THE WEIGHT IN A SERRE-TYPE CONJECTURE FOR TAME n-DIMENSIONAL GALOIS REPRESENTATIONS

FLORIAN HERZIG

Contents

1. Introduction

Serre ([Ser75]) conjectured that every continuous, odd, irreducible representation

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{F}}_p)$$

is modular. That is, there should exist positive integers N and $k \geq 2$, and a modular form $f \neq 0$ (over $\overline{\mathbb{F}}_p$) of weight k and level $\Gamma_1(N)$ which is an eigenvector of all the Hecke operators such that

$$\mathrm{tr}\,\rho(\mathrm{Frob}_l^{-1}) = \theta_f(T_l),$$
$$\det\rho(\mathrm{Frob}_l^{-1}) = \theta_f(S_l)l,$$

for all primes $l \nmid pN$ at which ρ is unramified. Here, Frob_l is a geometric Frobenius element at l and, whenever T is a Hecke operator, $\theta_f(T)$ denotes the eigenvalue of T on f. The representation ρ is said to be *attached to* f.

Date: November 18, 2010.

Later Serre ([Ser87]) refined his conjecture, predicting which "minimal" level $N = N(\rho)$ prime to p and weight $k = k(\rho)$ admit such an eigenform f. He defined $N(\rho)$ in terms of the ramification of ρ outside p (it is the Artin conductor) and gave a combinatorial recipe for $k(\rho)$ in terms of $\rho|_{I_p}$ where I_p is the inertia group at p. Due to the work of a lot of people it is now known that the original conjecture implies the refined form when $p \neq 2$. Until very recently, the modularity of ρ was known only in very special cases. The work of Khare and Khare-Wintenberger ([KWa], [Kha] and [KWb]) shows that ρ is modular if either p is odd and ρ is unramified at 2 or $p = 2$ and $k(\rho) = 2$.

1.1. **Reformulation of Serre's weight recipe.** After Serre's original papers, several people noticed that Serre's original prescription of the weight simplifies if a different notion of weight is used. A *Serre weight* is an isomorphism class of (absolutely) irreducible representations of $GL_2(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$. Let $Y_1(N)/\mathbb{Q}$ be the affine modular curve associated to $\Gamma_1(N)$ and $\mathcal{L}_k$ the locally constant sheaf defined by the representation of $\pi_1(Y_1(N))$ on $\mathrm{Sym}^{k-2}\,\overline{\mathbb{F}}_p^2$ via $\pi_1(Y_1(N)) \twoheadrightarrow GL_2(\mathbb{F}_p)$ (coming from an auxiliary full level p structure). Given f as above, there is an eigenclass in $H^1_{et}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathcal{L}_k)$ with the same Hecke eigenvalues θ_f. If $\{F_1, \dots, F_n\}$ are the Jordan-Hölder constituents of $\mathrm{Sym}^{k-2}\,\overline{\mathbb{F}}_p^2$ as $GL_2(\mathbb{F}_p)$-representation, then it is not hard to see that the existence of a Hecke eigenclass of type θ_f in $H^1_{et}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathcal{L}_k)$ is equivalent to the existence of a Hecke eigenclass of type θ_f in $H^1_{et}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathcal{L}_{F_i})$ for some i (it is essential that ρ is irreducible). Here, the $\mathcal{L}_{F_i}$ are defined in the same way as $\mathcal{L}_k$.

Let $N = N(\rho)$ and define

$$W(\rho) = \{F : \rho \text{ is attached to an eigenclass in } H^1_{et}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathcal{L}_F)\}.$$

Using that ρ is attached to an eigenform f of weight k if and only if $F_i \in W(\rho)$ for some i and that $W(\rho \otimes \omega) \cong W(\rho) \otimes \det$, the original weight recipe of Serre can be translated into a prediction $W_{conj}(\rho)$ for $W(\rho)$ (ω is the mod p cyclotomic character). For example, if $\rho|_{I_p} \sim \left(\begin{smallmatrix} \omega^i & \\ & 1 \end{smallmatrix}\right)$,

$$W_{conj}(\rho) = \begin{cases} F(i-1,0), F(p-2,i) & \text{if } 1 < i < p-2, \\ F(p-2,0) & \text{if } i = 0, \\ F(0,0), F(p-2,1), F(p-1,0) & \text{if } i = 1, \\ F(p-3,0), F(p-2,p-2), F(2p-3,p-2) & \text{if } i = p-2. \end{cases}$$

where ω is the mod p cyclotomic character and $F(a,b) = \mathrm{Sym}^{a-b}\,\overline{\mathbb{F}}_p^2 \otimes \det^b$ ($0 \le a-b \le p-1$, $0 \le b < p-1$). If $\rho|_{I_p} \sim \left(\begin{smallmatrix} \omega^i & * \\ & 1 \end{smallmatrix}\right)$ with $* \neq 0$, then $W_{conj}(\rho)$ is a subset of the above set of Serre weights (generically, it is a proper subset). There is a similar recipe in case $\rho|_{G_p}$ is irreducible, where G_p denotes the decomposition group at p (the "supersingular case").

1.2. **A Serre-type conjecture for GL_3.** We give a conjecture for the weights in which a given *three-dimensional* Galois representations over $\overline{\mathbb{F}}_p$ can arise (Conjecture A) and present some theoretical evidence (Theorem B). Ash, Sinnott, Doud and Pollack were the first to make a conjecture in this setting ([AS00], [ADP02]). In contrast to the case of two-dimensional Galois representations, it is essential to work with the second notion of weight. A *Serre weight* now is an isomorphism class of (absolutely) irreducible representations of $GL_3(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$. Serre weights are parametrised by triples (a, b, c) of integers with $0 \leq a-b,\ b-c \leq p-1,\ 0 \leq c < p-1$. The Serre weight corresponding to (a, b, c) is denoted by $F(a, b, c)$.

If N is a positive integer, let $\Gamma_1(N) \subset SL_3(\mathbb{Z})$ be the subgroup of matrices with last row congruent to $(0, 0, 1)$ modulo N. If, moreover, F is a Serre weight and $e \geq 0$ then $\Gamma_1(N)$ acts on F via reduction mod p and $H^e(\Gamma_1(N), F)$ inherits a Hecke action. A Hecke eigenclass $\alpha \in H^e(\Gamma_1(N), F)$ is said to have *attached Galois representation* $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_3(\overline{\mathbb{F}}_p)$ if for all primes $l \nmid pN$, ρ is unramified at l and

$$\sum_{i=0}^{3} (-1)^i l^{i(i-1)/2} a(l, i) X^i = \det(1 - \rho(\mathrm{Frob}_l^{-1})X),$$

where the $a(l, i)$ are the eigenvalues of Hecke operators at l and ρ is assumed to be continuous, semisimple, and odd (i.e., $\rho(c) \neq \pm 1$ if $p > 2$ and c is a complex conjugation). Note that these conditions determine ρ, if it exists. Any eigenclass α is conjectured to have an attached Galois representation ([Ash92]). Very little is known about this conjecture at present.

Conversely, start with an continuous, *irreducible*, and odd

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_3(\overline{\mathbb{F}}_p).$$

The "minimal" level $N(\rho)$ is again defined to be the Artin conductor of ρ (in particular, it is prime to p). The weight set $W(\rho)$ is defined to be the set of Serre weights F such that ρ is attached to a Hecke eigenclass in $H^e(\Gamma_1(N(\rho)), F)$ for some e. (Note that $W(\rho)$ for two-dimensional ρ can be defined analogously.)

From now on, suppose that ρ is *tamely ramified at* p, or equivalently, that $\rho|_{I_p}$ is diagonalisable. It is important to understand this case first since it is expected that $W(\rho) \subset W(\rho')$ if $\rho|_{I_p}^{ss} \cong \rho'|_{I_p}$ (as in the example in §1.1). Under this assumption, $\rho|_{I_p}$ is isomorphic to the sum of three characters, which are permuted under the action of Frobenius. If the action is trivial, ρ is said to be of *niveau 1*; if Frobenius fixes precisely one character, ρ is said to be of *niveau 2*; otherwise it is said to be of *niveau 3*.

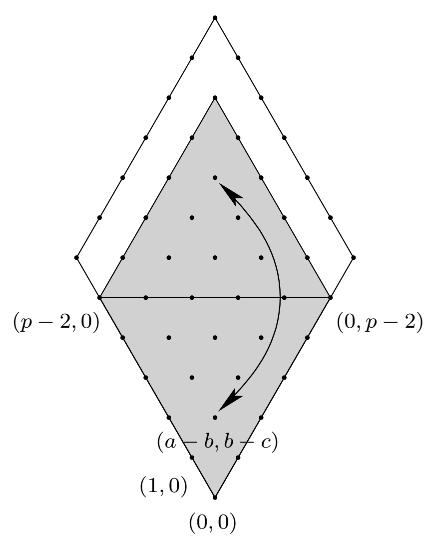
There is a simple and natural way to associate to ρ a Deligne-Lusztig representation $V(\rho|_{I_p})$ of $GL_3(\mathbb{F}_p)$ over $\overline{\mathbb{Q}}_p$ (see definition 6.4). It only depends on $\rho|_{I_p}$. For example, if $\rho|_{I_p}$ is of niveau 1, then $V(\rho|_{I_p})$ is the induction of a character on the Borel subgroup of upper-triangular matrices. Reducing a $\overline{\mathbb{Z}}_p$-lattice in $V(\rho|_{I_p})$ modulo the maximal ideal of $\overline{\mathbb{Z}}_p$ and semisimplifying gives rise to a representation $\overline{V(\rho|_{I_p})}$ of $GL_3(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$. It is independent of the choice of lattice and isomorphic to the direct sum of a collection of Serre weights (since all irreducible representations of $GL_3(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$ can be defined over $\mathbb{F}_p$).

A Serre weight $F(a,b,c)$ is called *regular* if $a-b \neq p-1$ and $b-c \neq p-1$. The set of regular Serre weights is denoted by W_{reg}. Define an operation on the set of Serre weights by

$$\mathcal{R} : F(a,b,c) \mapsto F(c-2, b-1, a)_{reg}, \tag{1.1}$$

where the right-hand side denotes the unique regular Serre weight $F(x,y,z)$ with $(x,y,z) \equiv (c-2, b-1, a)$ modulo $p-1$.

In fact, by construction, $F(a,b,c)$ arises from a representation of the algebraic group GL_3 of highest weight (a,b,c). Restricting to SL_3, and assuming $a-b$, $b-c < p-1$, the action of $\mathcal{R}$ on highest weights can be visualised as a reflection inside the weight space of SL_3:



Here, $(a-b, b-c)$ indicates the restriction of the GL_3-weight (a,b,c) to SL_3.

Conjecture A. *Suppose that* $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_3(\overline{\mathbb{F}}_p)$ *is continuous, irreducible, odd, and tame at* p. *Then*

$$W(\rho) \cap W_{reg} = \mathcal{R}(JH(\overline{V(\rho|_{I_p})})),$$

where $JH(W)$ *denotes the set of Jordan-Hölder constituents of* W.

Conjecture A improves upon the conjecture of Ash, Doud, Pollack and Sinnott—denoted by ADPS for brevity. Their weight conjecture consists of a *combinatorial* recipe for the weights suggested by theoretical results, heuristics, and computations ([AS00], [ADP02]). Conjecture A, however, is *conceptual* in relating the weights of ρ with modular representation theory. It predicts at least as many, and in many cases even more, weights in W_{reg}

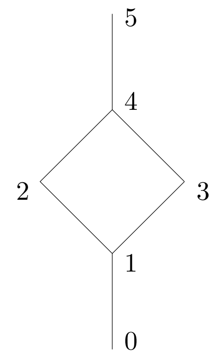


FIGURE 1.1. The partial order on the six regions of GL_4.

than the ADPS conjecture. In all examples covered in [ADP02] with irreducible, tame ρ, these extra weights were confirmed (up to Hecke eigenvalues at $l = 47$) by calculations of D. Doud and D. Pollack. Here is a chart of the generic numbers of weights $F(a, b, c)$ predicted with $a - c \leq p - 2$ (resp. $a - c \geq p - 2$), depending on the niveau of ρ. Note that these two regions correspond to the upper and lower gray triangles in the figure above.

	ADPS	Conj. A
niveau 1	3/6	3/6
niveau 2	3/5	3/6
niveau 3	3/3*	3/6

(*Doud, in unpublished work, independently generalised the combinatorial recipe of ADPS in the niveau 3 case so as to predict the same weights in W_{reg} as Conjecture A.) In particular, as $\mathcal{R}$ switches the two regions, this provides an explanation why an operator like $\mathcal{R}$ is needed in order to relate the weight recipe to the decomposition of a characteristic zero representation of $GL_3(\mathbb{F}_p)$.

At present it is not completely clear what non-regular weights should be predicted by the conjecture. Computational evidence suggests that the answer is more involved. Note however that a proportion of $(1 - 1/p)^2$ of the Serre weights are regular and thus covered by Conjecture A. The ADPS conjecture does predict non-regular weights and covers ρ that are not tame at p, but these predictions are based on little supporting evidence. The computations mentioned show that their predictions outside W_{reg} are incomplete, even in the niveau 1 case.

1.3. **Generalisation to GL_n.** The formalism of Conjecture A carries over immediately to GL_n. Yet the bigger n gets, the more complicated the modular representation theory of GL_n becomes. For $n = 4$ it is still tractable. Instead of the two regions discussed above, there are now $(4 - 1)! = 6$ and there is a natural partial order on them (see figure 1.1).

Using automorphic induction of Hecke characters over totally complex CM quartic fields with Galois group D_8, we are able to produce many eigenclasses in $H^e(\Gamma_1(N), V)$ for some $(N, p) = 1$ and some e with attached Galois representations ρ that are odd, irreducible, and tame at p. Here, V is a $GL_4(\mathbb{F}_p)$-module which is not necessarily irreducible but of small length.

Even though the eigenclasses produced generally have prime-to-p level bigger than $N(\rho)$, it is still expected—in analogy with the GL_2-case—that ρ can be attached only to eigenclasses in predicted weights. Thus the above collection of eigenclasses can be considered as evidence that ρ should predict one of the Jordan-Hölder constituents of V. The result obtained can best be stated if the conjecture is turned around, i.e., considered as predicting which tame $\rho|_{I_p}$ occur among ρ attached to an eigenclass in a given weight F. It turns out that for F lying sufficiently deep inside a region, all $\rho|_{I_p}$ obtained are predicted. The proportion of predicted $\rho|_{I_p}$ found, depending on the niveau, is summarised in the following table:

	F lies in region					
	0	1^*	2^*	3^*	4^*	5^*
niveau 1	1	$\frac{1}{2}$	0	0	$\frac{1}{5}$	$\frac{1}{8}$
niveau (2,2)	1	$\frac{1}{2}$	0	0	$\frac{1}{5}$	$\frac{1}{8}$
niveau (2,1,1)	$\frac{1}{3}$	$\frac{1}{6}$	0	0	$\frac{1}{15}$	$\frac{1}{24}$
niveau 4	$\frac{1}{3}$	$\frac{1}{6}$	0	0	$\frac{1}{15}$	$\frac{1}{24}$
niveau (3,1)	0	0	0	0	0	0

For F outside region 0, the evidence is weaker in the sense that it depends on assuming that the conjecture is correct in all regions below the one containing F (whence the "$*$").

1.4. **Serre's Conjecture for Hilbert modular forms.** In [BDJ] Buzzard, Diamond and Jarvis formulated a generalisation of Serre's Conjecture to Hilbert modular forms, a significant part of which was proved recently by Gee, assuming residual modularity ([Gee]). Our next result shows that their weight recipe, in the tame case, is indeed related to the modular representation theory of $GL_2(\mathbb{F}_q)$, analogous to Conjecture A.

Suppose that K is a totally real field that is unramified at p (a necessary assumption in [BDJ]). Pick a prime $\mathfrak{p}|p$ of K of residue degree f and residue field $k_\mathfrak{p}$. A *Serre weight at* $\mathfrak{p}$ is an isomorphism class of (absolutely) irreducible representations of $GL_2(k_\mathfrak{p})$ over $k_\mathfrak{p}$. Suppose

$$\rho : \mathrm{Gal}(\overline{K}/K) \to GL_2(\overline{\mathbb{F}}_p)$$

is continuous, irreducible and totally odd. In [BDJ], §3 a set $W_{conj,\mathfrak{p}}(\rho)$ of Serre weights at $\mathfrak{p}$ is specified with respect to which ρ is conjectured to be modular in some level prime to p. (Strictly speaking, to talk about modularity, a Serre weight $F_{\mathfrak{p}}$ needs to be fixed for each $\mathfrak{p}|p$, and ρ is said to be modular if it occurs in the component cut out by $\bigotimes F_{\mathfrak{p}}$ of the p-torsion of the Jacobian of an appropriate Shimura curve. The weights at different $\mathfrak{p}$ are predicted independently of one another, and so we can focus on one $\mathfrak{p}$.)

From now on, suppose that ρ is *tame at* $\mathfrak{p}$. Again there is a simple and natural way to associate to ρ a Deligne-Lusztig representation $V_{\mathfrak{p}}(\rho)$ of $GL_2(k_{\mathfrak{p}})$ over $\overline{\mathbb{Q}}_p$ which only depends on $\rho|_{I_{\mathfrak{p}}}$ (see definition 14.1). If $\rho|_{I_{\mathfrak{p}}}$ is reducible, then it is the induction of a character on the Borel subgroup of upper-triangular matrices; otherwise it is a cuspidal representation. The representation $V_{\mathfrak{p}}(\rho)$ has a well-defined semisimplified reduction $\overline{V_{\mathfrak{p}}(\rho)}$ over $\overline{\mathbb{F}}_p$ which is a direct sum of Serre weights at $\mathfrak{p}$ (since all irreducible representations of $GL_2(k_{\mathfrak{p}})$ over $\overline{\mathbb{F}}_p$ can be defined over $k_{\mathfrak{p}}$).

Serre weights at $\mathfrak{p}$ are parametrised by pairs (a, b) of integers with

$$0 \le a - b \le p^f - 1,\ 0 \le b < p^f - 1.$$

The Serre weight corresponding to (a, b) is denoted by $F(a, b)$. In fact, if we write $a - b = \sum_{i=0}^{f-1} m_i p^i$, $b = \sum_{i=0}^{f-1} b_i p^i$ with $0 \le m_i,\ b_i \le p - 1$ then

$$F(a,b) \cong \bigotimes_{i=0}^{f-1} (\mathrm{Sym}^{m_i} k_{\mathfrak{p}}^2 \otimes \det{}^{b_i}) \otimes_{k_{\mathfrak{p}},\phi^i} k_{\mathfrak{p}}$$

where $\phi(x) = x^p$. If $m_i \ne p - 1$ for all i, then $F(a, b)$ is said to be a *regular* Serre weight at $\mathfrak{p}$. The set of Serre weights at $\mathfrak{p}$ (resp. regular Serre weights at $\mathfrak{p}$) is denoted by $W_{Ser,\mathfrak{p}}$ (resp. $W_{reg,\mathfrak{p}}$). Define an operation on $W_{reg,\mathfrak{p}}$ by

$$\mathcal{R}_{\mathfrak{p}} : F(a,b) \mapsto F(b - \textstyle\sum_{i=0}^{f-1} p^i, a)_{reg},$$

where the right-hand side is the unique regular Serre weight $F(x, y)$ with $(x, y) \equiv (b - \sum_{i=0}^{f-1} p^i, a)$ modulo $p^f - 1$. Observe the analogy with $\mathcal{R}$ in (1.1). In particular, in the special case $f = 1$, which is Serre's original conjecture, $\mathcal{R}_{\mathfrak{p}} : F(a, b) \mapsto F(b - 1, a)_{reg}$.

