is a pro-p group.

Proof. We prove the claim for N; the proof for N' is identical. By Lemma 50, we have

$$\begin{aligned} v &= (*)\mathcal{X}^r t\mathcal{X}^{p-1-r}t.v \\ &= (*)\mathcal{X}^{r+p(p-1-r)}t^2.v, \end{aligned}$$

where $(*)$ denotes an unspecified nonzero constant. Iterating this, we obtain

$$v = (*)\mathcal{X}^{e_n}t^{2n}.v,$$

where $e_n = (r + p(p-1-r))(1 + p^2 + \ldots + p^{2n-2})$. Since $v \in \pi^{I(1)}$, we have $\mathcal{X}.v = 0$; hence, $\mathcal{X}^{e_n}t^{2n}.v \neq 0$, but $\mathcal{X}^{e_n+1}t^{2n}.v = 0$.

Let $N_n := \langle I^+t^{2n}.v\rangle_E \subset N$. As $I^+ \cong \mathbb{Z}_p$, we have

$$I^+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}_p} = (1+\mathcal{X})^{\mathbb{Z}_p}.$$

Since we may express the action of I^+ in terms of the operator $\mathcal{X}$, we have an isomorphism

$$\begin{aligned} E[\mathcal{X}]/(\mathcal{X}^{e_n+1}) &\cong N_n \\ 1 &\longmapsto t^{2n}.v \end{aligned}$$

that is compatible with the action of $\mathcal{X}$. Note that $N_n^{I^+} = N_n[\mathcal{X}]$, the $\mathcal{X}$-torsion elements. The $\mathcal{X}$-torsion elements of $E[\mathcal{X}]/(\mathcal{X}^{e_n+1})$ are one-dimensional (spanned by $\mathcal{X}^{e_n}$), and correspond to the subspace $E\mathcal{X}^{e_n}t^{2n}.v = Ev$ of N_n.

Now, since $t^{2n}.v = (*)\mathcal{X}^{p^{2n}e_1}t^{2n+2}.v \in N_{n+1}$, we obtain a series of inclusions

$$N_0 \subset N_1 \subset \ldots \subset N = \bigcup_{n\geq 0} N_n.$$

This, combined with the fact that $N_n^{I^+} = Ev$ shown above, proves that $N^{I^+} = Ev$. $\square$

Remark 53. The identity $I^+ = (1+\mathcal{X})^{\mathbb{Z}_p}$ relies crucially on the fact that $\mathbb{Z}_p$ is *pro-cyclic*. There is no analogous such statement for the ring of integers in a finite extension of $\mathbb{Q}_p$.

Lemma 54. *The subspaces $M, N,$ and N' are I-stable.*

Proof. Recall that $M = \langle I^+t^{\mathbb{N}}.(Ev \oplus Ev')\rangle_E$. Therefore, by the Iwahori Decomposition (Lemma 48), we have

$$I.M = \langle I^+I^0I^-t^{\mathbb{N}}.(Ev \oplus Ev')\rangle_E.$$

By Lemma 48, we have $I^0I^-t^n \subset t^nI^0I^-$ for every $n \geq 0$, and moreover, the space $Ev \oplus Ev'$ is I-stable by definition. Hence

$$I.M \subset \langle I^+t^{\mathbb{N}}.(Ev \oplus Ev')\rangle_E = M.$$

The same proof applies to N and N'. $\square$

Lemma 55. *We have $t.v \in N'$ and $t.v' \in N$.*

Proof. By Lemma 50, we have $t.v = (*)\mathcal{X}^{pr}t^2.v' \in N'$. The other claim follows by symmetry. □

Lemma 56. *The K-subrepresentation generated by v is contained in M; that is,*

$$V = \langle K.v\rangle_E \subset M.$$

Proof. The Bruhat Decomposition for $G_p = \mathrm{GL}_2(\mathbb{F}_p)$ states

$$G_p = B_p \sqcup B_p s B_p,$$

which we inflate to a decomposition of $K = \mathrm{GL}_2(\mathbb{Z}_p)$:

$$K = I \sqcup IsI.$$

Since I acts on v by a character, we obtain

$$\begin{aligned}
\langle K.v\rangle_E &= \langle I.v\rangle_E + \langle IsI.v\rangle_E \\
&= Ev + \langle Is.v\rangle_E \\
&= Ev + \langle It\Pi^{-1}.v\rangle_E \\
&= Ev + \langle It.v'\rangle_E.
\end{aligned}$$

By Lemma 55, $t.v'$ is contained in $N \subset M$, and by Lemma 54, $It.v'$ is contained in M. □

Lemma 57. *The space M decomposes as*

$$M = N \oplus N'.$$

Proof. We clearly have $N + N' \subset M$; we prove the opposite inclusion. This follows easily from

$$I^+t^{\mathbb{N}} = I^+t^{2\mathbb{N}} \cup I^+t^{2\mathbb{N}}t,$$

along with Lemma 55 and that N and N' are $I^+t^{2\mathbb{N}}$-stable.