Theorem B. *Suppose that* $\rho : \mathrm{Gal}(\overline{K}/K) \to GL_2(\overline{\mathbb{F}}_p)$ *is continuous, irreducible, totally odd, and tame at* $\mathfrak{p}$*.*

(i) $W_{conj,\mathfrak{p}}(\rho) \cap W_{reg,\mathfrak{p}} = \mathcal{R}_{\mathfrak{p}}(JH(\overline{V_{\mathfrak{p}}(\rho)}) \cap W_{reg,\mathfrak{p}})$.

(ii) *There is a* multi-valued *function* $\mathcal{R}_{ext,\mathfrak{p}} : W_{Ser,\mathfrak{p}} \to W_{Ser,\mathfrak{p}}$ *that extends* $\mathcal{R}_{\mathfrak{p}}$ *such that*

$$W_{conj,\mathfrak{p}}(\rho) = \mathcal{R}_{ext,\mathfrak{p}}(JH(\overline{V_{\mathfrak{p}}(\rho)})).$$

A function $\mathcal{R}_{ext,\mathfrak{p}}$ as in (ii) is described on page 68. It is bijective on $W_{reg,\mathfrak{p}}$, with the exception of taking two values on weights F with $\mathcal{R}_{\mathfrak{p}}(F)$ one-dimensional. Note that it is not unique.

Diamond [Dia] found a different way of relating the predicted Serre weights at $\mathfrak{p}$ for ρ to the decomposition of a characteristic zero representation of $GL_2(k_{\mathfrak{p}})$. In his case, the weight recipe is directly compared to the reduction of such a representation—a point of view that does not generalise to

GL_3 (as mentioned at the end of §1.2). Another advantage of Theorem B is that the characteristic zero representation $V_{\mathfrak{p}}(\rho)$ is very naturally defined in terms of $\rho|_{I_{\mathfrak{p}}}$. In particular, if ρ is of niveau 1, then $V_{\mathfrak{p}}(\rho)$ is a "principal series" and if ρ is of niveau 2, then $V_{\mathfrak{p}}(\rho)$ is a cuspidal representation. For the correspondence in [Dia], this is the case only if f is even, whereas the types are switched if f is odd. Combining Theorem B with the theorem of [Dia] shows that $\mathcal{R}_{\mathfrak{p}}$ (generically) maps the constituents of an irreducible $GL_2(k_{\mathfrak{p}})$-representation to the constituents of another such—an *a priori* unexpected fact.

1.5. **Decomposition of $GL_3(\mathbb{F}_p)$-representations.** In order to formulate Conjecture A, we required the decomposition of irreducible $GL_3(\mathbb{F}_p)$-representations in characteristic zero when reduced modulo p. This is apparently lacking in the literature.

Theorem C. *The explicit determination of the constituents of $\overline{V}$ for any irreducible representation V of $GL_3(\mathbb{F}_p)$ over $\overline{\mathbb{Q}}_p$.*

The proof is an exercise, once the correct decomposition formulae are spotted, in comparing the (known) characters of ordinary and modular irreducible representations.

As we learnt later, Jantzen [Jan81] had already established the decomposition of Deligne-Lusztig representations of $G(\mathbb{F}_q)$ for G semisimple, simply connected, defined and split over $\mathbb{F}_q$; in particular for $SL_3(\mathbb{F}_p)$. When asked about $GL_3(\mathbb{F}_p)$, he generalised his result to G reductive with simply connected derived group, defined and split over $\mathbb{F}_q$ ([Jan05]). This yields an alternative proof of Theorem C, at least in the case of Deligne-Lusztig representations.

1.6. **Organisation of this paper.** In sections 3 and 4 we recall the ordinary and modular representation theory of $GL_3(\mathbb{F}_p)$. These are combined in section 5 to establish Theorem C. Conjecture A is stated in section 6; it is compared with the ADPS conjecture in section 7. The following section lists some computational evidence supporting Conjecture A. Sections 9–13 discuss the conjecture for GL_n, obtaining in particular the results of §1.3. Finally, Theorem B is proved in section 14.

2. Notation

Throughout this thesis let p denote a prime number. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and denote by $\overline{\mathbb{F}}_p$ its residue field. Fix an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$, and let G_p (resp. I_p) denote the corresponding decomposition group (resp. inertia group) in $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $\tilde{}: \overline{\mathbb{F}}_p^\times \to \overline{\mathbb{Q}}_p^\times$ denote the Teichmüller lift and let χ_n be its restriction to $\mathbb{F}_{p^n}^\times$.

2.1. **Cuspidal representations of $GL_n(\mathbb{F}_q)$.** Fix q, a p-power.

Call a homomorphism from $\mathbb{F}_{q^n}^\times$ into an abelian group *primitive* if it does not factor through the norm map $\mathbb{F}_{q^n}^\times \to \mathbb{F}_{q^m}^\times$ for any $m|n$, $m \neq n$. We have a bijection for any $n \geq 1$:

$$\left\{\begin{matrix}\text{cuspidal representations}\\ \text{of } GL_n(\mathbb{F}_q) \text{ over } \overline{\mathbb{Q}}_p\end{matrix}\right\}/\cong \stackrel{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{primitive characters}\\ \mathbb{F}_{q^n}^\times \xrightarrow{\chi} \overline{\mathbb{Q}}_p^\times\end{matrix}\right\}/(\chi \sim \chi^q)$$

$$\Theta(\chi, \chi^q, \dots) \longleftrightarrow [\chi] \tag{2.1}$$

$\Theta(\chi, \chi^q, \dots)$ is characterized by demanding that it is cuspidal and has character value

$$(-1)^{n-1}(\chi(\alpha) + \chi(\alpha)^q + \dots + \chi(\alpha)^{q^{n-1}})$$

on matrices with eigenvalues $\alpha, \dots, \alpha^{q^{n-1}}$ ($\alpha \in \mathbb{F}_{q^n}^\times$ such that $\mathbb{F}_q(\alpha) = \mathbb{F}_{q^n}$).

There is an alternative characterisation. An $\mathbb{F}_q$-basis of $\mathbb{F}_{q^n}$ determines an algebra homomorphism $i : \mathbb{F}_{q^n} \to M_n(\mathbb{F}_q)$; on the other hand, any homomorphism $\mathbb{F}_{q^n} \to M_n(\mathbb{F}_q)$ is a $GL_n(\mathbb{F}_q)$-conjugate of i by the Skolem-Noether theorem. Moreover, those i with a fixed image form a q-power orbit. Thus for any primitive $\chi : \mathbb{F}_{q^n}^\times \to \overline{\mathbb{Q}}_p^\times$, $\operatorname{Ind}_{\mathbb{F}_{q^n}^\times}^{GL_n(\mathbb{F}_q)}(\chi)$ is well-defined up to isomorphism and only depends on the q-power orbit of χ, where $\mathbb{F}_{q^n}^\times$ is embedded using any i. Let St denote the Steinberg representation. It is an irreducible constituent of $\operatorname{Ind}_{B_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)$ of dimension $q^{\binom{n}{2}}$ ($B_n(\mathbb{F}_q)$ is the subgroup of upper-triangular matrices).

Then $\Theta(\chi, \chi^q, \dots)$ is characterised among cuspidal representations by

$$\Theta(\chi, \chi^q, \dots) \otimes St \cong \operatorname{Ind}_{\mathbb{F}_{q^n}^\times}^{GL_n(\mathbb{F}_q)}(\chi).$$

To see that this is equivalent to the previous description, use the character of St ([DL76], §7) and prop. 7.3 in [DL76] together with lemma 10.1 below (note that even though that lemma is only stated for $GL_n(\mathbb{F}_p)$, it works equally well for $GL_n(\mathbb{F}_q)$).

A proof of the above correspondence can be obtained from [Spr70a]. First note that a character is in the discrete series in Springer's nomenclature ([Spr70b], §4.3) if and only if it is cuspidal (see [Car85], cor. 9.1.2). Theorems 8.6 and 7.12 in [Spr70a] show that the cuspidal characters are precisely the ones denoted there by $\chi_n(\phi)$, for primitive characters $\phi : \mathbb{F}_{q^n}^\times \to \overline{\mathbb{Q}}_p^\times$ (and $\mathbb{F}_{q^n}^\times$ embedded in $GL_n(\mathbb{F}_q)$ as above; this is denoted by T_n in Springer's notes), with $\chi_n(\phi) = \chi_n(\phi')$ if and only if ϕ is in the q-power orbit of ϕ'. Combining thm. 7.12(i) and thm. 6.3(iii) in [Spr70a], we obtain the values of these characters on $\mathbb{F}_{q^n}^\times$. This establishes the correspondence above (with $\phi = \chi$).

It is interesting that there is a simple formula computing the character of a cuspidal representation.

Lemma 2.1. *The character of $\Theta(\chi, \chi^q, \dots)$ on $g \in GL_n(\mathbb{F}_q)$ is zero unless the characteristic polynomial of g is the power of an irreducible polynomial*

$f \in \mathbb{F}_q[x]$. In that case, suppose that the elementary divisors of g are f^{n_1}, f^{n_2}, ..., f^{n_r} with $(\sum n_i)(\deg f) = n$, and let $\alpha \in \mathbb{F}_{q^n}^\times$ be a root of f. Then the character of $\Theta(\chi, \chi^q, \dots)$ on g is

$$(-1)^{n-r} \prod_{i=1}^{r-1} (q^i - 1) \cdot (\chi(\alpha) + \chi(\alpha)^q + \dots \chi(\alpha)^{q^{\deg(f)-1}}).$$

Proof. In Springer's notation, $\chi_n(\phi) = (-1)^{n-1}\chi_n(n;\theta)$, where θ is an extension of ϕ to $\overline{\mathbb{F}}_p^\times$ (see his proof of thm. 7.12; he incorrectly states that θ can be chosen to be injective; but this does not seem to be relevant to his argument). Now we claim that Springer's character $\chi_n(d;\theta)$ of $GL_n(\mathbb{F}_q)$ (for $d|n$) is isomorphic to Green's character $I_d^k[v]$ ([Gre55], thm. 5 on p. 419), if we set $v = n/d$ and k is any integer such that $\phi(x) = \widetilde{x}^k$ (then we can also assume that $\theta(x) = \widetilde{x}^k$). This can be seen by a tedious comparison of Springer's proof of thm. 7.16 and [Gre55], pp. 415–420. Note in particular the following errors in Springer's proof. (i) The induction has to be not just over n alone, but also over d. Thus, whenever it says "all v such that $sv(s,B) < n$," it should say instead "all v such that $sv(s,B) < n$ *and* $v(s,B) < d$." (ii) In the third-last sentence in the proof "$\chi_n(d;\theta)$" has to be replaced by "$\chi_n(n/d;\theta)$".

Thus $\Theta(\chi, \chi^q, \dots)$ has character $(-1)^{n-1}I_n^k[1]$. Finally this is computed in [Gre55], example (ii), p. 430f. □

2.2. **Hecke pairs and Hecke algebras.** We will use the same terminology as Ash-Stevens ([AS86]), but prefer left actions for our modules. Thus a *Hecke pair* is a pair (Γ, S) consisting of a subgroup Γ and a subsemigroup S of a fixed ambient group G such that

(i) $\Gamma \subset S$.
(ii) Γ and $s\Gamma s^{-1}$ are commensurable for all $s \in S$.

The *Hecke algebra* $\mathcal{H}(\Gamma, S)$ consists of left Γ-invariant elements in the free abelian group of left cosets $s\Gamma$ ($s \in S$), with the usual multiplication law:

$$\sum a_i(s_i\Gamma) \sum b_j(t_j\Gamma) = \sum a_ib_j(s_it_j\Gamma),$$

where a_i, $b_j \in \mathbb{Z}$, s_i, $t_j \in S$. In particular, any double coset $\Gamma s\Gamma = \coprod_i s_i\Gamma$ (a finite disjoint union) becomes a Hecke operator in $\mathcal{H}(\Gamma, S)$ in the natural way; it is denoted by $[\Gamma s\Gamma]$. If M is a left S-module (over any ring), the group cohomology modules $H^\cdot(\Gamma, M)$ inherit a natural linear action of $\mathcal{H}(\Gamma, S)$. This action is δ-functorial, i.e., long exact sequences associated to short exact sequences of S-modules are $\mathcal{H}(\Gamma, S)$-equivariant. It is thus determined by demanding that

$$[\Gamma s\Gamma]m = \sum_i s_im$$

for all $s \in S$, $m \in H^0(\Gamma, M)$. It is also possible to explicitly describe the action on cocyles in any degree (see [AS86], p. 194).

A Hecke pair (Γ_0, S_0) is *compatible* with (Γ, S) if $\Gamma_0 \subset \Gamma$, $S_0 \subset S$, $S_0\Gamma = S$, and $\Gamma \cap S_0^{-1}S_0 = \Gamma_0$. In this case, it is easy to check that there is a natural injection

$$\mathcal{H}(\Gamma, S) \hookrightarrow \mathcal{H}(\Gamma_0, S_0)$$

induced by restriction from the map on left cosets sending $s_0\Gamma \mapsto s_0\Gamma_0$ ($s_0 \in S_0$).

3. Representations of $GL_3(\mathbb{F}_q)$ in characteristic zero

The irreducible finite-dimensional representations of $GL_3(\mathbb{F}_q)$ over $\mathbb{C}$ were first determined by Steinberg in [Ste51]. The notations here are adapted from his paper. Here is a list of (the Jordan normal forms of) representatives of the conjugacy classes of $GL_3(\mathbb{F}_q)$:

$$A_1\colon \begin{pmatrix} a & & \\ & a & \\ & & a \end{pmatrix},\ A_2\colon \begin{pmatrix} a & 1 & \\ & a & \\ & & a \end{pmatrix},\ A_3\colon \begin{pmatrix} a & 1 & \\ & a & 1 \\ & & a \end{pmatrix},\ A_4\colon \begin{pmatrix} a & & \\ & a & \\ & & b \end{pmatrix},$$

$$A_5\colon \begin{pmatrix} a & 1 & \\ & a & \\ & & b \end{pmatrix},\ A_6:\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix},\ B_1:\begin{pmatrix} a & & \\ & \beta & \\ & & \beta^q \end{pmatrix},\ C_1\colon \begin{pmatrix} \alpha & & \\ & \alpha^q & \\ & & \alpha^{q^2} \end{pmatrix},$$

with a, b, c running over $\mathbb{F}_q^\times$, β over $\mathbb{F}_{q^2}^\times \setminus \mathbb{F}_q^\times$, and α over $\mathbb{F}_{q^3}^\times \setminus \mathbb{F}_q^\times$ (the latter two up to the action of the q-power map).

The character table that Steinberg calculated, with values in $\overline{\mathbb{Q}}_p$, can be found in figure 3.1. The ranges of the superscripts are listed in figure 3.2. The representation space corresponding to any of these characters will be denoted by V with the same sub- and superscript (strictly speaking, a choice is made). For example, $V_1^{(i)} \cong \overline{\mathbb{Q}}_p(\chi^i \circ \det)$.

Finally define $G_3 = GL_3(\mathbb{F}_q)$ and denote by P_3 and B_3 two standard "parabolic" subgroups:

$$P_3 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \qquad B_3 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$

Proposition 3.1. (i) *The "principal series" representation* $\mathrm{Ind}_{B_3}^{G_3}(\chi^i \otimes \chi^j \otimes \chi^k)$ (*which is independent of the order of* i, j, $k \in \mathbb{Z}/(q-1)$) *is isomorphic to*

$$\begin{aligned} V_1^{(i)} \oplus 2V_{q^2+q}^{(i)} \oplus V_{q^3}^{(i)} &\quad \text{if } i = j = k \\ V_{q^2+q+1}^{(i,j)} \oplus V_{q(q^2+q+1)}^{(i,j)} &\quad \text{if } i \neq j = k \\ V_{(q+1)(q^2+q+1)}^{(i,j,k)} &\quad \text{if } i,\ j,\ k \text{ are distinct} \end{aligned}$$

(*Adding up the corresponding entries in the character table shows that the character of this induction has a uniform shape for all* i, j, k.)

	$\chi_1^{(n)}$	$\chi_{q^2+q}^{(n)}$	$\chi_{q^3}^{(n)}$
A_1	$\tilde{a}^{3n}$	$(q^2+q)\tilde{a}^{3n}$	$q^3\tilde{a}^{3n}$
A_2	$\tilde{a}^{3n}$	$q\tilde{a}^{3n}$	0
A_3	$\tilde{a}^{3n}$	0	0
A_4	$\tilde{a}^{2n}\tilde{b}^n$	$(q+1)\tilde{a}^{2n}\tilde{b}^n$	$q\tilde{a}^{2n}\tilde{b}^n$
A_5	$\tilde{a}^{2n}\tilde{b}^n$	$\tilde{a}^{2n}\tilde{b}^n$	0
A_6	$\tilde{a}^n\tilde{b}^n\tilde{c}^n$	$2\tilde{a}^n\tilde{b}^n\tilde{c}^n$	$\tilde{a}^n\tilde{b}^n\tilde{c}^n$
B_1	$\tilde{a}^n\tilde{\beta}^{n(q+1)}$	0	$-\tilde{a}^n\tilde{\beta}^{n(q+1)}$
C_1	$\tilde{\alpha}^{n(q^2+q+1)}$	$-\tilde{\alpha}^{n(q^2+q+1)}$	$\tilde{\alpha}^{n(q^2+q+1)}$

	$\chi_{q^2+q+1}^{(m,n)}$	$\chi_{q(q^2+q+1)}^{(m,n)}$
A_1	$(q^2+q+1)\tilde{a}^{m+2n}$	$q(q^2+q+1)\tilde{a}^{m+2n}$
A_2	$(q+1)\tilde{a}^{m+2n}$	$q\tilde{a}^{m+2n}$
A_3	$\tilde{a}^{m+2n}$	0
A_4	$(q+1)\tilde{a}^{m+n}\tilde{b}^n+\tilde{a}^{2n}\tilde{b}^m$	$(q+1)\tilde{a}^{m+n}\tilde{b}^n+q\tilde{a}^{2n}\tilde{b}^m$
A_5	$\tilde{a}^{m+n}\tilde{b}^n+\tilde{a}^{2n}\tilde{b}^m$	$\tilde{a}^{m+n}\tilde{b}^n$
A_6	$\sum_{(a,b,c)}\tilde{a}^m\tilde{b}^n\tilde{c}^n$	$\sum_{(a,b,c)}\tilde{a}^m\tilde{b}^n\tilde{c}^n$
B_1	$\tilde{a}^m\tilde{\beta}^{n(q+1)}$	$-\tilde{a}^m\tilde{\beta}^{n(q+1)}$
C_1	0	0

	$\chi_{(q+1)(q^2+q+1)}^{(l,m,n)}$	$\chi_{(q-1)(q^2+q+1)}^{(m,n)}$	$\chi_{(q-1)^2(q+1)}^{(n)}$
A_1	$(q+1)(q^2+q+1)\tilde{a}^{l+m+n}$	$(q-1)(q^2+q+1)\tilde{a}^{m+n}$	$(q-1)^2(q+1)\tilde{a}^n$
A_2	$(2q+1)\tilde{a}^{l+m+n}$	$-\tilde{a}^{m+n}$	$-(q-1)\tilde{a}^n$
A_3	$\tilde{a}^{l+m+n}$	$-\tilde{a}^{m+n}$	$\tilde{a}^n$
A_4	$(q+1)\sum_{(l,m,n)}\tilde{a}^{l+m}\tilde{b}^n$	$(q-1)\tilde{a}^n\tilde{b}^m$	0
A_5	$\sum_{(l,m,n)}\tilde{a}^{l+m}\tilde{b}^n$	$-\tilde{a}^n\tilde{b}^m$	0
A_6	$\sum_{(l,m,n)}\tilde{a}^l\tilde{b}^m\tilde{c}^n$	0	0
B_1	0	$-\tilde{a}^m(\tilde{\beta}^n+\tilde{\beta}^{nq})$	0
C_1	0	0	$\tilde{\alpha}^n+\tilde{\alpha}^{nq}+\tilde{\alpha}^{nq^2}$

FIGURE 3.1. The character table of $GL_3(\mathbb{F}_q)$ with values in $\overline{\mathbb{Q}}_p$. Here, $\sum_{(a,b,c)} f(a,b,c)$ denotes the symmetric polynomial in a, b, c with typical term $f(a,b,c)$. (So there are one, two, three or six terms depending on the symmetries of f.)

family	range of the parameters
$\chi_1^{(n)}$	$n \in \mathbb{Z}/(q-1)$
$\chi_{q^2+q}^{(n)}$	$n \in \mathbb{Z}/(q-1)$
$\chi_{q^3}^{(n)}$	$n \in \mathbb{Z}/(q-1)$
$\chi_{q^2+q+1}^{(m,n)}$	$m \neq n \in \mathbb{Z}/(q-1)$
$\chi_{q(q^2+q+1)}^{(m,n)}$	$m \neq n \in \mathbb{Z}/(q-1)$
$\chi_{(q+1)(q^2+q+1)}^{(l,m,n)}$	$\{l,m,n\} \subset \mathbb{Z}/(q-1)$ distinct
$\chi_{(q-1)(q^2+q+1)}^{(m,n)}$	$m \in \mathbb{Z}/(q-1),\ \{n,nq\} \subset \mathbb{Z}/(q^2-1) - \mathbb{Z}/(q-1)$
$\chi_{(q-1)^2(q+1)}^{(n)}$	$\{n,nq,nq^2\} \subset \mathbb{Z}/(q^3-1) - \mathbb{Z}/(q-1)$

FIGURE 3.2. Parameter ranges for the characters in figure 3.1. Writing $\{l,m,n\}$ makes sense because the corresponding character only depends on this set. Similarly, $\{n,nq\}$ makes sense because (m,n) and (m,nq) index the same character in the second-last line.

(ii) $\mathrm{Ind}_{P_3}^{G_3}(\Theta(\chi_2^m,\chi_2^{qm})\otimes\chi^i)$ $(q+1\nmid m)$ *is isomorphic to*

$$V_{(q-1)(q^2+q+1)}^{(i,m)}.$$

(iii) *The cuspidal representation* $\Theta(\chi_3^m,\chi_3^{qm},\chi_3^{q^2m})$ $(q^2+q+1\nmid m)$ *is isomorphic to*

$$V_{(q-1)^2(q+1)}^{(m)}.$$

Remark 3.2. *Note that part* (*iii*) *is a special case of lemma 2.1.*

Proof. (i) In each case the dimensions agree on both sides, as B_3 has index $(q+1)(q^2+q+1)$, so it suffices to check that the (non-zero) multiplicities on the right-hand side are correct. Suppose now that i, j, k are distinct (the other cases follow in exactly the same way). By Frobenius reciprocity it suffices to show

$$\begin{aligned}1 &= \langle \chi_{(q+1)(q^2+q+1)}^{(i,j,k)}, \chi^i\otimes\chi^j\otimes\chi^k\rangle_{B_3}\\ &= (\#B_3)^{-1}\sum_{h\in B_3}\chi_{(q+1)(q^2+q+1)}^{(i,j,k)}(h)(\chi^i\otimes\chi^j\otimes\chi^k)(h^{-1}).\end{aligned}$$

The sum over B_3 splits up into six parts depending on the G_3-conjugacy class of h (B_1, C_1 cannot occur). The contribution of each is a polynomial in q (even each term is, in case of A_1, A_2 and A_3), and counting (resp. trivial estimates of the size of the terms) show that

	# of terms	each term
A_1	$O(q)$	$O(q^3)$
A_2	$O(q^3)$	$O(q)$
A_3	$O(q^4)$	$O(1)$
A_4	$O(q^4)$	$O(q)$
A_5	$O(q^5)$	$O(1)$
A_6	$O(q^6)$	$O(1)$

As $\#B_3 \sim q^6$, it follows that we only need to show that the sum over the A_6 terms is also asymptotically equivalent to q^6. Explicitly this sum is (with a, b, $c \in \mathbb{F}_q^\times$)

$$\begin{aligned} &= q^3 \sum_{\substack{a,b,c \\ \text{distinct}}} \Big\{ \sum_{(i,j,k)} \tilde{a}^i \tilde{b}^j \tilde{c}^k \Big\} \tilde{a}^{-i} \tilde{b}^{-j} \tilde{c}^{-k} \\ &= q^3 \sum_{\substack{a,b,c \\ \text{distinct}}} \Big(1 + \tilde{a}^{j-i} \tilde{b}^{i-j} + \tilde{b}^{k-j} \tilde{c}^{j-k} + \tilde{c}^{i-k} \tilde{a}^{k-i} \\ &\qquad\qquad + \tilde{a}^{j-i} \tilde{b}^{k-j} \tilde{c}^{i-k} + \tilde{a}^{k-i} \tilde{b}^{i-j} \tilde{c}^{j-k} \Big) \\ &= q^6 + O(q^5) \end{aligned}$$

as all the terms in the sum, except the first, only contribute $O(q^5)$ to the total (it is essential that i, j, k are distinct, because this implies $\sum_a \tilde{a}^{i-j} = 0$, etc.).

(ii) It suffices to show

$$\langle \chi^{(i,m)}_{(q-1)(q^2+q+1)}, \Theta(\chi_2^m, \chi_2^{qm}) \otimes \chi^i \rangle_{P_3} = 1.$$

Here is the relevant table for the number of terms for each G_3-conjugacy class, and the estimate for each term, split up into the contribution of the term on the left-hand side (resp. right-hand side) of the inner product:

	# of terms	1^{st} factor	2^{nd} factor
A_1	$O(q)$	$O(q^3)$	$O(q)$
A_2	$O(q^4)$	$O(1)$	$O(q)$
A_3	$O(q^5)$	$O(1)$	$O(1)$
A_4'	$O(q^4)$	$O(q)$	$O(q)$
A_4''	$O(q^5)$	$O(q)$	$O(1)$ [even 0]
A_5	$O(q^6)$	$O(1)$	$O(1)$
A_6	$\sim \frac{1}{2}q^7$	0	$O(1)$
B_1	$\sim \frac{1}{2}q^7$	$O(1)$	$O(1)$

Here A_4' denotes those matrices of type A_4 whose top left 2×2 matrix is a scalar, and A_4'' denotes the others.

Analogously to the proof of (i), it suffices to show that the contribution of the B_1 terms is asymptotically equivalent to q^7. This follows easily, as in part (i), because $\Theta(\chi_2^m, \chi_2^{qm}) \otimes \chi^i$ has character $-\tilde{a}^i(\tilde{\beta}^m + \tilde{\beta}^{mq})$ on matrices in P_3 with eigenvalues a, β, β^q (see (2.1)). The point this time is that

$$\sum_{\beta \in \mathbb{F}_{q^2}^\times \setminus \mathbb{F}_q^\times} \tilde{\beta}^i = \begin{cases} q^2 - q & \text{if } i \equiv 0 \pmod{q^2-1}, \\ O(q) & \text{otherwise.} \end{cases}$$

(iii) It is well-known that every character has to occur as constituent of some "parabolic induction" ([Bum97], ex. 4.1.17). The first two parts thus show that the irreducible characters of dimension $(q-1)^2(q+1)$ are cuspidal. The characterization of Θ in (2.1) completes the proof. □

4. Representations of $GL_3(\mathbb{F}_p)$ in characteristic p

Let G be a connected split reductive group over the finite field $\mathbb{F} = \mathbb{F}_q$ ($q = p^r$). Let $T \subset G$ be a maximal torus defined and split over $\mathbb{F}$. Let $R \subset X(T)$ be the root system of G. Choose a set of positive roots R^+ and a corresponding Borel subgroup $B \supset T$. Denote by B^- the opposite Borel. Denote by W the Weyl group of (G, T) and by $X(T)_+$ the dominant weights with respect to this choice of positive roots.

It will be useful in the following to also use a modified action of the Weyl group on $X(T)$. Let $\rho' \in X(T) \otimes \mathbb{Q}$ be any weight in $\frac{1}{2}\sum_{\alpha \in R^+} \alpha + (X(T) \otimes \mathbb{Q})^W$ and define

$$w \cdot \lambda = w(\lambda + \rho') - \rho'. \tag{4.1}$$

Of course, this is independent of the choice of ρ'. (In the literature, usually $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ is used for ρ', but in the case of $G = GL_n$ it will later be convenient to fix a different choice of ρ.)

For $\lambda \in X(T)_+$ the *(dual) Weyl module* $W(\lambda)$ is defined as algebraic induced module:

$$\begin{aligned} W(\lambda) &= \operatorname{ind}_{B^-}^G(\mathbb{F}(\lambda)) \\ &= \{f \in \operatorname{Mor}_{\mathbb{F}}(G, \mathbb{F}) : f(gb) = \lambda(b)^{-1} f(g) \\ &\qquad \forall\, \mathbb{F}\text{-algebras } A,\ g \in G(A),\ b \in B^-(A)\}. \end{aligned}$$

This is in fact a finite-dimensional $\mathbb{F}$-vector space, which becomes a left G-module in the natural way:

$$(xf)(g) = f(x^{-1}g) \quad \text{for all } \mathbb{F} \to A,\ f \in W(\lambda) \otimes A,\ g,\ x \in G(A).$$

Define $F(\lambda) = \operatorname{soc}_G W(\lambda)$ (the socle of the Weyl module, as G-module).

Theorem 4.1. *The set of simple G-modules is $\{F(\lambda) : \lambda \in X(T)_+\}$. If $F(\lambda) \cong F(\mu)$ (λ, $\mu \in X(T)_+$) then $\lambda = \mu$.*

More generally, considered *in the Grothendieck group of G-modules* we can extend the definition of Weyl module to all of $X(T)$, as an Euler characteristic (see [Jan03], II.5.7):

$$W(\lambda) = \sum_i (-1)^i (R^i \operatorname{ind}_{B^-}^G)(\mathbb{F}(\lambda)).$$

(If λ is dominant, only the $i = 0$ term is non-zero, so this agrees with the previous definition.) The context should always make it clear whether $W(\lambda)$ refers to a genuine representation (and λ dominant) or to an element of the Grothendieck group. The formal character is given by the Weyl character formula ([Jan03], II.5.10):

$$\operatorname{ch} W(\lambda) = \frac{\sum_{w\in W} \det w \cdot e(w(\lambda+\rho'))}{\sum_{w\in W} \det w \cdot e(w(\rho'))} \in \mathbb{Z}[X(T)]^W. \tag{4.2}$$

Here $e(\lambda) \in \mathbb{Z}[X(T)]$ denotes the weight λ considered in the group algebra. In particular it follows that

$$W(w\cdot\lambda) = \det(w) W(\lambda), \tag{4.3}$$

and in turn that $W(\lambda) = 0$ if and only if $\lambda + \rho'$ lies on the wall of a Weyl chamber, whereas in all other cases, this formula allows to express $W(\lambda)$ as $\pm W(\lambda_+)$ with λ_+ dominant.

Note also that the map

$$\operatorname{ch} : \{G\text{-modules}\} \to \mathbb{Z}[X(T)]^W$$

induces an isomorphism between the Grothendieck group of G-modules and $\mathbb{Z}[X(T)]^W$ (see [Jan03], II.5.8).

Definition 4.2.

$$X^0(T) = \{\lambda \in X(T) : \langle \lambda, \alpha^\vee\rangle = 0 \quad \forall \alpha \in R\}.$$

The (r-)restricted region is defined to be:

$$X_r(T) = \{\lambda \in X(T) : 0 \le \langle \lambda, \alpha^\vee\rangle < q = p^r \quad \textit{for all simple roots } \alpha\}.$$

Remark 4.3.

(i) *Note that $X^0(T) = X(T)^W$, by looking at the basic reflections s_α ($\alpha \in R$) generating W.*

(ii) *Note that if $\mu \in X(T)_+$, $\nu \in X^0(T)$, then*

$$W(\mu+\nu) \cong W(\mu) \otimes W(\nu), \ F(\mu+\nu) \cong F(\mu)\otimes F(\nu)$$

and $W(\nu) = F(\nu)$ is a one-dimensional representation with character $e(\nu)$. This follows immediately from the Weyl character formula.

Proposition 4.4 (Brauer's formula)**.** *If $\sum_{\mu\in X(T)} a_\mu e(\mu) \in \mathbb{Z}[X(T)]^W$, then for all $\lambda \in X(T)$,*

$$W(\lambda) \otimes \sum_{\mu \in X(T)} a_\mu e(\mu) \cong \sum_{\mu\in X(T)} a_\mu W(\lambda+\mu)$$

in the Grothendieck group of G-modules.

For the proof, see for example [Jan77], §2(1).

As G is split reductive over $\mathbb{F}$, thm. 3.6.6 in Demazure's thesis ([Dem65]) shows that G can be canonically defined over $\mathbb{Z}$, in particular over $\mathbb{F}_p$. In particular, there is a Frobenius endomorphism $F : G \to G$. For any $i \ge 0$

and any G-module V, corresponding to a homomorphism $r : G \to GL(V)$, define a new G-module $V^{(i)}$ which equals V abstractly but whose G-action is obtained by composing r with F^i. (**Correction:** not true that G is canonically defined over $\mathbb{Z}$—still have automorphisms. Better to fix a $\mathbb{Z}$- or even just $\mathbb{F}_p$-structure at the start.)

Theorem 4.5 (Steinberg)**.** *Suppose $\lambda = \sum_{i=0}^r \lambda_i p^i$ with $\lambda_i \in X_1(T)$. Then*

$$F(\lambda) \cong F(\lambda_0) \otimes F(\lambda_1)^{(1)} \otimes \ldots \otimes F(\lambda_r)^{(r)}.$$

For the proof, see for example [Jan03], cor. II.3.17.

Now we can state the classification theorem for irreducible modular representations of $G(\mathbb{F})$, under a further condition on G.

Theorem 4.6 ([Jan05])**.** *Suppose that G has simply connected derived group (e.g., $G = GL_n$).*

(i) *If $\lambda \in X_r(T)$, $F(\lambda) \otimes \overline{\mathbb{F}}$ is irreducible as representation of $G(\mathbb{F})$. Any irreducible representation of $G(\mathbb{F})$ over $\overline{\mathbb{F}}$ arises in this way.*
(ii) *$F(\lambda) \otimes \overline{\mathbb{F}} \cong F(\mu) \otimes \overline{\mathbb{F}}$ as representation of $G(\mathbb{F})$ if and only if $\lambda - \mu \in (q-1)X^0(T)$.*

Remark 4.7. *In particular this result shows that $\mathbb{F}$ is a splitting field for representations of $G(\mathbb{F})$. Since Schur's lemma together with Hilbert's theorem 90 shows that*

$$F(\lambda) \cong F(\mu) \iff F(\lambda) \otimes \overline{\mathbb{F}} \cong F(\mu) \otimes \overline{\mathbb{F}},$$

isomorphism classes of representations of $G(\mathbb{F})$ over $\mathbb{F}$ and over $\overline{\mathbb{F}}$ (or over any intermediate field) are in natural bijection with one another.

To apply these results to $G = GL_{3/\mathbb{F}_p}$, let T be the diagonal matrices and B the upper-triangular matrices. Then $X(T) \cong \mathbb{Z}^3$ naturally: for $(a, b, c) \in \mathbb{Z}^3$ let (a, b, c) denote the weight

$$\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \mapsto t_1^a t_2^b t_3^c$$

in $X(T)$. Then $X^0(T) = (1, 1, 1)\mathbb{Z}$, $X_1(T) = \{(a, b, c) : 0 \le a - b, b - c \le p - 1\}$, (a, b, c) is dominant if and only if $a \ge b \ge c$, and choose $\rho' = (2, 1, 0)$.

Corollary 4.8.

(i) *The irreducible algebraic GL_3-modules are the $F(a, b, c)$, $a \ge b \ge c$.*
(ii) *The irreducible modular representations of $GL_3(\mathbb{F}_p)$ are the $F(a, b, c)$, $0 \le a - b$, $b - c \le p - 1$. $F(a, b, c) \cong F(a', b', c')$ if and only if $(a, b, c) - (a', b', c') \in (p - 1, p - 1, p - 1)\mathbb{Z}$.*

It will be convenient to use the terminology of "alcoves" in the following. They are the complements of a family of hyperplanes in weight space $X(T) \otimes \mathbb{R}$. In the case of GL_3, these hyperplanes are (for each $n \in \mathbb{Z}$)

$$\{a - b = np - 1\}, \ \{b - c = np - 1\}, \ \{a - c = np - 2\}.$$

The corollary concerns precisely (the dominant weights in the closures of) two alcoves: the *lower alcove*

$$\{(a,b,c) \in \mathbb{R}^3 : -1 < a-b, b-c \quad \text{and} \quad a-c < p-2\}$$

and the *upper alcove*

$$\{(a,b,c) \in \mathbb{R}^3 : p-2 < a-c \quad \text{and} \quad a-b, b-c < p-1\}.$$

Proposition 4.9 (Jantzen)**.** *Suppose that (a,b,c) is in the restricted region $X_1(T)$.*

(i) *If (a,b,c) is in the upper alcove then there is an exact sequence*

$$0 \to F(a,b,c) \to W(a,b,c) \to F(c+p-2,b,a-p+2) \to 0.$$

(ii) *Otherwise, i.e., if (a,b,c) is in the lower alcove or on the boundary of the upper alcove, $F(a,b,c) = W(a,b,c)$.*

Notation: ${}^r(a,b,c) = (c+p-2,b,a-p+2)$.

Proof. (ii) follows from the strong linkage principle (Prop. II.6.13 in [Jan03]). (i) is a consequence of prop. II.7.11 and lemma II.7.15 ([Jan03]): Let $\lambda = (a,b,c)$. Then ${}^r\lambda = (c+p-2,b,a-p+2)$ is the unique weight which is strictly smaller than λ in the $\uparrow$-ordering of $X(T)$. Pick a weight μ in the upper closure of the lower alcove, but not in the lower alcove itself (e.g. $\mu = (p-2,0,0)$), and apply the translation functor $T^{\mu}_{{}^r\lambda}$ to the identity of formal characters

$$\operatorname{ch} W(\lambda) = \operatorname{ch} F(\lambda) + m \operatorname{ch} F({}^r\lambda),$$

which holds for some integer m by the strong linkage principle, to deduce that $m = 1$. □

5. Decomposition of $GL_3(\mathbb{F}_p)$-representations

Suppose that $V/\overline{\mathbb{Q}}_p$ is a finite-dimensional representation of a finite group G. Then we can define the (semisimplified) reduction of V "modulo p" to be $\overline{V} = (M/\mathfrak{m}_{\overline{\mathbb{Z}}_p} M)^{ss}$ for any G-stable $\overline{\mathbb{Z}}_p$-lattice $M \subset V$. This is a semisimple representation over $\overline{\mathbb{F}}_p$ which, by the Brauer-Nesbitt theorem, is independent of the choice of M.