Assume now that $N \cap N' \neq 0$. On the one hand, I^+ is a pro-p group, and therefore we obtain $(N \cap N')^{I^+} \neq 0$; on the other hand, Lemma 51 gives $(N \cap N')^{I^+} = N^{I^+} \cap (N')^{I^+} = Ev \cap Ev' = 0$, a contradiction. □

We are now ready to finish the proof of Proposition 49.

Proof of Proposition 49. Part (ii) of the proposition follows from Lemma 56. We prove part (i). It follows from Lemma 48 that M is $I^0I^+t^{\mathbb{N}}$-stable. Let M' be a nonzero submodule of M, stable by $I^0I^+t^{\mathbb{N}}$. By Lemmas 51 and 57, we have

$$M^{I^+} = Ev \oplus Ev'.$$

Note that $(M')^{I^+} \neq 0$. We consider two cases.

- If $0 < r < p-1$, then I^0 acts on v and v' by distinct characters. This implies that either v or v' is contained in $(M')^{I^+} \subset M'$, and the relation $v = (*)\mathcal{X}^r t.v'$ (resp. $v' = (*)\mathcal{X}^{p-1-r}t.v$) implies that both vectors must be contained in M'. Hence $M' = M$.
- If $r = 0$ or $r = p-1$, then $\lambda v + \mu v' \in (M')^{I^+} \subset M'$ for some $\lambda, \mu \in E$, not both zero. We assume $\lambda\mu \neq 0$, else we may proceed as above to conclude $M' = M$. Lemma 50 shows that applying $\mathcal{X}^{p-1}t$ to $\lambda v + \mu v'$ gives an element of M', equal to either $(*)v$ or $(*)v'$. Hence, we proceed as above and conclude $M' = M$.

$\square$

All that remains is to show π is admissible, which will occupy the remainder of these notes. To do this, we'll use the following lemmas.

Lemma 58. *Any quotient of an admissible I^+-representation is admissible.*

Sketch of proof. This comes down to the fact that $E[\![I^+]\!] \cong E[\![\mathcal{X}]\!]$ is noetherian, where $E[\![I^+]\!]$ denotes the completed group ring of I^+. Suppose that M is an admissible I^+-representation. As M is smooth, M is naturally an $E[\![\mathcal{X}]\!]$-module, where $\mathcal{X} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) - 1$, as above. Hence $M^* := \mathrm{Hom}_E(M, E)$ inherits an $E[\![\mathcal{X}]\!]$-module structure as well. It is easy to show that

$$M^*/\mathcal{X}M^* \xrightarrow{\sim} (M[\mathcal{X}])^*.$$

Therefore M is admissible as an I^+-representation if and only if $M[\mathcal{X}] = M^{I^+}$ is finite dimensional (by Proposition 36), if and only if $M^*/\mathcal{X}M^*$ is finite-dimensional. As M^* is separated for the $\mathcal{X}$-adic topology (exercise), the latter condition is equivalent to M^* being finitely generated as an $E[\![\mathcal{X}]\!]$-module, by a variant of Nakayama's Lemma (see [20], Theorem 8.4). Thus, if $M \twoheadrightarrow N$ with M admissible, then $N^* \hookrightarrow M^*$ and N^* is finitely generated, as $E[\![\mathcal{X}]\!]$ is noetherian. $\square$

Let $\sigma := N + \Pi.N'$. Then σ is G^+-stable by Proposition 4.12 of [21], where $G^+ := \{g \in G : \det g \in p^{2\mathbb{Z}} \cdot \mathbb{Z}_p^\times\}$.

Lemma 59. *The G^+-representation σ is admissible.*

Proof. Consider the exact sequence of I-representations

$$0 \longrightarrow \Pi.N' \longrightarrow \sigma \longrightarrow N/(N \cap \Pi.N') \longrightarrow 0. \tag{13}$$

We know that $(\Pi.N')^{I(1)} \cong (N')^{I(1)}$ is finite-dimensional by Lemma 51. On the other hand, N is I^+-admissible; hence $N/(N \cap \Pi.N')$ is I^+-admissible by Lemma 58, and thus also $I(1)$-admissible. It follows from (13) that $\sigma^{I(1)}$ is finite-dimensional, and we are done by Proposition 36. $\square$

Proof of Theorem 44 (admissibility). Note that $(G : G^+) = 2$ and that $G = \langle G^+, \Pi\rangle$. It follows that $\sigma + \Pi.\sigma = \pi$, as this is a nonzero G-subrepresentation and we've already shown that π is irreducible. Similarly, $\sigma \cap \Pi.\sigma$ equals either 0 or π, hence $\pi = \sigma \oplus \Pi.\sigma$ or $\pi = \sigma$. In either case, we have $\dim_E \pi^{I(1)} < \infty$ by Lemma 59. (In fact, $\sigma \cap \Pi.\sigma = 0$ by Corollary 6.5 in [21].) $\square$

We remark that this is *not* Breuil's original proof; his method relies on computing

$$(\mathrm{ind}_{KZ}^{G}(V)/(\mathrm{T}))^{I(1)}$$

using explicit calculations with the Bruhat-Tits tree $\mathfrak{X}$.

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