Theorem 5.1. *The following identities hold in the Grothendieck group of representations of $GL_3(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$:*

$$\begin{aligned}
\overline{V}_1^{(i)} &= W(i,i,i) \\
\overline{V}_{p^2+p}^{(i)} &= W(p-1+i,i,i) + W(p-1+i,p-1+i,i) \\
\overline{V}_{p^3}^{(i)} &= W(2p-2+i,p-1+i,i) \\
\overline{V}_{p^2+p+1}^{(i,j)} &= W(i,j,j) + W(p-1+j,p-1+j,i) + F(p-1+j,i,j)
\end{aligned}$$

$$(0 < i - j < p - 1)$$

$$\begin{aligned}\overline{V}^{(i,j)}_{p(p^2+p+1)} &= W(p-2+j, i, j+1) + W(p-1+j, i, j)\\ &+ W(2p-2+j, p-1+j, i) + W(p-1+i, p-1+j, j)\end{aligned}$$

$$(0 < i - j < p - 1)$$

$$\begin{aligned}\overline{V}^{(i,j,k)}_{(p+1)(p^2+p+1)} &= W(i,j,k) + W(p-1+j, p-1+k, i)\\ &+ W(p-1+k, i, j) + W(2p-2+k, p-1+j, i)\\ &+ W(p-1+i, p-1+k, j) + W(p-1+j, i, k)\end{aligned}$$

$$(i > j > k,\ i - k \leq p-2)$$

$$\begin{aligned}\overline{V}^{(i,j+pk)}_{(p-1)(p^2+p+1)} &= W(i, j-1, k+1) + W(p-2+k, i, j+1)\\ &+ W(p-2+j, p+k, i) + W(p-2+j, i, k+1)\\ &+ W(p-1+i, p-1+k, j) + W(2p-2+k, p-1+j, i)\end{aligned}$$

$$(i \geq j > k,\ i - k \leq p-1)$$

$$\begin{aligned}\overline{V}^{(i+jp+kp^2)}_{(p-1)^2(p+1)} &= W(i-2, j+1, k+1) + W(p-2+k, i, j+1)\\ &+ W(p-2+j, p+k, i) + W(2p-2+k, p-1+j, i)\\ &+ W(p-1+j, i-1, k+1) + W(p-2+i, p-1+k, j+1)\end{aligned}$$

$$(i > j \geq k,\ i - k \leq p, \text{ resp. } i < j \leq k,\ k - i \leq p)$$

Remark 5.2. (i) *The identities hold for all integers i, j, k, as long as the left-hand side is defined (see fig. 3.2). The ranges listed in brackets beneath the formulae (modulo $\sim$ defined below) are chosen so as to cover all possible cases exactly once while at the same time (in all but a few exceptional cases) keeping the highest weights of the Weyl modules in the restricted region. The equivalence relation $\sim$ is generated by the relations (depending on the family):*

Remark (Nov 2010): when $i < j \leq k$, $k - i \leq p$ the highest weights don't generally stay inside the restricted region. The easiest way to obtain a nice formula in this case is to dualise the formula for $i > j \geq k$, $i - k \leq p$.

$$\begin{aligned}&(i,j,k) \sim (i+p-1, j+p-1, k+p-1), && (\textit{all cases})\\ &(i,j,k) \sim (k+p-1, i, j), && (\chi^{(i,j,k)}_{(p+1)(p^2+p+1)})\\ &(i,j,k) \sim (k+p, i-1, j), && \\ &\quad \textit{resp. } (i,j,k) \sim (j, k-1, i+p). && (\chi^{(i+jp+kp^2)}_{(p-1)^2(p+1)})\end{aligned}$$

In the stated ranges, zero Weyl modules occur only in the last two decomposition formulae, and Weyl modules outside the closures of lower and upper alcove can only appear in the last (i.e., cuspidal) decomposition formula, in which case one has to keep in mind that $W(a,b,c) = -W(b-1, a+1, c)$

(use if $a-b=-2$*) and* $W(a,b,c)=-W(a,c-1,b+1)$ *(use if* $b-c=-2$*); c.f.* (4.3)*.*

(ii) *Decomposing Weyl modules in the upper alcove into its two irreducible constituents (proposition 4.9) shows that* generically *there are the following number of irreducible constituents in each family (if* p *is not too small):*

dimension of rep.	*# of constituents*
1	1
p^2+p	2
p^3	1
p^2+p+1	3
$p(p^2+p+1)$	5
$(p+1)(p^2+p+1)$	9
$(p-1)(p^2+p+1)$	9
$(p-1)^2(p+1)$	9

Proof. It suffices to check that, in each case, the Brauer character on the right-hand side agrees with the character on the left-hand side on semisimple conjugacy classes. The Brauer character of a Weyl module $W(a_1,a_2,a_3)$ is computed in the following way (see [Jan87]). Each semisimple element $g\in GL_3(\mathbb{F}_p)$ is diagonalisable, hence is conjugate (over $\overline{\mathbb{F}}_p$) to an element t of a fixed maximal torus $T\subset GL_3$. If the formal character of the GL_3-module $W(a_1,a_2,a_3)$ is

$$\sum_{\mu\in X(T)} m(\mu)\mu$$

then the Brauer character evaluated at g equals

$$\sum_{\mu\in X(T)} m(\mu)\widetilde{\mu(t)}$$

(Here $\widetilde{\ }$ denotes the Teichmüller lift again.) Finally, the formal character of $W(a_1,a_2,a_3)$ is obtained from (4.2): it is the Schur polynomial

$$s_{(a_1,a_2,a_3)}(x_1,x_2,x_3)=\frac{\det(x_i^{a_j+3-j})}{\det(x_i^{3-j})},$$

where x_i is the weight $\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3\end{pmatrix}\mapsto t_i$ (denoted above by $(0,\dots,1,\dots,0)$ with a 1 in the i-th position) considered as element of $\mathbb{Z}[X(T)]$.

By proposition 4.9, we also know the Brauer character of $F(a_1,a_2,a_3)$ if $0\le a_1-a_2$, $a_2-a_3\le p-1$.

To check the claim for $\chi_1^{(i)}$, note that

$$s_{(i,i,i)}(x,y,z)=(xyz)^i.$$

Thus the Brauer character of $W(i,i,i)$ on the semisimple conjugacy classes, as listed above, equals $\tilde{a}^{3i}$ on A_1, $\tilde{a}^{2i}\tilde{b}^i$ on A_4, $\tilde{a}^i\tilde{b}^i\tilde{c}^i$ on A_6, $\tilde{\alpha}^i\tilde{\beta}^{i(p+1)}$ on B_1 and $\tilde{\alpha}^{i(p^2+p+1)}$ on C_1, precisely as claimed.

Similarly for $\chi^{(i)}_{p^3}$, we can calculate

$$s_{(2p-2+i,p-1+i,i)}(x,y,z) = (xyz)^i \frac{(x^p-y^p)(x^p-z^p)(y^p-z^p)}{(x-y)(x-y)(y-z)}.$$

So the Brauer character of $W(2p-2+i,p-1+i,i)$ is again immediately evaluated, and agrees with the ordinary character on semisimple conjugacy classes.

For $\chi^{(i,j)}_{(p^2+p+1)}$: the Brauer character of

$$W(i,j,j) + W(p-1+j,p-1+j,i) + F(p-1+j,i,j)$$

is computed by evaluating

$$s_{(i,j,j)} + s_{(p-1+j,p-1+j,i)} + s_{(p-1+j,i,j)} - s_{(j+p-2,i,j+1)} \tag{5.1}$$

on eigenvalues. For matrices of type A_1 note first that in general

$$s_{(a,b,c)}(x,x,x) = \frac{\begin{vmatrix} x^{a+2} & x^{b+1} & x^c \\ (a+2)x^{a+1} & (b+1)x^b & cx^{c-1} \\ (a+2)(a+1)x^a & (b+1)bx^{b-1} & c(c-1)x^{c-2} \end{vmatrix}}{\begin{vmatrix} x^2 & x & 1 \\ 2x & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix}}$$

$$= \frac{1}{2}(a-b+1)(b-c+1)(a-c+2)x^{a+b+c}$$

where the first equality follows from two applications of de l'Hôpital's rule. (Observe that this recovers the Weyl dimension formula for GL_3, which we could have used alternatively.)

Thus (5.1) evaluated at $(\tilde a,\tilde a,\tilde a)$ becomes $(p^2+p+1)\tilde a^{i+2j}$ after a few lines of calculation.

For matrices of type A_4 again de l'Hôpital gives in general

$$s_{(a,b,c)}(x,x,y) = \frac{\begin{vmatrix} x^{a+2} & x^{b+1} & x^c \\ (a+2)x^{a+1} & (b+1)x^b & cx^{c-1} \\ y^{a+2} & y^{b+1} & y^c \end{vmatrix}}{\begin{vmatrix} x^2 & x & 1 \\ 2x & 1 & 0 \\ y^2 & y & 1 \end{vmatrix}}.$$

Evaluating (5.1) at $(\tilde a,\tilde a,\tilde b)$ yields a common denominator $-(\tilde a-\tilde b)^2$ and a numerator

$$\begin{vmatrix} \tilde a^{i+2} & \tilde a^{j+1} & \tilde a^j \\ (i+2)\tilde a^{i+1} & (j+1)\tilde a^j & j\tilde a^{j-1} \\ \tilde b^{i+2} & \tilde b^{j+1} & \tilde b^j \end{vmatrix} + \begin{vmatrix} \tilde a^i & \tilde a^{j+2} & \tilde a^{j+1} \\ i\tilde a^{i-1} & (j+p+1)\tilde a^{j+1} & (j+p)\tilde a^j \\ \tilde b^i & \tilde b^{j+2} & \tilde b^{j+1} \end{vmatrix} +$$

$$\begin{vmatrix} \tilde a^{i+1} & \tilde a^j & \tilde a^{j+2} \\ (i+1)\tilde a^i & j\tilde a^{j-1} & (j+p+1)\tilde a^{j+1} \\ \tilde b^{i+1} & \tilde b^j & \tilde b^{j+2} \end{vmatrix} - \begin{vmatrix} \tilde a^{i+1} & \tilde a^{j+1} & \tilde a^{j+1} \\ (i+1)\tilde a^i & (j+1)\tilde a^j & (j+p)\tilde a^j \\ \tilde b^{i+1} & \tilde b^{j+1} & \tilde b^{j+1} \end{vmatrix}$$

by cyclically permuting the columns. Note that we can subtract integer multiples of $\tilde{a}^{-1}$ times the first row from the second row. By taking out scalars from rows, moreover, and letting $n = i - j$ we get

$$\tilde{a}^{2j}\tilde{b}^{j}\left\{\begin{vmatrix} \tilde{a}^{n+2} & \tilde{a} & 1 \\ (n+2)\tilde{a}^{n+1} & 1 & 0 \\ \tilde{b}^{n+2} & \tilde{b} & 1 \end{vmatrix} + \tilde{a}^2\tilde{b}\begin{vmatrix} \tilde{a}^{n-1} & \tilde{a} & 1 \\ (n-p)\tilde{a}^{n-2} & 1 & 0 \\ \tilde{b}^{n-1} & \tilde{b} & 1 \end{vmatrix}\right.$$
$$\left. + \tilde{a}\begin{vmatrix} \tilde{a}^{n+1} & 1 & \tilde{a}^2 \\ (n+1)\tilde{a}^{n-1} & 0 & p+1 \\ \tilde{b}^{n+1} & 1 & \tilde{b}^2 \end{vmatrix} - \tilde{a}\tilde{b}\begin{vmatrix} \tilde{a}^{n} & 1 & 1 \\ n\tilde{a}^{n} & 0 & p-1 \\ \tilde{b}^{n} & 1 & 1 \end{vmatrix}\right\}$$

which expands out to the desired answer (keeping in mind the denominator from before)

$$-(\tilde{a} - \tilde{b})^2((p+1)\tilde{a}^{i+j}\tilde{b}^{j} + \tilde{a}^{2j}\tilde{b}^{i}).$$

For type A_6, evaluate (5.1) at $(\tilde{a}, \tilde{b}, \tilde{c})$, noting that one of the terms vanishes. The numerator equals

$$\begin{vmatrix} \tilde{a}^{i+2} & \tilde{a}^{j+1} & \tilde{a}^{j} \\ \tilde{b}^{i+2} & \tilde{b}^{j+1} & \tilde{b}^{j} \\ \tilde{c}^{i+2} & \tilde{c}^{j+1} & \tilde{c}^{j} \end{vmatrix} + \begin{vmatrix} \tilde{a}^{i} & \tilde{a}^{j+2} & \tilde{a}^{j+1} \\ \tilde{b}^{i} & \tilde{b}^{j+2} & \tilde{b}^{j+1} \\ \tilde{c}^{i} & \tilde{c}^{j+2} & \tilde{c}^{j+1} \end{vmatrix} + \begin{vmatrix} \tilde{a}^{i+1} & \tilde{a}^{j} & \tilde{a}^{j+2} \\ \tilde{b}^{i+1} & \tilde{b}^{j} & \tilde{b}^{j+2} \\ \tilde{c}^{i+1} & \tilde{c}^{j} & \tilde{c}^{j+2} \end{vmatrix}.$$

Expanding the matrices (with π running through the permutations of $\tilde{a}$, $\tilde{b}$, $\tilde{c}$) this becomes:

$$\begin{aligned} &\sum_{\pi}(-1)^{\pi}\pi\big(\tilde{a}^{i}\tilde{b}^{j}\tilde{c}^{j}(\tilde{a}^2\tilde{b} + \tilde{b}^2\tilde{c} + \tilde{c}^2\tilde{a})\big) \\ &= \Big\{\sum_{\pi \text{ even}} \pi(\tilde{a}^{i}\tilde{b}^{j}\tilde{c}^{j})\Big\}(\tilde{a}^2\tilde{b} + \tilde{b}^2\tilde{c} + \tilde{c}^2\tilde{a} - \tilde{a}\tilde{b}^2 - \tilde{b}\tilde{c}^2 - \tilde{c}\tilde{a}^2) \end{aligned}$$

which is as required, since the second factor is the denominator.

For type B_1, evaluate (5.1) at $(\tilde{a}, \tilde{\beta}, \tilde{\beta}^{p})$ which gives a numerator of

$$\begin{vmatrix} \tilde{a}^{i+2} & \tilde{a}^{j+1} & \tilde{a}^{j} \\ \tilde{\beta}^{i+2} & \tilde{\beta}^{j+1} & \tilde{\beta}^{j} \\ \tilde{\beta}^{ip+2p} & \tilde{\beta}^{jp+p} & \tilde{\beta}^{jp} \end{vmatrix} + \begin{vmatrix} \tilde{a}^{i} & \tilde{a}^{j+2} & \tilde{a}^{j+1} \\ \tilde{\beta}^{i} & \tilde{\beta}^{j+p+1} & \tilde{\beta}^{j+p} \\ \tilde{\beta}^{ip} & \tilde{\beta}^{jp+p+1} & \tilde{\beta}^{jp+1} \end{vmatrix}$$
$$+ \begin{vmatrix} \tilde{a}^{i+1} & \tilde{a}^{j} & \tilde{a}^{j+2} \\ \tilde{\beta}^{i+1} & \tilde{\beta}^{j} & \tilde{\beta}^{j+p+1} \\ \tilde{\beta}^{ip+p} & \tilde{\beta}^{jp} & \tilde{\beta}^{jp+p+1} \end{vmatrix} - \begin{vmatrix} \tilde{a}^{i+1} & \tilde{a}^{j+1} & \tilde{a}^{j+1} \\ \tilde{\beta}^{i+1} & \tilde{\beta}^{j+1} & \tilde{\beta}^{j+p} \\ \tilde{\beta}^{ip+p} & \tilde{\beta}^{jp+p} & \tilde{\beta}^{jp+1} \end{vmatrix}.$$

Note that the negative terms in these determinants (i.e., those involving an odd permutation) are, up to sign, the conjugates over $\mathbb{F}_p$ of the positive ones. So, denoting by $\sum' A$ the difference $A - \overline{A}$ of A and its conjugate $\overline{A}$

we get

$$
\begin{aligned}
&= {\sum}' \Big[\tilde{a}^i\tilde{\beta}^{j(p+1)}(\tilde{a}^2\tilde{\beta} + \tilde{\beta}^{p+2} + \tilde{a}\tilde{\beta}^{p+1} - \tilde{a}\tilde{\beta}^2) \\
&\qquad + \tilde{a}^j\tilde{\beta}^{i+jp}\big((\tilde{\beta}^{p+2} - \tilde{a}\tilde{\beta}^2) + (\tilde{a}\tilde{\beta}^{p+1} - \tilde{a}^2\tilde{\beta}) \\
&\qquad\qquad + (\tilde{a}^2\tilde{\beta} - \tilde{\beta}^{p+2}) - (\tilde{a}\tilde{\beta}^{p+1} - \tilde{a}\tilde{\beta}^2)\big)\Big] \\
&= \tilde{a}^i\tilde{\beta}^{j(p+1)}(\tilde{a} - \tilde{\beta})(\tilde{a} - \tilde{\beta}^p)(\tilde{\beta} - \tilde{\beta}^p)
\end{aligned}
$$

as required (the product again being the denominator).

For type C_1, evaluate (5.1) at $(\tilde{\alpha}, \tilde{\alpha}^p, \tilde{\alpha}^{p^2})$. Here again is the numerator:

$$
\begin{vmatrix}
\tilde{\alpha}^{i+2} & \tilde{\alpha}^{j+1} & \tilde{\alpha}^j \\
\tilde{\alpha}^{ip+2p} & \tilde{\alpha}^{jp+p} & \tilde{\alpha}^{jp} \\
\tilde{\alpha}^{ip^2+2p^2} & \tilde{\alpha}^{jp^2+p^2} & \tilde{\alpha}^{jp^2}
\end{vmatrix}
+
\begin{vmatrix}
\tilde{\alpha}^{i} & \tilde{\alpha}^{j+p+1} & \tilde{\alpha}^{j+p} \\
\tilde{\alpha}^{ip} & \tilde{\alpha}^{jp+p^2+p} & \tilde{\alpha}^{jp+p^2} \\
\tilde{\alpha}^{ip^2} & \tilde{\alpha}^{jp^2+p^2+1} & \tilde{\alpha}^{jp^2+1}
\end{vmatrix}
+
\begin{vmatrix}
\tilde{\alpha}^{i+1} & \tilde{\alpha}^{j} & \tilde{\alpha}^{j+p+1} \\
\tilde{\alpha}^{ip+p} & \tilde{\alpha}^{jp} & \tilde{\alpha}^{jp+p^2+p} \\
\tilde{\alpha}^{ip^2+p^2} & \tilde{\alpha}^{jp^2} & \tilde{\alpha}^{jp^2+p^2+1}
\end{vmatrix}
-
\begin{vmatrix}
\tilde{\alpha}^{i+1} & \tilde{\alpha}^{j+1} & \tilde{\alpha}^{j+p} \\
\tilde{\alpha}^{ip+p} & \tilde{\alpha}^{jp+p} & \tilde{\alpha}^{jp+p^2} \\
\tilde{\alpha}^{ip^2+p^2} & \tilde{\alpha}^{jp^2+p^2} & \tilde{\alpha}^{jp^2+1}
\end{vmatrix}.
$$

Observe then that as the rows in each determinant are mutual conjugates over $\mathbb{F}_p$, each of the four terms is the trace of just two terms (one even, one odd). We obtain:

$$
\begin{aligned}
&= \mathrm{tr}_{\mathbb{F}_{p^3}/\mathbb{F}_p}\Big\{\tilde{\alpha}^{i+j(p+p^2)}\big((\tilde{\alpha}^{p+2} - \tilde{\alpha}^{p^2+2}) + (\tilde{\alpha}^{p^2+p+1} - \tilde{\alpha}^{2p^2+1}) \\
&\qquad\qquad + (\tilde{\alpha}^{p^2+2} - \tilde{\alpha}^{p^2+p+1}) - (\tilde{\alpha}^{p+2} - \tilde{\alpha}^{2p^2+1})\big)\Big\} \\
&= 0,
\end{aligned}
$$

as we needed.

This completes the proof for the decomposition of the characters of dimension $p^2 + p + 1$.

Claim: It remains to check that $\overline{\mathrm{Ind}_{B_3}^{G_3}(\chi^i \otimes \chi^j \otimes \chi^k)}$ equals

$$
\begin{aligned}
&W(i, j,k) + W(p - 1 + j, p - 1 + k, i) \\
&\quad + W(p - 1 + k, i, j) + W(2p - 2 + k, p - 1 + j, i) \\
&\quad + W(p - 1 + i, p - 1 + k, j) + W(p - 1 + j, i, k)
\end{aligned}
\tag{5.2}
$$

(in the Grothendieck group, for *all* i, j, k) and to prove the decomposition for the families of characters of dimensions $(p-1)(p^2+p+1)$ and $(p-1)^2(p+1)$.

The claim holds simply by proposition 3.1 together with the decompositions we have already checked.

There are two ways to proceed:

Method 1: Continue along the same lines, in each case evaluating the Brauer character of the sum of Weyl modules in question. Note that in the case of (5.2) the character of the induction is uniformly of the same shape for all values of i, j, k—as observed in proposition 3.1. The calculations are lengthier, but not more difficult, than the ones above.

Method 2: It is a simple consequence of Jantzen's result (thm. 10.3) on the decomposition of Deligne-Lusztig representations—we will use the notation of section 10. In the case of GL_3, Jantzen's result becomes (again in the Grothendieck group):

$$\begin{aligned}\overline{R_w(i,j,k)} = W((i,j+1,k+2) - w(0,1,2))& \\ + W((j+p-1,i,k+1) - (1\ 2)w(0,0,1))& \\ + W((i+p-1,k+p,j+1) - (2\ 3)w(0,1,1))& \\ + W((k+p-1,i,j+1) - (1\ 2\ 3)w(0,1,0))& \\ + W((j+p-1,k+p,i+1) - (1\ 3\ 2)w(1,0,1))& \\ + W((k+2p-2,j+p-1,i)).&\end{aligned}$$

Set now $\mu = (i,j,k)$. If $w = 1$, $T_w = T \subset B$. By lemmas 10.1 and 10.2,

$$R_1(i,j,k) \cong \mathrm{Ind}_{B_3}^{G_3}(\chi^i \otimes \chi^j \otimes \chi^k).$$

If $w = (2\ 3)$ and $p+1 \nmid j + pk$, the same two lemmas together with prop. 3.1 give

$$R_{(2\ 3)}(i,j,k) \cong V_{(p-1)(p^2+p+1)}^{(i,j+pk)}.$$

If $w = (1\ 2\ 3)$ and $p^2+p+1 \nmid i+pj+p^2k$, lemma 10.1 and prop. 3.1 yield

$$R_{(1\ 2\ 3)}(i,j,k) \cong V_{(p-1)^2(p+1)}^{(i+pj+p^2k)}.$$

Finally just plug these into Jantzen's formula. $\square$

6. Statement of a conjecture for GL_3

Definition 6.1. *A* Serre weight *is an isomorphism class of irreducible representations of $GL_3(\mathbb{F}_p)$ over $\mathbb{F}_p$ (note that by remark 4.7 we could equally well replace the base field by $\overline{\mathbb{F}}_p$). Let W_{Ser} denote the set of all Serre weights:*

$$W_{Ser} = \{F(a,b,c) : 0 \le a-b, b-c \le p-1\}.$$

A regular Serre weight *is an element of*

$$W_{reg} = \{F(a,b,c) : 0 \le a-b, b-c < p-1\} \subset W_{Ser}.$$

Note that $\#W_{Ser} = p^2(p-1)$, $\#W_{reg} = (p-1)^3$.

Define a surjective map $\mathcal{R} : W_{Ser} \to W_{reg}$ by

$$F(a,b,c) \mapsto F(c+2(p-2), b+(p-2), a)_{reg}, \tag{6.1}$$

where $F(x_1,x_2,x_3)_{reg}$ is the regular Serre weight $F(x_1',x_2',x_3')$ with $x_i' \equiv x_i$ (mod $p-1$) for all i (note that this is well defined, even though the x_i' are not). On regular Serre weights, the subscript "reg" can be omitted in the definition.

Fix for now a continuous Galois representation $\rho : G_{\mathbb{Q}} \to GL_3(\overline{\mathbb{F}}_p)$ that is *irreducible* and *odd* (i.e., if $p > 2$, $\rho(c) \neq \pm 1$ for any complex conjugation c).

For any positive integer N which is prime to p, define $\Gamma_0(N)$ to be the group of matrices in $SL_3(\mathbb{Z})$ with last row congruent to $(0, 0, *)$ modulo N. Also let $S_0(N)$ be the group of matrices in $GL_3^+(\mathbb{Z}_{(N)})$ with last row congruent to $(0, 0, *)$ modulo N and let $S_0'(N) = S_0(N) \cap GL_3^+(\mathbb{Z}_{(Np)})$. Here $\mathbb{Z}_{(N)}$ is the ring of rational numbers with denominators prime to N.

Then $(\Gamma_0(N), S_0'(N))$, $(\Gamma_0(N), S_0(N))$ are Hecke pairs (see §2.2). The corresponding Hecke algebras over the integers are denoted by $\mathcal{H}_0'(N)$, $\mathcal{H}_0(N)$; clearly $\mathcal{H}_0'(N) \subset \mathcal{H}_0(N)$. Whenever M is an $\overline{\mathbb{F}}_p[S_0'(N)]$-module and for any e, $\mathcal{H}_0'(N)$ acts on the group cohomology group $H^e(\Gamma_0(N), M)$.

For primes $l \nmid N$ and $0 \leq k \leq 3$ define the Hecke operator $T_{l,k} = [\Gamma_0(N)\begin{pmatrix} l & & \\ & \ddots & \\ & & 1\end{pmatrix}\Gamma_0(N)]$ in $\mathcal{H}_0(N)$ (k entries equal l). Note that:

$$\begin{array}{l} \mathcal{H}_0(N) = \mathbb{Z}[T_{l,1}, T_{l,2}, T_{l,3}, T_{l,3}^{-1} : l \nmid N] \\ \quad\cup \\ \mathcal{H}_0'(N) = \mathbb{Z}[T_{l,1}, T_{l,2}, T_{l,3}, T_{l,3}^{-1} : l \nmid Np] \end{array}$$

We say that ρ is *attached to* an $\mathcal{H}_0'(N)$-eigenvector $\alpha \in H^e(\Gamma_0(N), M)$ if for all $l \nmid Np$, ρ is unramified at l and

$$\sum_{i=0}^{3} (-1)^i l^{i(i-1)/2} a(l, i) X^i = \det(1 - \rho(\mathrm{Frob}_l^{-1})X), \tag{6.2}$$

where $a(l, i) \in \overline{\mathbb{F}}_p$ is the eigenvalue of $T_{l,i}$ on α, and $\mathrm{Frob}_l \in G_{\mathbb{Q}}$ is any *geometric* Frobenius element at l.

If F is a $GL_3(\mathbb{F}_p)$-representation and ϵ as above, then let $F(\epsilon) = F \otimes \overline{\mathbb{F}}_p(\epsilon)$ with $S_0'(N)$ acting on the first factor via reduction modulo p ($S_0'(N) \to GL_3(\mathbb{F}_p)$), and on the second factor via reduction modulo N of the $(3, 3)$-entry ($S_0'(N) \to (\mathbb{Z}/N)^\times$).

The given Galois representation ρ has an associated level $N = N(\rho)$, prime to p, defined in terms of the ramification of ρ away from p, and nebentype $\epsilon = \epsilon(\rho)$, defined in terms of the determinant of ρ, much as in Serre's original conjecture (see [ADP02]).

Definition 6.2. *Let $W(\rho)$ be the set of all $GL_3(\mathbb{F}_p)$-representations F such that ρ is attached to a non-zero element of $H^e(\Gamma_0(N), F(\epsilon))$ (for some e) with $N = N(\rho)$ and $\epsilon = \epsilon(\rho)$.*

Remark 6.3. *As discussed in* [ADP02]*, rk. 3.2, e can be replaced with 3 in this definition.*

Definition 6.4.

If $\rho|_{I_p} \sim \begin{pmatrix} \omega^i & & \\ & \omega^j & \\ & & \omega^k \end{pmatrix}$ *then*

$$V(\rho|_{I_p}) = \mathrm{Ind}_{B_3}^{G_3}(\chi^i \otimes \chi^j \otimes \chi^k).$$

If $\rho|_{I_p} \sim \begin{pmatrix} \omega_2^m & & \\ & \omega_2^{pm} & \\ & & \omega^i \end{pmatrix}$, $p+1 \nmid m$ *then*

$$V(\rho|_{I_p}) = \mathrm{Ind}_{P_3}^{G_3}(\Theta(\chi_2^m, \chi_2^{pm}) \otimes \chi^i).$$

If $\rho|_{I_p} \sim \begin{pmatrix} \omega_3^m & & \\ & \omega_3^{pm} & \\ & & \omega_3^{p^2 m} \end{pmatrix}$, $p^2+p+1 \nmid m$ *then*

$$V(\rho|_{I_p}) = \Theta(\chi_3^m, \chi_3^{pm}, \chi_3^{p^2 m}).$$

Remark 6.5. *ρ is tamely ramified at p if and only if $\rho|_{I_p}$ is diagonalisable. Because of the Frobenius action on $\rho|_{I_p}$, only the above three cases occur in this case.*

Conjecture 6.6. *Suppose that $\rho : G_{\mathbb{Q}} \to GL_3(\overline{\mathbb{F}}_p)$ (as above) is tamely ramified at p. Then*

$$W(\rho) \cap W_{reg} = \mathcal{R}(JH(\overline{V(\rho|_{I_p})})).$$

7. Comparison with the ADPS conjecture

The framework of the conjecture as stated here differs slightly from that of [ADP02], so here is how to translate between them. Let $\Gamma_0'(N)$ be the matrices with first row congruent to $(*, 0, 0) \mod N$, and $S_0'(N)$ the matrices in $GL_3^+(\mathbb{Z}_{(Np)})$ with first row congruent to $(*, 0, 0) \mod N$. Then $\mathcal{H}_0'(N)$ is the Hecke algebra defined by this Hecke pair, but now it is the subgroup of the free abelian group on all *right* cosets $\Gamma_0'(N)s$ ($s \in S_0'(N)$) that are right invariant under $\Gamma_0'(N)$, with corresponding multiplication. $T_{l,i}'$ is defined as $T_{l,i}$ with $\Gamma_0(N)$ replaced by $\Gamma_0'(N)$. Let $F'(a,b,c)$ be the simple *right* GL_3-module with highest weight (a,b,c) over $\mathbb{F}_p$.

Lemma 7.1. *Under the natural isomorphism*

$$H^e(\Gamma_0(N), F(a,b,c)) \cong H^e(\Gamma_0'(N), F'(-c,-b,-a)),$$

if an eigenclass on the left has the Galois representation ρ attached, then the corresponding eigenclass on the right has $\rho^\vee \otimes \omega^2$ attached.

Proof. Let $\eta = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$. Note that $g \mapsto {}^\eta g^{-1}$ defines isomorphisms $\Gamma_0'(N) \to \Gamma_0(N)$, $S_0'(N) \to S_0(N)$. Let $M = F(a,b,c)$, and let M' be the right $S_0'(N)$-module whose underlying vector space is M and if $s \in S_0'(N)$ then $ms = {}^\eta s^{-1} m$. So we get a natural isomorphism $H^e(\Gamma_0(N), M) \cong H^e(\Gamma_0'(N), M')$ such that the action of $[\Gamma_0(N) s \Gamma_0(N)]$ on the left corresponds to that of $[\Gamma_0'(N) {}^\eta s^{-1} \Gamma_0'(N)]$ under it. So for an eigenclass the $T_{l,i}$ eigenvalues $a(l,i)$ on the left give $T_{l,i}'$ eigenvalues $a(l, 3-i)/a(l,3)$ on the right. This easily implies the statement about the attached Galois representations. Finally, looking at the torus action, we see that $M' \cong F'(-c,-b,-a)$. $\square$

Remark 7.2. *Note that the conjecture for the weights formally looks the same in both formulations.*

Proposition 7.3. *Suppose that ρ is as above. Consider the set $\mathcal{A}$ of all $F(x'-2, y'-1, z')_{reg}$ where (x', y', z') is a restricted weight (i.e., $0 \le x' - y', y' - z' \le p-1$) that has a permutation (x, y, z) satisfying the following conditions.*

If $\rho|_{I_p} \sim \begin{pmatrix} \omega^i & & \\ & \omega^j & \\ & & \omega^k \end{pmatrix}$ *then*

$$x \equiv i, \quad y \equiv j, \quad z \equiv k \pmod{p-1}.$$

If $\rho|_{I_p} \sim \begin{pmatrix} \omega_2^m & & \\ & \omega_2^{pm} & \\ & & \omega^i \end{pmatrix}$, $p+1 \nmid m$ *then*

$$x \equiv i \pmod{p-1},$$
$$y + pz \equiv m \pmod{p^2-1}.$$

If $\rho|_{I_p} \sim \begin{pmatrix} \omega_3^m & & \\ & \omega_3^{pm} & \\ & & \omega_3^{p^2 m} \end{pmatrix}$, $p^2+p+1 \nmid m$ *then*

$$x + py + p^2 z \equiv m \pmod{p^3-1}.$$

Then

$$W_{reg,conj}(\rho) = \mathcal{A} \cup \{F({}^r\lambda) : F(\lambda) \in \mathcal{A},\ \lambda \text{ in the lower alcove}\}.$$

Recall that the reflection ^{r} arose in the context of decomposing Weyl modules into simples (p. 18). For simplicity, we will also use the notation ${}^rF(\lambda) = F({}^r\lambda)$ if λ and ${}^r\lambda$ are both dominant (equivalently, $F(\lambda)$ is a regular Serre weight). The proof of the proposition depends on the following lemma.

Lemma 7.4. (i) *Let $i \ge j \ge k$, $i - k \le p-1$. Suppose that $(x', y', z') \in \mathbb{Z}^3$ is a restricted weight and has a permutation (x, y, z) satisfying*

$$x \equiv i, \quad y \equiv j, \quad z \equiv k \pmod{p-1}.$$

Then, modulo $(p-1, p-1, p-1)\mathbb{Z}$, (x', y', z') is among

$$(i,j,k), (j,k,i-p+1), (k+p-1,i,j),$$
$$(j+p-1,i,k), (i,k,j-p+1), (k+p-1,j,i-p+1).$$

All of these do occur.

(ii) *Let $i \ge j > k$, $i - k \le p-1$. Suppose that $(x', y', z') \in \mathbb{Z}^3$ is a restricted weight and has a permutation (x, y, z) satisfying*

$$x \equiv i \pmod{p-1},$$
$$y + pz \equiv j + pk \pmod{p^2-1}.$$

Then, modulo $(p-1,p-1,p-1)\mathbb{Z}$, (x',y',z') *is among*

$$(i,j,k),(j,k,i-p+1),(k+p,i,j-1),$$
$$(j+p,i,k-1),(i,k+1,j-p),(k+p,j-1,i-p+1),$$
$$(i+p-1,j,k),(j,k,i-2p+2).$$

All of these that are restricted do occur.

(iii) *Let* $i>j\geq k$, $i-k\leq p$. *Suppose that* $(x',y',z')\in\mathbb{Z}^3$ *is a restricted weight and has a permutation* (x,y,z) *satisfying*

$$x+py+p^2z\equiv i+pj+p^2k \pmod{p^3-1}.$$

Then, modulo $(p-1,p-1,p-1)\mathbb{Z}$, (x',y',z') *is among*

$$(i,j,k),(j+1,k,i-p),(k+p,i-1,j),$$
$$(j+p,i,k-1),(i,k+1,j-p),(k+p,j+1,i-p-1).$$

All of these that are restricted do occur.

Proof. (i) Note that lemma is invariant under the coordinate change

$$\theta:(x,y,z;i,j,k)\mapsto(z,x,y;k+p-1,i,j).$$

Case 1: i, j, k are distinct modulo $p-1$: Without loss of generality (using θ), $x>y>z$ or $x<y<z$, and $y=j$. In the first case $(x',y',z')=(i,j,k)$, as it is restricted. Similarly, $(x',y',z')=(k+p-1,j,i-p+1)$ in the second case.

Case 2: Precisely two of i, j, k are congruent modulo $p-1$: Without loss of generality, $j\equiv k$. It follows that $j=k$ and $0<i-j<p-1$. Then it is easily seen that, modulo $(p-1,p-1,p-1)\mathbb{Z}$, the only solutions for (x',y',z') are (i,j,j), $(j,j,i-p+1)$, $(j,j,i-p+1)$, $(j+p-1,i,j)$, $(i,j,j-p+1)$, $(j+p-1,j,i-p+1)$, as claimed.

Case 3: $i\equiv j\equiv k$: Without loss of generality, $i=j=k$. It is easily seen that, modulo $(p-1,p-1,p-1)\mathbb{Z}$, the only solutions for (x',y',z') are (i,i,i), $(i,i,i-p+1)$, $(i+p-1,i,i)$, $(i+p-1,i,i-p+1)$, as claimed.

(ii) Without loss of generality, $y+pz=j+pk$. Note that $|y-z|\leq 2p-2$. Thus $(y,z)=(j,k)+n(p,-1)$ with $-2\leq n\leq 1$.

If $n=-2$: since $j-2p<i-2p+2<i-p+1<k+2$, this can't happen.

If $n=-1$: at most the possibilities $(i+p-1,k+1,j-p)$, $(i,k+1,j-p)$, $(k+1,i-p+1,j-p)$ and $(k+1,j-p,i-2p+2)$, $(k+1,j-p,i-3p+3)$ can arise. But note that the first and last can never be restricted, so will not occur.

If $n=0$, at most (i,j,k), $(i+p-1,j,k)$, $(j,k,i-p+1)$, $(j,k,i-2p+2)$ arise.

If $n=1$, the only possibility is $(j+p,i,k-1)$, since $(j+p)-(k-1)>p-1$ and $j+p>i>k-1$.

(iii) The lemma is invariant under the coordinate change

$$(x,y,z;i,j,k)\mapsto(z,x,y;k+p,i-1,j).$$

So, without loss of generality, either $x \geq y \geq z$ or $x < y < z$.

In the first case, $(x', y', z') = (x, y, z)$. Without loss of generality, $A + pB + p^2C = 0$, with $A = x - i$, $B = y - j$, $C = z - k$. Noting that

$$\begin{aligned} |A - C| &= |(x - z) - (i - k)| \\ &\leq \max(p, 2p - 3) \leq 2p - 2, \\ |B - C| &\leq p - 1, \end{aligned}$$

it follows that

$$|(1 + p + p^2)C| = |(A - C) + p(B - C)| \leq p^2 + p - 2.$$

Thus $C = 0$, and $A + pB = 0$ implies

$$|(1 + p)B| = |A - B| \leq p$$

and hence $B = A = 0$. So, if $x \geq y \geq z$, $(x', y', z') = (i, j, k)$.

In the second case, $(x', y', z') = (z, y, x)$. Letting $A = x - (i - p - 1)$, $B = y - (j + 1)$, $C = z - (k + p)$, note that

$$|C - A| \leq 2p - 2, \qquad |C - B| \leq p - 1,$$

so that again $C = 0$. Also, $|B - A| \leq p$ once more, so that $A = B = 0$ and $(x', y', z') = (k + p, j + 1, i - p - 1)$. $\square$

Proof of Prop. 7.3. First note that, for λ a restricted weight,

$$\mathcal{R}(JH(W(\lambda)))$$

consists of $F = \mathcal{R}(F(\lambda))$ and, if the highest weight of F lies in the lower alcove, also ${}^rF$.

If $\rho|_{I_p} \sim \begin{pmatrix} \omega^i & & \\ & \omega^j & \\ & & \omega^k \end{pmatrix}$, without loss of generality, $i \geq j \geq k$, $i - k \leq p - 1$. Now combine the above observation with the lemma and thm. 5.1.

If $\rho|_{I_p} \sim \begin{pmatrix} \omega_2^m & & \\ & \omega_2^{pm} & \\ & & \omega^i \end{pmatrix}$, we can write $m \equiv j + pk \pmod{p^2 - 1}$ with $i \geq j > k$, $i - k \leq p - 1$ (replacing m with mp if necessary). The argument is only slightly more complicated than the previous one. Note on the one hand that the last two triples in the lemma, part (ii) do not contribute to $\mathcal{A}$ (e. g., for $(i+p-1, j, k)$ to occur we need $i = j$ in which case $F(i+p-3, j-1, k)_{reg} = F(i - 2, j - 1, k)_{reg}$). On the other hand, of the remaining six triples in the lemma, only the fourth through the sixth can fail to be restricted. In each case the failure is caused by two consecutive coordinates differing by p. But this happens, in each instance, if and only if the corresponding Weyl module in thm. 5.1 vanishes, so we are done.

If $\rho|_{I_p} \sim \begin{pmatrix} \omega_3^m & & \\ & \omega_3^{pm} & \\ & & \omega_3^{p^2m} \end{pmatrix}$, assume first that there are $i > j \geq k$, $i - k \leq p$ such that $m \equiv i + pj + p^2k \pmod{p^3 - 1}$ (up to replacing m by mp or mp^2). We know from the previous case that no problem arises if two consecutive coordinates in a triple differ by p. But this time, the fourth through sixth triple of the lemma can fail to be restricted by having their second and third coordinate differ by $p+1$. This happens, respectively, when $i = k+p$, $j = k$, $i = j+1$. Using the cyclic symmetry exploited in the lemma, we may assume without loss of generality that $i = k+p$ and $i \neq j+1$ (because not all three equalities can hold simultaneously).

Observe first that $(i, k+1, j-p)$ from the lemma fails to be restricted iff $j - k \le 1$ iff $W(j+p-2, k+p, i) = 0$ in thm. 5.1 (by (4.3)). The only other triple which can (and does) have this coordinate difference of $p+1$ is $(j+p, i, k-1)$. The corresponding Weyl module in thm. 5.1 equals

$$W(k+p-2, i, j+1) = -W(i-1, k+p-1, j+1),$$

which cancels the second irreducible constituent of $W(j+p-1, i-1, k+1)$. It remains to check that the loss of this constituent does not affect $W_{reg,conj}(\rho)$. Indeed, just note that

$$\begin{aligned}\mathcal{R}(F(i-1, k+p-1, j+1)) &= F(j+p-2, k+p-2, i-p)_{reg} \\ &= \mathcal{R}(F(i+p-2, k+p-1, j+1)).\end{aligned}$$

Finally, the remaining half of ρ of niveau 3 are duals of the above. It thus suffices to show that

$$W_{reg,conj}(\rho^\vee) = \{F^\vee \otimes \det^{-2} : F \in W_{reg,conj}(\rho)\} \tag{7.1}$$

and

$$\mathcal{A}(\rho^\vee) = \{F^\vee \otimes \det^{-2} : F \in \mathcal{A}(\rho)\}, \tag{7.2}$$

and that ^{r} and ${}^\vee$ commute when applied to a regular Serre weight F (here ${}^\vee$ denotes the dual). First note that $F(a,b,c)^\vee \cong F(-c,-b,-a)$ if $a \ge b \ge c$ (consider the highest weight) which also implies the last claim. Equation (7.1) holds because the characteristic zero representations associated to ρ and $\rho^\vee$ are dual to each other (for example by the character table), hence have dual constituents after reduction mod p and $\mathcal{R}(F^\vee) = \mathcal{R}(F)^\vee \otimes \det^{-2}$. On the other hand, in prop. 7.3 each (x', y', z') can be replaced by $(-z', -y', -x')$ when passing from ρ to its dual. This demonstrates (7.2). $\square$

Theorem 7.5. *Suppose that ρ is as above.*

(i) *If ρ is of niveau 1, the regular Serre weights predicted in* [ADP02] *agree exactly with the ones here.*

(ii) *If ρ is of niveau 2, we can write* $\rho|_{I_p} \sim \begin{pmatrix} \omega_2^m & & \\ & \omega_2^{pm} & \\ & & \omega^i \end{pmatrix}$, *with* $m \equiv j+pk \pmod{p^2-1}$ *and* $i \ge j > k$, $i-k \le p-1$. *Then the regular Serre weights predicted in* [ADP02] *are precisely the ones given by proposition 7.3 and lemma 7.4(ii), with* $(j+p, i, k-1)$ *excluded from the list.*

(iii) *If ρ is of niveau 3, then (up to taking a dual) we can write* $\rho|_{I_p} \sim \begin{pmatrix} \omega_3^m & & \\ & \omega_3^{pm} & \\ & & \omega_3^{p^2m} \end{pmatrix}$ *with* $m \equiv i+pj+p^2k \pmod{p^3-1}$ *and* $i > j \ge k$, $i-k \le p$. *Then the regular Serre weights predicted in* [ADP02] *are precisely the ones* *given by proposition 7.3 when the following weights are removed from the list in lemma 7.4(iii): the last three and those among the first three of the form (x', y', z') with $x' - z' = p$ and $x' - 1 > y' > z'$.*

Proof. (i) This is obvious.

(ii) Note that according to [ADP02] we write $m \equiv j + pk$ (note that $0 \leq j-k \leq p-1$) and $mp \equiv (k+p)+p(j-1)$ (note that $0 \leq (k+p)-(j-1) \leq p-1$ unless $j = k+1$, in which case mp cannot be so expressed). So the regular weights predicted there are $F(i-2, j-1, k)_{reg}$, $F(j-2, i-1, k)_{reg}$, $F(j-2, k-1, i)_{reg}$ and, if $j \neq k+1$, $F(i-2, k+p-1, j-1)_{reg}$, $F(k+p-2, i-1, j-1)_{reg}$, $F(k+p-2, j-2, i)_{reg}$ together with the reflections ${}^rF$ for any F in this list that is in the lower alcove. Since $F(k+p-2, i-1, j-1)$ lies in the lower alcove, and applying ^{r} gives $F(j-2, i-1, k)_{reg}$, the latter weight can be omitted whenever $j \neq k+1$. But if $j = k+1$, these two weights coincide. Also, $j \neq k+1$ is precisely the condition for $(i, k+1, j-p)$ and $(k+p, j-1, i-p+1)$ to be restricted. Now put all this together and compare with lemma 7.4(ii).

(iii) Using the cyclic symmetry employed in the proof of lemma 7.4(iii), this is straightforward. □

Remark 7.6. *Doud independently extended the conjecture of* [ADP02] *to include the remaining weights in niveau 3 predicted here (see* [Dou]*).*

8. Computational evidence for the conjecture

In [ADP02], Ash, Doud and Pollack verify the ADPS conjecture for some irreducible, odd ρ that are tame at p and have small Artin conductor $N(\rho)$. In particular, they considered seven examples of such ρ of niveau 2, in which case conjecture 6.6 predicts one further weight F than the ADPS conjecture. There is one further such example in [Dou02], §3. Upon my request, Doud and Pollack verified with their computer programs that there is indeed an eigenclass in $H^3(\Gamma_0(N), F(\epsilon))$ which appears to have attached Galois representation ρ in the sense that (6.2) is satisfied for all $l \leq 47$ (except in the one case of level $N = 144$, which was too large for their programs).

To summarise, here is a table of the further weight confirmed in each case:

p	level(s) N	$\rho\|_{I_p}$	weight
5	73, 83, 89, 151, 157	$\begin{pmatrix} \omega_2^8 & & \\ & \omega_2^{16} & \\ & & 1 \end{pmatrix}$	$F(6,3,0)$
7	67	$\begin{pmatrix} \omega_2^{12} & & \\ & \omega_2^{36} & \\ & & \omega^3 \end{pmatrix}$	$F(13,8,3)$
11	17	$\begin{pmatrix} \omega_2^{40} & & \\ & \omega_2^{80} & \\ & & 1 \end{pmatrix}$	$F(16,9,2)$

The image of ρ in these cases is either S_4 ($N = 17$, 67, 73), A_5 ($N = 89$, 151, 157) or a suitable semi-direct product $(\mathbb{Z}/3 \times \mathbb{Z}/3) \rtimes S_3$ when $N = 83$ ([ADP02], [Dou02]).

9. Representations of $GL_n(\mathbb{F}_p)$ in characteristic p

We keep the notations from section 4. Consider $G = GL_{n/\mathbb{F}_p}$ or $G = SL_{n/\mathbb{F}_p}$. Now define

$$\rho = (n-1, n-2, \dots, 1, 0) \in \frac{1}{2} \sum_{\alpha \in R^+} \alpha + (X(T) \otimes \mathbb{Q})^W.$$

As before, let T be the diagonal matrices and B the upper-triangular matrices. Denote by $\epsilon_i \in X(T)$ the character

$$\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \mapsto t_i,$$

and we identify $X(T)$ with $\mathbb{Z}^n$, also writing $(a_1, a_2, \dots, a_n)$ for $\sum a_i \epsilon_i$. Then $R = \{\epsilon_i - \epsilon_j : i \neq j\}$ and the simple roots are given by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \le i \le n-1$. The coroots $(\epsilon_i - \epsilon_j)^\vee$ for $i \neq j$ then sends t to a diagonal matrix whose only entries are 1's except for a t in the (i,i)-entry and a t^{-1} in the (j,j)-entry. We will identify W with S_n so that $w(\epsilon_i) = \epsilon_{w(i)}$.

Consider the following collection of hyperplanes in $X(T) \otimes \mathbb{R}$ for $n \in \mathbb{Z}$, $\alpha \in R^+$:

$$\{\lambda : \langle \lambda + \rho, \alpha^\vee \rangle = np\}.$$

Definition 9.1. *An* alcove *is a connected component of the complement of these hyperplanes.*

Thus each alcove is determined by a collection of inequalities

$$n_\alpha p < \langle \lambda + \rho, \alpha^\vee \rangle < (n_\alpha + 1)p \quad \forall \alpha \in R^+$$

for some $n_\alpha \in \mathbb{Z}$. In particular there is the *lowest alcove*

$$C_0 : 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \quad \forall \alpha \in R^+.$$

In fact, this is an open n-simplex (times $\mathbb{R}$ if $G = GL_{n/\mathbb{F}_p}$) determined by the $n+1$ inequalities

$$0 < \langle \lambda + \rho, \alpha_i^\vee \rangle \quad \forall i; \qquad \langle \lambda + \rho, (\epsilon_1 - \epsilon_n)^\vee \rangle < p.$$

The *affine Weyl group* $W_p = \mathbb{Z}R \rtimes W$ is defined with respect to the natural action of W on $\mathbb{Z}R$. It acts faithfully on $X(T) \otimes \mathbb{R}$ as follows, using the notation introduced in (4.1):

$$(\nu, w) : \lambda \mapsto w \cdot \lambda + p\nu.$$

Often we will identify W_p with its image in the group of affine linear automorphisms of $X(T) \otimes \mathbb{R}$.

It is easy to check that W_p preserves the set of hyperplanes above, and hence maps alcoves to alcoves. (This is also true for the translations $pX(T)$, a fact that will be used in the sequel.) Moreover, it is an important (basic) fact that an alcove is a fundamental domain for the W_p-action.

Definition 9.2. *An alcove C is* restricted *if it is contained in*

$$A_{res} = \{\lambda : 0 < \langle \lambda + \rho, \alpha_i^\vee \rangle < p \quad \forall i\}. \tag{9.1}$$

An alcove C is dominant *if it is contained in*

$$\{\lambda : 0 < \langle \lambda + \rho, \alpha_i^\vee \rangle \quad \forall i\}.$$

Note that A_{res} compares with the restricted region (def. 4.2) as follows:

$$X(T) \cap A_{res} \subset X_1(T) \subset X(T) \cap \bar{A}_{res}.$$

Also, it is clear from the definition that $\bar{A}_{res}$ is a union of closures of alcoves.

* * *

Let us now describe the relation between the representation theory for GL_n and SL_n. For the remainder of this section, $G = GL_{n/\mathbb{F}_p}$ and its derived group $G' = SL_{n/\mathbb{F}_p}$. Let $T_1 = T \cap G'$ be the subgroup of diagonal matrices in G'. Then the restriction map $X(T) \to X(T_1)$ is surjective and has kernel $X^0(T)$. Every irreducible G-module yields, by restriction to G', an irreducible G'-module (see [Jan03], II.2.10(2)). Thus by thm. 4.6 every irreducible G'-module arises by restriction from an irreducible G-module. Also, two irreducible G-modules $F(\mu_1)$, $F(\mu_2)$ restrict to isomorphic G'-modules if and only if $\mu_1 - \mu_2 \in X^0(T)$.

We can identify root systems, Weyl groups and affine Weyl groups of G, G'. Then $X(T) \to X(T_1)$ is equivariant for the actions of W, W_p. Also, $X(T) \otimes \mathbb{R} \to X(T_1) \otimes \mathbb{R}$ projects alcoves of G to alcoves G' and induces a bijection between them.

* * *

For $G = GL_{4/\mathbb{F}_p}$ (equivalently, for $G' = SL_{4/\mathbb{F}_p}$), we will need an explicit list of the following dominant alcoves, consisting of all $(a, b, c, d) - \rho \in X(T) \otimes \mathbb{R}$ satisfying respectively:

$$\begin{aligned}
C_0 :&\quad 0 < a - b,\ b - c,\ c - d;\ a - d < p,\\
C_1 :&\quad 0 < b - c;\ p < a - d;\ a - c,\ b - d < p,\\
C_2 :&\quad 0 < c - d;\ p < a - c;\ a - b,\ b - d < p,\\
C_3 :&\quad 0 < a - b;\ p < b - d;\ c - d,\ a - c < p,\\
C_4 :&\quad p < a - c,\ b - d;\ b - c < p;\ a - d < 2p,\\
C_5 :&\quad 2p < a - d;\ a - b,\ b - c,\ c - d < p,\\
C_{0'} :&\quad 0 < b - c,\ c - d;\ p < a - b;\ a - d < 2p,\\
C_{0''} :&\quad 0 < a - b,\ b - c;\ p < c - d;\ a - d < 2p.
\end{aligned}$$

C_i will also be called *alcove i*. As C_0 is the lowest alcove, the notation is compatible with the above.

The first six alcoves in this list are all the restricted ones. In general there is a partial ordering of weights and of alcoves, denoted by $\uparrow$ (see [Jan03],

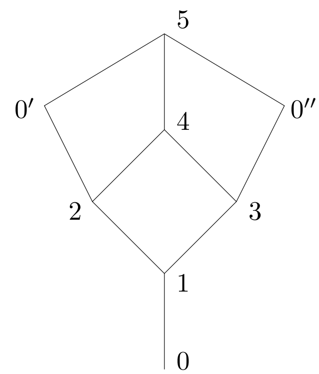


FIGURE 9.1. The $\uparrow$-relation for certain alcoves of GL_4.

§II.6.4, §II.6.5). In particular,

$$\lambda \uparrow \mu \Rightarrow \lambda \leq \mu \text{ and } \lambda \in W_p \cdot \mu$$

and if $\lambda \in C$, $\mu \in C'$ (two alcoves), then

$$\lambda \uparrow \mu \iff C \uparrow C'.$$

We will say that alcove C lies *below* alcove C' and C' *above* C if $C \uparrow C'$. The $\uparrow$-ordering on the above eight alcoves of GL_4 is visualised in figure 9.1: the partial order is generated by $i \uparrow j$ whenever alcoves i and j are connected by a line and j lies above i. In fact, alcoves $0'$, $0''$ are the only non-restricted, dominant alcoves lying below a restricted alcove.

Proposition 9.3 ([Jan74])**.** *Suppose that λ_0 lies in alcove 0 and for each alcove i, denote by λ_i the unique W_p-translate of λ_0 in alcove i. Then in the Grothendieck group of G-modules,*

$$\begin{aligned}
W(\lambda_0) &= F(\lambda_0), \\
W(\lambda_1) &= F(\lambda_1) + F(\lambda_0), \\
W(\lambda_2) &= F(\lambda_2) + F(\lambda_1), \\
W(\lambda_3) &= F(\lambda_3) + F(\lambda_1), \\
W(\lambda_4) &= F(\lambda_4) + F(\lambda_3) + F(\lambda_2) + F(\lambda_1) + F(\lambda_0), \\
W(\lambda_5) &= F(\lambda_5) + F(\lambda_{0'}) + F(\lambda_{0''}) + F(\lambda_4) \\
&\qquad + F(\lambda_3) + F(\lambda_2) + F(\lambda_1).
\end{aligned}$$

Remark 9.4. *Note that it is a general phenomenon, called the "linkage principle" that all constituents $F(\nu)$ of $W(\mu)$ satisfy $\nu \uparrow \mu$. That the multiplicities are independent of p for (p not too small) is by now understood to some extent (there is a general conjecture due to Lusztig on these, which was proved for $p \gg 0$ using quantum groups; see* [Jan03]*, §8.22).*

If $p > 3$ it follows from results of Jantzen how all $W(\mu)$ with μ lying on the boundary of a restricted alcove decompose (see [Jan03] prop. II.7.11 and lemma II.7.15). For $p \leq 3$, when alcoves are devoid of weights, the decomposition formulas were shown to be compatible with the ones for general p in [Jan74].

It will also be useful to write down explicitly certain elements of the affine Weyl group. Let us denote by r_{ij} the unique element of W_p mapping alcove i to alcove j. It is easy to check that:

$$\begin{aligned} r_{01} &: \lambda \mapsto (1\,4)\cdot\lambda + p(1,0,0,-1),\\ r_{13} = r_{24} = r_{0'5} &: \lambda \mapsto (2\,4)\cdot\lambda + p(0,1,0,-1),\\ r_{12} = r_{34} = r_{0''5} &: \lambda \mapsto (1\,3)\cdot\lambda + p(1,0,-1,0),\\ r_{45} &: \lambda \mapsto (1\,4)\cdot\lambda + p(2,0,0,-2). \end{aligned} \tag{9.2}$$

Moreover all of these are reflections.

10. Decomposition of $GL_n(\mathbb{F}_p)$-representations

We keep the notation from last section; in particular, $G = GL_{n/\mathbb{F}_p}$ and $G' = SL_{n/\mathbb{F}_p}$. In either G or G', let w_0 denote the longest Weyl group element: it is the unique element sending positive roots to negative roots. Concretely, $w_0(i) = n+1-i$.

Consider for now only G'. As G' is semisimple, there is a fundamental weight $\omega_\alpha \in X(T_1)$ for each simple root α. It is determined by $\langle \omega_\alpha, \beta^\vee \rangle = \delta_{\alpha\beta}$ for all simple roots α, β. (Note that $\omega_{\alpha_i} = \epsilon_1 + \cdots + \epsilon_i$.) Hence, A_{res}, defined in (9.1), is a fundamental domain for the translation action of $pX(T_1)$ on $X(T_1) \otimes \mathbb{R}$. Thus, for any $\sigma \in W$ there is a unique $\rho_\sigma \in X(T_1)$ such that $\sigma \cdot C_0 + p\rho_\sigma$ is a restricted alcove. A simple lemma shows that

$$\rho_\sigma = \sum_{\substack{\alpha \text{ simple} \\ \sigma^{-1}(\alpha) < 0}} \omega_\alpha \tag{10.1}$$

([Jan77], lemma 1). Also let $\varepsilon_\sigma = \sigma^{-1}\rho_\sigma$ and define

$$W_1 = \{\sigma \in W : \sigma \cdot C_0 + p\rho_\sigma = C_0\}.$$

W acts on the set of alcoves modulo translations by elements of $pX(T_1)$ (this set is in natural bijection with the set of restricted alcoves). The stabiliser of C_0 is W_1, and we see that the number of restricted alcoves is $(W : W_1)$. It is not hard to see that W_1 is generated by $(1\,2\,\ldots\,n)$ (with the notations of section 9), so that there are $(n-1)!$ restricted alcoves. (More generally, for a root system of a simply connected group, W_1 is isomorphic to the root lattice modulo the weight lattice; see [Jan77], lemmas 3 and 2.)

Now consider again G. For each simple root α, choose a lift $\omega'_\alpha \in X(T)$ of ω_α, and define ρ'_σ by (10.1) with ω_α replaced by ω'_α. Let $\varepsilon'_\sigma = \sigma^{-1}\rho'_\sigma$ and let (compatibly with notation of section 4),

$$\rho' = \rho'_{w_0} = \sum_{\alpha \text{ simple}} \omega'_\alpha \in \frac{1}{2} \sum_{\alpha \in R^+} \alpha + (X(T) \otimes \mathbb{Q})^W.$$

It is known, due to a theorem of Hulsurkar, that the matrix

$$(\det(\tau) \operatorname{ch} W(-\varepsilon'_{w_0\sigma} + \varepsilon'_\tau - \rho'))_{\sigma,\tau \in W}$$

with entries in $\mathbb{Z}[X(T)]^W$ is upper triangular with respect to some ordering of W (not the Bruhat ordering!). It is easy to see that its diagonal entries are invertible, as the highest weight of the Weyl module is in $X^0(T)$ if $\sigma = \tau$. Denote by $(\gamma'_{\sigma,\tau})_{\sigma,\tau \in W}$ the inverse matrix. It depends on the choice of the ω'_α. By restriction to $X(T_1)$ (or by the analogous definition), we also get $\gamma_{\sigma,\tau} \in \mathbb{Z}[X(T_1)]^W$.

Not very much seems to be known about the $\gamma'_{\sigma,\tau}$; for $n \leq 3$ this matrix is diagonal, but this is no longer true when $n = 4$ (see the proof of lemma 12.11 below).

Let $F : G \to G$ be the (absolute) Frobenius morphism, and also denote by F its base change to $G_{\overline{\mathbb{F}}_p}$. We will now use the language of classical algebraic geometry for varieties over $\overline{\mathbb{F}}_p$, as is common in the representation theory of algebraic groups. For every F-stable maximal torus $T' \subset G_{\overline{\mathbb{F}}_p}$ and every character $\theta : (T')^F \to \overline{\mathbb{Q}}_p^\times$, Deligne and Lusztig define a virtual representation $R^\theta_{T'}$ (see [DL76]). These can also be parametrised by elements of $W \times X(T)$(see [Jan81], §3.1 or [Jan05], §4.1): Suppose that (w, μ) in $W \times X(T)$. Let $n_w \in N(T_{\overline{\mathbb{F}}_p})$ be a representative of $w \in W = N(T_{\overline{\mathbb{F}}_p})/T_{\overline{\mathbb{F}}_p}$. By Lang's theorem it is possible to find a $g_w \in G_{\overline{\mathbb{F}}_p}$ such that $g_w^{-1} \cdot F(g_w) = n_w$. Then the maximal torus $T_w = g_w T_{\overline{\mathbb{F}}_p} g_w^{-1}$ is F-stable. Recalling that $\tilde{}$ denotes the Teichmüller lift, we let

$$\begin{aligned} \theta_\mu : T_w^F &\to \overline{\mathbb{Q}}_p^\times \\ t_w &\mapsto \tilde{\mu}(g_w^{-1} t_w g_w). \end{aligned}$$

With these notations,

$$R_w(\mu) = (-1)^{n - N(w)} R^{\theta_\mu}_{T_w},$$

where $N(w)$ is the number of orbits of w in its action on $\{1, 2, \dots, n\}$ (which also equals the $\mathbb{F}_p$-rank of T_w). This virtual representation has the property that its character at 1 is positive. It is moreover independent of the choice of g_w and n_w.

With the natural action of W on $X(T)$ form the semi-direct product $X(T) \rtimes W$: that is, $(\nu_1, \sigma_1)(\nu_2, \sigma_2) = (\nu_1 + \sigma_1\nu_2, \sigma_1\sigma_2)$. The group $X(T) \rtimes W$ acts on $W \times X(T)$ as follows:

$${}^{(\nu,\sigma)}(w, \mu) = (\sigma w \sigma^{-1}, \sigma\mu + (p - \sigma w \sigma^{-1})\nu).$$

We will also use the notation $(w,\mu)\sim(w',\mu')$ if these elements are conjugate under the above action. Then

$$\langle R_w(\mu), R_{w'}(\mu')\rangle = \#\{(\nu,\sigma)\in X(T)\rtimes W : {}^{(\nu,\sigma)}(w,\mu)=(w',\mu')\}.$$

(see [Jan05], §A.7–9 for the deduction from Deligne-Lusztig's results). In particular,

$$\begin{aligned} R_w(\mu) &\cong R_{w'}(\mu') && \text{if } (w,\mu)\sim(w',\mu'),\\ \langle R_w(\mu), R_{w'}(\mu')\rangle &= 0 && \text{otherwise.}\end{aligned} \tag{10.2}$$

The following two lemmas will be useful.

Lemma 10.1. *Suppose $w=(1\,2\ldots n)$, $\mu=(i_1,\ldots,i_n)$ and $i_1+pi_2+\cdots+p^{n-1}i_n$ is not divisible by $\frac{p^n-1}{p^d-1}$ for all $d|n$, $0<d\neq n$. Then*

$$R_w(\mu)\cong\Theta(\chi_n^{i_1+pi_2+\cdots+p^{n-1}i_n},\ldots).$$

(This lemma thus explicitly identifies all cuspidal representations of $GL_n(\mathbb{F}_p)$ as Deligne-Lusztig representations.)

Proof. To see this, note that T_w is not contained in any proper F-stable parabolic subgroup. Also, no non-trivial element of $(N(T_w)/T_w)^F$ (a cyclic group of order n) fixes θ_μ, because of the condition imposed on μ. Thm. 8.3 of [DL76] then shows that $R_w(\mu)$ is cuspidal. Finally, cor. 7.2 of [DL76] allows the computation of the character value of this representation on $\begin{pmatrix}\alpha & & \\ & \ddots & \\ & & \alpha^{p^{n-1}}\end{pmatrix}$ for all $\alpha\in\mathbb{F}_{p^n}^\times$ such that $\mathbb{F}_p(\alpha)=\mathbb{F}_{p^n}$, and a comparison with the definition of Θ in (2.1) completes the proof of the lemma. □

Lemma 10.2. *Suppose there is a partition $\lambda: n_1\geq\cdots\geq n_r>0$ of n such that*

$$w=(1\ 2\ \cdots\ n_1)(n_1+1\ \cdots\ n_1+n_2)\cdots.$$

Write $\mu=(a_1,\ldots,a_n)$. Then

$$R_w(\mu)\cong \operatorname{Ind}_{P_\lambda}^{GL_n(\mathbb{F}_p)}\left(\bigotimes_{i=1}^r R_{(1\ 2\ \cdots\ n_i)}(\mu_i)\right),$$

where P_λ is the standard "parabolic" subgroup of type λ and μ_i is the weight for GL_{n_i} given by

$$\mu_i=(a_{n_1+\cdots+n_{i-1}+1},\ldots,a_{n_1+\cdots+n_i}).$$

Proof. This follows from prop. 8.2 of [DL76] together with the Künneth formula. □

Theorem 10.3 ([Jan05], Cor. 4.8)**.** *In the Grothendieck group of $GL_n(\mathbb{F}_p)$-modules,*

$$\overline{R_w(\mu)}=\sum_{\sigma,\tau\in W}\gamma'_{\sigma,\tau}W(\sigma(\mu-w\varepsilon'_{w_0\tau})+p\rho'_\sigma-\rho').$$

This theorem holds, in fact, for G as in theorem 4.6.

Remark 10.4. *Jantzen had in fact proved the analogue of this theorem in the case of simply-connected semisimple groups, defined and split over a finite field (in particular for $SL_{n/\mathbb{F}_p}$); this is the main theorem of* [Jan05]. *In fact, there is a unique lifting, modulo $(p-1)X^0(T)$, of each highest weight in $X(T_1)$ occurring in Jantzen's formula for $\overline{R_w(\mu)}$ over $SL_n(\mathbb{F}_p)$ to $X(T)$ which is, for fixed w, a polynomial over $\mathbb{Z}$ in p and the coordinates of μ. (For: the central character of any Weyl module that occurs has to agree with that of $\overline{R_w(\mu)}$; the lifts of $\nu_1 \in X(T_1)$ to $\nu \in X(T)$ with a fixed central character differ by an element in $\frac{p-1}{n}X^0(T)$. In particular, the choice of lift is even forced in case* $\gcd(p-1,n) = 1$.) *The content of the above theorem is that when all highest weights in Jantzen's formula for SL_n are lifted to GL_n in this natural way, the correct decomposition formula is obtained.*

Let us analyse the statement of Jantzen's formula a little. Notice first that a typical highest weight appearing, $\sigma(\mu - w\varepsilon'_{w_0\tau}) + p\rho'_\sigma - \rho'$, is a small perturbation of $\sigma\cdot\mu + p\rho'_\sigma$. If μ lies in alcove C, the latter weight is contained in alcove $\sigma\cdot C + p\rho'_\sigma$. This alcove is automatically restricted if $C = C_0$, which can always be achieved, up to a small error, by varying (w,μ) in its orbit (see lemma 12.1(ii)).

To use Jantzen's formula to find the complete decomposition of $R_w(\mu)$ into irreducible $GL_n(\mathbb{F}_p)$-modules, we use Brauer's formula (prop. 4.4) to express each $\gamma'_{\sigma,\tau}W(\lambda)$ as a linear combination of Weyl modules, thus

$$\overline{R_w(\mu)} = \sum_\nu a_\nu W(\nu), \text{ some } a_\nu \in \mathbb{Z}.$$

There is a small neighbourhood of the restricted region which contains all ν occurring in this expression. Any non-dominant $W(\nu)$ can be converted into a dominant one using (4.3). Next, we decompose each $W(\nu)$ as GL_n-module. This is a difficult problem which has not been solved in general (see however [Jan03], II.8.22).

For this, let us assume for simplicity that $n = 3$ or 4 and that ν lies in one of the restricted alcoves (this will be true if μ is sufficiently deep inside C_0). Then we can use prop. 4.9 or prop. 9.3.

Finally it can happen (for $n \geq 4$) that a constituent of a restricted $W(\nu)$ lies outside the restricted region, e.g., for $n = 4$ a Weyl module in alcove 5 has constituents in alcoves $0'$, $0''$ (see prop. 9.3). To decompose these over $GL_n(\mathbb{F}_p)$, use the Steinberg tensor product theorem, thm. 4.5, noting that the Frobenius endomorphism is trivial on $GL_n(\mathbb{F}_p)$. For example, if ν and $\nu + \epsilon_i$ ($1 \leq i \leq 4$) all lie in alcove $0'$, then $\nu' = \nu - p\epsilon_1$ and $\nu' + \epsilon_i$ lie in alcove 0 and

$$\begin{aligned} F(\nu) &= F(\nu' + p\epsilon_1) = F(\nu') \otimes F(\epsilon_1) \\ &= W(\nu') \otimes \sum_{i=1}^4 \epsilon_i \qquad (10.3) \\ &= F(\nu'+\epsilon_1) + F(\nu'+\epsilon_2) + F(\nu'+\epsilon_3) + F(\nu'+\epsilon_4). \end{aligned}$$

We used Brauer's formula (prop. 4.4), the fact that $W(\lambda) = F(\lambda)$ if λ is in alcove 0, and that $\operatorname{ch} W(\epsilon_1) = \sum \epsilon_i$.

11. The conjecture for GL_n

For $(w, \mu) \in W \times X(T)$, $\mu = \sum a_i \epsilon_i$, define $\rho(w, \mu)$ to be the following tame n-dimensional Galois representation of I_p which extends to G_p. Let I be a set of representatives for the orbits with respect to the action of (the powers of) w on $\{1, 2, \ldots, n\}$. For each $1 \le k \le n$, let n_k be the size of the orbit of k. Then

$$\rho(w, \mu) = \bigoplus_{k \in I} \bigoplus_{i \bmod n_k} \omega_{n_k}^{\sum_{j \bmod n_k} (a_{w^j(k)} p^{i+j})}. \tag{11.1}$$

It is easy to check that $\rho(w, \mu)$ only depends on the orbit of (w, μ).

Definition 11.1. *$(w, \mu) \in W \times X(T)$ is said to be* good *if for all $1 \le k \le n$,*

$$\sum_{j \bmod n_k} a_{w^j(k)} p^j \not\equiv 0 \pmod{\frac{p^{n_k} - 1}{p^d - 1}}$$

for all $d | n_k$, $d \neq n_k$.

Intuitively this means that we cannot express $\rho(w, \mu)$ in terms of tame fundamental characters of level smaller than n_k in (11.1). It is easy to check that this property only depends on the orbit of (w, μ).

For example, if $n = 4$, $w = (1\ 2)(3\ 4)$, $\mu = (a, b, c, d)$ then

$$\rho(w, \mu) = \begin{pmatrix} \omega_2^{a+pb} & & & \\ & ** & & \\ & & \omega_2^{c+pd} & \\ & & & ** \end{pmatrix},$$

where "$**$" stands for the p-th power of the previous diagonal matrix entry. Also, (w, μ) is good if and only if $p + 1$ does not divide either of $a - b$, $c - d$.

Definition 11.2. *A representation V of $GL_n(\mathbb{F}_p)$ is said to be a* parabolic induction *if it is of the form*

$$V \cong \operatorname{Ind}_P^{GL_n(\mathbb{F}_p)} (\sigma_1 \otimes \ldots \otimes \sigma_r) \tag{11.2}$$

where P is a standard "parabolic" subgroup of type $(d_1, \ldots, d_r)$ and σ_i is a cuspidal representation of $GL_{d_i}(\mathbb{F}_p)$.

That is, $\sum d_i = n$, P consists of matrices with $d_i \times d_i$ square blocks along the diagonal (in that order) with arbitrary entries above the blocks and zeroes below, and σ_i is considered as representation of P via projection to the i-th square block.

We have the following bijections:

$$\begin{tikzcd}[column sep=small] & \left\{\begin{matrix}\text{good } X(T)\rtimes W\text{-orbits} \\ \text{on } W\times X(T)\end{matrix}\right\} \arrow[dl, "\rho(-,-)"'] \arrow[dr, "R_-(-)"] & \\ \left\{\begin{matrix}\text{tame } \tau : I_p \to GL_n(\overline{\mathbb{F}}_p) \\ \text{that extend to } G_p\end{matrix}\right\}/\cong & & \left\{\begin{matrix}\text{parabolic inductions} \\ \text{of } GL_n(\mathbb{F}_p) \text{ over } \overline{\mathbb{Q}}_p\end{matrix}\right\}/\cong\end{tikzcd}$$

The map on the right is injective by (10.2) and surjective by (2.1) and lemmas 10.1, 10.2. As for the other map, note first that a parabolic induction V arises from a unique set of $\{\sigma_i\}$ as in (11.2), and the order of the σ_i is irrelevant (this follows e.g. from [Bum97], ex. 4.1.19). Thus domain and codomain of the map have the same cardinality, as both are in bijection with sets of primitive characters $\{F_i^\times \to \overline{\mathbb{Q}}_p^\times\}$, F_i a finite field of characteristic p and $\sum_i [F_i : \mathbb{F}_p] = n$ (this uses (2.1)). That the map is a bijection now follows from (2.1) and lemmas 10.1, 10.2.

Suppose $\rho : G_{\mathbb{Q}} \to GL_n(\overline{\mathbb{F}}_p)$ is continuous, irreducible, and tamely ramified at p. We also require it to be *odd*: that is, either $p = 2$ or $|n_+ - n_-| \leq 1$ where n_+ (resp. n_-) is the number of eigenvalues 1 (resp. -1) of a complex conjugation in $G_{\mathbb{Q}}$.

Definition 11.3. *As $\rho|_{I_p}$ is tame, we can write $\rho|_{I_p} \cong \rho(w, \mu)$ for a good (w, μ) (unique up to $X(T) \rtimes W$-action). Then let*

$$V(\rho|_{I_p}) = R_w(\mu)$$

be the parabolic induction defined above.

We define $\mathcal{R}$ both as operator on weights and as an operation on Serre weights.

Definition 11.4. *If $\mu \in X(T)$ define*

$$\mathcal{R}(\mu) = w_0 \cdot \mu - p w_0 \rho.$$

In particular, $\mathcal{R}$ preserves alcoves and sends restricted alcoves to restricted alcoves. Explicitly,

$$\mathcal{R}(a_1, \dots, a_n) = (a_n + (n-1)(p-2), \dots, a_2 + (p-2), a_1).$$

With the obvious generalisation of F_{reg} from (6.1) define for $\mu \in X_1(T)$,

$$\mathcal{R}(F(\mu)) = F(\mathcal{R}(\mu))_{reg}.$$

The notions of ρ being attached to an eigenclass in $H^e(\Gamma_0(N), M)$ for a $GL_n(\mathbb{F}_p)$-module M and of the predicted weight set $W(\rho)$ will be the obvious generalisations from section 6.

Conjecture 11.5. *Suppose $\rho : G_{\mathbb{Q}} \to GL_n(\overline{\mathbb{F}}_p)$ is continuous, odd, irreducible, and tamely ramified at p. Then*

$$W(\rho) \cap W_{reg} = \mathcal{R}(JH(\overline{V(\rho|_{I_p})})).$$

In case $F \in \mathcal{R}(JH(\overline{V(\rho|_{I_p})}))$, it will be convenient to say both that $\rho|_{I_p}$ predicts F and that F predicts $\rho|_{I_p}$.

12. Analysing the conjecture for GL_4

Lemma 12.1.

(i) *Every orbit on $W \times X(T)$ contains an element (w, μ) with μ restricted.*

(ii) *Every orbit on $W \times X(T)$ contains an element (w, μ) with μ close to the lowest alcove. That is, there is a $\delta > 0$, independent of p, such that μ can be chosen with*

$$-\delta < \langle \mu + \rho, \alpha^\vee \rangle < p + \delta \quad \forall \alpha \in R^+.$$

Proof. Suppose $(w, \mu) \in W \times X(T)$ and $\mu = \sum a_i \epsilon_i$. Its *span* is $\mathrm{span}(\mu) = \max_{i,j} |a_i - a_j|$. Suppose that $\mathrm{span}(\mu) \geq p$. Let $S = \{i : a_i \geq \min_j a_j + p\}$, $\nu = -\sum_{i \in S} \epsilon_i$. Then ${}^{(\nu,1)}(w, \mu) = (w, \mu + (p - w)\nu)$. By definition of S, $\mathrm{span}(\mu + p\nu) = \mathrm{span}(\mu) - p$, hence $\mathrm{span}(\mu + (p - w)\nu) \leq \mathrm{span}(\mu) - p + 1$. Thus, using the $(X(T) \rtimes W)$-action, we can always achieve $\mathrm{span}(\mu) < p$. Finally, using the action of $(0, w) \in X(T) \rtimes W$, we can also assume that $a_1 \geq \cdots \geq a_n$. Moreover, $a_1 - a_n \leq p - 1$. Thus on the one hand μ is restricted, which proves (i). On the other hand, (ii) follows from

$$1 \leq \langle \mu + \rho, \alpha^\vee \rangle \leq p + n - 2 \quad \forall \alpha \in R^+. \qquad \square$$

Definition 12.2. $\mu \in X(T)$ *(or $(w, \mu) \in W \times X(T)$) is said to lie* δ-deep *in a restricted alcove C for some $\delta > 0$ if*

$$n_\alpha p + \delta < \langle \mu + \rho, \alpha^\vee \rangle < (n_\alpha + 1)p - \delta \quad \forall \alpha \in R^+ \tag{12.1}$$

where C is the alcove determined by putting $\delta = 0$ in these inequalities.

A statement in which p varies is said to be true for all μ *sufficiently deep* in some alcove C if there is a $\delta > 0$, independent of p, such that the statement is true for all δ-deep $\mu \in C$.

Lemma 12.3. *Suppose that C, C' are restricted alcoves and $\sigma \cdot C + p\nu = C'$ ($(\nu, \sigma) \in X(T) \rtimes W$). Then if (w, μ) is sufficiently deep inside C, ${}^{(\nu,\sigma)}(w, \mu)$ lies as deep inside C' as we like.*

Proof. Recall that ${}^{(\nu,\sigma)}(w, \mu) = (w', \sigma\mu + (p - w')\nu)$ with $w' = \sigma w \sigma^{-1}$. By making (w, μ) lie sufficiently deep inside C, we can certainly achieve that $\sigma \cdot \mu + p\nu = \sigma(\mu + \rho) - \rho + p\nu$ lies as deep inside C' as we like. It thus suffices to consider the difference of this weight and $\sigma\mu + (p - w')\nu$. But clearly, $|\langle w'\nu + \sigma\rho - \rho, \alpha^\vee \rangle|$ is bounded for all $\alpha \in R^+$, independent of p. $\square$

Lemma 12.4. *Suppose (w,μ) lies sufficiently deep in a restricted alcove. If ${}^{(\nu,1)}(w,\mu) = (w',\mu')$ is restricted, then $\nu \in X^0(T)$. In particular, $F(\mu) \cong F(\mu')$ as representations of $GL_n(\mathbb{F}_p)$.*

Proof. Take any $\delta > n-1$ and assume that

$$\delta < \langle \mu + \rho, \alpha_i^\vee \rangle < p - \delta \quad \forall i$$

(i.e., we don't even assume that μ necessarily lies in an alcove, just that it lies sufficiently deep inside the restricted region). Suppose now that (w',μ') is restricted:

$$0 < \langle \mu + (p-w)\nu + \rho, \alpha_i^\vee \rangle \le p \quad \forall i.$$

Then $|\langle (p-w)\nu, \alpha_i^\vee \rangle| < p - \delta$ for all i. Writing $\nu = \sum b_i \epsilon_i$, this yields

$$|p(b_i - b_{i+1}) - (b_{w^{-1}(i)} - b_{w^{-1}(i+1)})| < p - \delta \quad \forall i.$$

Let $m = \max_i |b_i - b_{i+1}|$ and assume the maximum is achieved when $i = j$. Then we find

$$(p-n+1)m \le |\langle (p-w)\nu, \alpha_j^\vee \rangle| < p - \delta;$$

so by our choice of δ, $m = 0$ and $\nu \in X^0(T)$.

The last part follows by thm. 4.6 as W acts trivially on $X^0(T)$. □

For $\mu = \sum a_i \epsilon_i$, define its *norm* to be $\|\mu\| = \sum |a_i|$. Also let $\|(w,\mu)\| = \|\mu\|$.

Lemma 12.5. *Given $\delta > 0$ and suppose that (w,μ) lies sufficiently deep in a restricted alcove. Then ${}^{(\nu,\sigma)}(w,\mu) = (w',\mu')$ with $\|\mu - \mu'\| < \delta$ implies $(\nu,\sigma) = (0,1)$. In particular, $\# \operatorname{Stab}_{X(T) \rtimes W}(w,\mu) = 1$.*

Proof. Without loss of generality, we may assume that $\mu \in C_0$. For suppose C is the alcove containing μ and that $\sigma_0 \cdot C + p\nu_0 = C_0$. Then ${}^{(\nu_0,\sigma_0)}(w,\mu)$ can be assumed to lie in the lowest alcove (by lemma 12.3). Also,

$$\begin{aligned} \|{}^{(\nu_0,\sigma_0)}(w,\mu) - {}^{(\nu_0,\sigma_0)}(w',\mu')\| &= \|\sigma_0(\mu - \mu') - \sigma_0(w - w')\sigma_0^{-1}\nu_0\| \\ &< \delta + \max_{w,w'} \|\sigma_0(w-w')\sigma_0^{-1}\nu_0\| \end{aligned}$$

is bounded independent of p. So if we replace $((w,\mu),(w',\mu'))$ by $({}^{(\nu_0,\sigma_0)}(w,\mu), {}^{(\nu_0,\sigma_0)}(w',\mu'))$ we are reduced to the lowest alcove. Note that the new (ν,σ) is a conjugate of the original one.

Now suppose $\mu \in C_0$. Letting $w' = \sigma w \sigma^{-1}$, we have

$$(p - w')\nu = \mu' - \sigma\mu. \tag{12.2}$$

Write $\mu' - \sigma\mu = \sum c_i \epsilon_i$. Given $\delta' > 1$. If μ lies sufficiently deep in the lowest alcove,

$$|c_i| \le \delta + |(\mu - \sigma\mu)_i| < p - \delta'$$

for all i. Suppose that $\nu = \sum b_i \epsilon_i$. Let $m = \max_i |b_i|$ and suppose $m = |b_j|$. Then, looking at the j-th coordinate in (12.2) shows that

$$(p-1)m \le |pb_j - b_{(w')^{-1}(j)}| < p - \delta'.$$

Hence $m = 0$, $\nu = 0$, $\mu' = \sigma\mu$. This implies $\sigma = 1$ if μ lies sufficiently deep in the lowest alcove (even if μ lies just one layer inside the lowest alcove, it will be inside the (open) dominant Weyl chamber and thus not fixed by any element of the Weyl group). $\square$

Definition 12.6. *$(w, \mu) \in W \times X(T)$ is* special *if $\mu \in C$, a restricted alcove and for all $(\nu, \sigma) \in X(T) \rtimes W$,*

$$\begin{aligned} {}^{(\nu,\sigma)}(w, \mu) = (w', \mu') &\text{ with } \mu' \text{ restricted} \\ \Leftrightarrow\ \sigma \cdot C + p\nu &\text{ is a restricted alcove.} \end{aligned}$$

Moreover, in each such case we demand that $(w', \mu') \in \sigma \cdot C + p\nu$.

It is easy to check that this property only depends on the orbit of (w, μ) (in the sense that it doesn't matter for which (w', μ') in the orbit with μ' restricted the above condition is checked; by lemma 12.1 there always is such a (w', μ')). Note that for each $\sigma \in W$ there is a unique $\nu_\sigma \in X(T)$ (modulo $X^0(T)$) such that $\sigma \cdot C + p\nu_\sigma$ is a restricted alcove (see p. 35).

Lemma 12.7.

(i) *If (w, μ) lies sufficiently deep in a restricted alcove then it is good.*
(ii) *If (w, μ) lies sufficiently deep in a restricted alcove then it is special.*
(iii) *If (w, μ) lies sufficiently deep in a restricted alcove then $R_w(\mu)$ is irreducible.*
(iv) *If μ lies sufficiently deep in a restricted alcove then $F(\mu - \rho)$ is not predicted by any non-special or non-good $\rho(w, \mu)$.*

Proof. (i) By lemma 12.3 we may assume without loss of generality that (w, μ) is in the lowest alcove. If $\mu = \sum a_i \epsilon_i$, note that $\sum_{j=1}^r a_{i_j} \epsilon_j$ is in the lowest alcove for GL_r whenever $1 \le i_1 < \cdots < i_r \le n$. We are thus reduced to the case when w is an n-cycle. We need to show that if μ is sufficiently deep in the lowest alcove,

$$\sum_{i=0}^{n-1} a_{w^i(1)} p^i \not\equiv 0 \pmod{\frac{p^n - 1}{p^d - 1}} \tag{12.3}$$

for all $d|n$, $d \ne n$. Fix $n = de$ with $d < n$. Using

$$\frac{p^n - 1}{p^d - 1} = \sum_{j=0}^{e-1} p^{dj},$$

equation (12.3) becomes

$$\sum_{i=0}^{d(e-1)-1} (c_i - c_{d(e-1)+d\{\frac{i}{d}\}}) p^i \not\equiv 0 \pmod{\sum_{j=0}^{e-1} p^{dj}}, \tag{12.4}$$

where $c_i = a_{w^i(1)}$ and $\{x\} \in [0, 1)$ denotes the fractional part of a real number x. As μ is in the lowest alcove, $|c_i - c_j| \le p - 1$ for all i, j. So if μ

lies sufficiently deep in the lowest alcove then $c_i \neq c_j$ for all $i \neq j$ and (12.4) is automatic as $(p-1)(1+p+\cdots+p^i) < p^{i+1}$ for all i.

(ii) Suppose μ lies in alcove C. For each $\sigma \in W$, fix a choice of $\nu_\sigma \in X(T)$ as above. By lemma 12.3 we can certainly achieve that if μ lies sufficiently deep in C, ${}^{(\nu_\sigma,\sigma)}(w,\mu)$ lies as deep as we wish in the restricted alcove $\sigma \cdot C + p\nu_\sigma$ for all σ. Moreover, suppose that ${}^{(\nu,\sigma)}(w,\mu)$ is restricted. Applying lemma 12.4 to

$$ {}^{(\nu_\sigma,\sigma)}(w,\mu) = {}^{(\nu_\sigma-\nu,1)}\big({}^{(\nu,\sigma)}(w,\mu)\big), $$

we see that $\nu_\sigma - \nu \in X^0(T)$. Thus $\sigma \cdot C + p\nu$ is a restricted alcove, as required.

(iii) By (i), we may assume (w,μ) to be good. By lemma 12.5 we may assume that $\#\operatorname{Stab}_{X(T)\rtimes W}(w,\mu) = 1$. Thus the claim follows by (10.2). (Alternatively, by [Bum97], ex. 4.1.19, for example, a parabolic induction is irreducible if and only if it is induced from a collection of distinct cuspidal representations. This easily leads to a more elementary proof.)

(iv) A weight μ is said to lie *δ-close to an alcove boundary* for some $\delta > 0$ if (12.1) does not hold for any alcove (not necessarily restricted). If moreover

$$ n_\alpha p - \delta < \langle \mu+\rho, \alpha^\vee\rangle < (n_\alpha+1)p + \delta \quad \forall \alpha \in R^+, $$

where C is the alcove determined by putting $\delta = 0$, then μ is said to lie *δ-close to the boundary of C*.

It suffices to show that all constituents of $\overline{R_w(\mu)}$ for (w,μ) non-special or non-good can be made to lie δ-close to an alcove boundary for some $\delta > 0$. Note that it makes sense to talk about a simple module F of $GL_n(\mathbb{F}_p)$ lying in an alcove (or being δ-close to an alcove boundary) as $F = F(\nu)$ for a restricted weight ν which is uniquely determined up to $(p-1)X^0(T)$ (see thm. 4.6).

We may assume that μ satisfies the conclusion of lemma 12.1(ii). By increasing δ if necessary, μ lies δ-close to the boundary of the lowest alcove (otherwise it would be good/special by parts (i), (ii)). Now consider any term in Jantzen's formula for $\overline{R_w(\mu)}$:

$$ \gamma'_{\sigma,\tau} W(\sigma(\mu - w\varepsilon'_{w_0\tau}) + p\rho'_\sigma - \rho'). $$

Note that

$$ \sigma(\mu - w\varepsilon'_{w_0\tau}) + p\rho'_\sigma - \rho' = (\sigma\cdot\mu + p\rho'_\sigma) - (\sigma(\rho + w\varepsilon'_{w_0\tau})) $$

where the first term is δ-close to the boundary of the restricted alcove $\sigma \cdot C_0 + p\rho'_\sigma$, whereas the second term is bounded independent of p. Also, $\gamma'_{\sigma,\tau}$ is bounded independent of p. The result is a consequence of the following facts:

- Brauer's formula (prop. 4.4) for decomposing tensor products.
- Equation (4.3) for Weyl modules with non-dominant highest weights.
- All constituents of a Weyl module $W(\mu)$ (as GL_n-representation) are of the form $F(\nu)$ with ν related to μ by the affine Weyl group

and lie in lower alcoves (see prop. 9.3 for $n = 4$ and μ restricted; in general see [Jan03], II.6.13). Thus if μ is δ-close to an alcove boundary, then so is any of the ν.

- Steinberg's tensor product formula, prop. 4.5, for decomposing a non-restricted $F(\nu)$ over $GL_n(\mathbb{F}_p)$.

(Compare with the discussion after thm. 10.3.) □

Definition 12.8.

(i) *Suppose that (w, μ) is good and special. Using the $X(T) \rtimes W$-action, we may assume that μ is in the lowest alcove. Then the* diagonal predictions *for $\rho(w, \mu)$ is the set of all*

$$\mathcal{R}(\gamma'_{\sigma,\sigma} F(\sigma(\mu - w\varepsilon'_{w_0\sigma}) + p\rho'_\sigma - \rho')) \\ \cong \mathcal{R}(F(\sigma(\mu - w\varepsilon'_{w_0\sigma}) + (p-1)\rho' + pw_0\rho'_{w_0\sigma})).$$

(ii) *Suppose that F is a regular Serre weight which is not predicted by any non-special or non-good $\rho(w, \mu)$ (this can be achieved by requiring F to lie sufficiently deep in a restricted alcove; see lemma 12.7(iv)). Then the* diagonal predictions *for F is the set of all good, special $\rho(w, \mu)$ such that F is a diagonal prediction of $\rho(w, \mu)$.*

Remark 12.9. (i) *We need to check in part (i) of the definition that the set of diagonal predictions is independent of the choice of $(X(T) \rtimes W)$-translate lying in the lowest alcove. This is an exercise using rk. 4.3 and* (12.5).

(ii) *The reason for the terminology is as follows. Given $\rho(w, \mu)$ as in part (i) of the lemma, with μ lying in the lowest alcove. Then the Jantzen formula (thm. 10.3) contains the following "diagonal" terms (i.e., $\sigma = \tau$):*

$$\sum_{\sigma \in W} \gamma'_{\sigma,\sigma} W(\sigma(\mu - w\varepsilon'_{w_0\sigma}) + p\rho'_\sigma - \rho').$$

These Weyl modules will decompose further, and the off-diagonal terms might cancel some of the resulting simple modules. But in generic cases, at least all $\gamma'_{\sigma,\sigma} F(\sigma(\mu - w\varepsilon'_{w_0\sigma}) + p\rho'_\sigma - \rho')$ will survive as constituents of $\overline{R_w(\mu)}$; thus applying $\mathcal{R}$ to this collection will give a subset of the predicted weights for $\rho(w, \mu)$.

Proposition 12.10. *Suppose that (w, μ) is good and special (e.g., μ is sufficiently deep in a restricted alcove). Then the set of diagonal predictions for $\rho(w, \mu)$ is*

$$\{F(\mu' - \rho) : (w', \mu') \sim (w, \mu),\ \mu' \text{ restricted}\}.$$

There are $\#W = n!$ diagonal predictions, $\#W_1 = n$ in each restricted alcove.

Proof. As (w, μ) is special, we may assume without loss of generality that it lies in the lowest alcove. For $\sigma \in W$ consider

$$\begin{aligned}
&\mathcal{R}(F(\sigma(\mu - w\varepsilon'_{w_0\sigma}) + (p-1)\rho' + pw_0\rho'_{w_0\sigma})) \\
&\qquad = F(w_0\sigma(\mu - w\varepsilon'_{w_0\sigma}) + (p-1)w_0\rho' + p\rho'_{w_0\sigma} + (p-2)\rho)) \\
&\qquad \cong F(w_0\sigma\mu + (p-w')\rho'_{w_0\sigma} - \rho)
\end{aligned}$$

where $w' = w_0\sigma w(w_0\sigma)^{-1}$, using that $w_0\rho' + \rho \in X^0(T)$. Now note that

$$^{(\rho'_{w_0\sigma}, w_0\sigma)}(w, \mu) = (w', w_0\sigma\mu + (p-w')\rho'_{w_0\sigma}).$$

The first claim of the proposition follows by observing that $\lambda \mapsto \tau \cdot \lambda + p\nu$ sends C_0 to a restricted alcove if and only if $\nu \equiv \rho'_\tau \pmod{X^0(T)}$.

The proof of lemma 12.5 shows that $\#\operatorname{Stab}_{X(T)\rtimes W}(w, \mu) = 1$ for all special (w, μ). This implies the second claim. □

Lemma 12.11. *Suppose that $n = 4$. If μ is sufficiently deep in a restricted alcove, the off-diagonal elements in Jantzen's formula are irrelevant in the following sense: an irreducible representation of $GL_n(\mathbb{F}_p)$ occurs in $\overline{R_w(\mu)}$ if and only if it occurs in one of the diagonal terms of Jantzen's formula for this representation.*

Proof. For the reason discussed in remark 10.4, it will be enough to show this result for $SL_n(\mathbb{F}_p)$-representations. Throughout this proof, ϖ will denote an arbitrary element of W_1.

The following formulae will be useful, which are easily verified from the definition of ρ_σ (or see [Jan77], lemma 2).

$$\begin{gathered}
\rho_{\sigma\varpi} = \sigma\rho_\varpi + \rho_\sigma,\ \varepsilon_{\sigma\varpi} = \varepsilon_\varpi + \varpi^{-1}\varepsilon_\sigma, \\
\gamma_{\sigma\varpi,\tau\varpi} = \gamma_{\sigma,\tau}
\end{gathered} \qquad \forall \sigma \in W. \tag{12.5}$$

Note that $\gamma_{\sigma,\sigma} = 1$ for all $\sigma \in W$. It will be necessary to know when $\gamma_{\sigma,\tau} \neq 0$ for $\sigma \neq \tau$. It can be checked (by hand) that the only such cases are given by

$$\gamma_{\varpi,(1\,2\,4\,3)\varpi} = -1,\ \ \gamma_{(1\,4)\varpi,(1\,2)(3\,4)\varpi} = 1.$$

The corresponding off-diagonal terms in these cases are, respectively,

$$\begin{gathered}
-W(\varpi(\mu - w\varepsilon_{w_0(1\,2\,4\,3)\varpi}) + p\rho_\varpi - \rho), \\
W((1\,4)\varpi(\mu - w\varepsilon_{w_0(1\,2)(3\,4)\varpi}) + p\rho_{(1\,4)\varpi} - \rho).
\end{gathered}$$

The corresponding highest weights are contained in alcove 0, respectively alcove 1. By prop. 9.3, the first Weyl module is irreducible whereas the second has another irreducible constituents in alcove 0. Using $r_{01} \in W_p$ from (9.2) and simplifying with the help of (12.5) these become, respectively,

$$-F(\varpi\mu + p\rho_\varpi - \varpi w\varpi^{-1}(\rho_\varpi + \varepsilon_{w_0(1\,2\,4\,3)}) - \rho), \tag{12.6a}$$

$$F((1\,4)\varpi(\mu - w\varepsilon_{w_0(1\,2)(3\,4)\varpi}) + p\rho_{(1\,4)\varpi} - \rho) \tag{12.6b}$$

$$+F(\varpi\mu + p\rho_\varpi - \varpi w\varpi^{-1}(\rho_\varpi + \varepsilon_{w_0(1\,2)(3\,4)}) - \rho). \tag{12.6c}$$

To prove the proposition, it will in fact suffice to analyse the diagonal terms in Jantzen's formula whose highest weight is in alcove 5. Since $C_5 = (2\ 4) \cdot C_0 + p\rho_{(2\ 4)}$, these terms are $W(\nu)$ with

$$\nu = (2\ 4)\varpi(\mu - w\varepsilon_{w_0(2\ 4)\varpi}) + p\rho_{(2\ 4)\varpi} - \rho.$$

By prop. 9.3 and using $r_{0'5}$ from (9.2), the irreducible constituent of $W(\nu)$ in alcove $0'$ has highest weight

$$(2\ 4) \cdot \nu + p(0, 1, 0, -1).$$

This constituent decomposes further over $SL_4(\mathbb{F}_p)$ by Steinberg's tensor product theorem, yielding four irreducible constituents in alcove 0 ($1 \le i \le 4$):

$$F((2\ 4) \cdot \nu + p(-1, 1, 0, -1) + \epsilon_i)$$

(see the discussion after thm. 10.3). Plugging in ν, using (12.5) and the fact that $\rho_{(2\ 4)} = (1, 1, 0, -1)$, this becomes

$$F(\varpi\mu + p\rho_\varpi - \varpi w \varpi^{-1}(\rho_\varpi + \varepsilon_{(1\ 4\ 3\ 2)}) - \rho + \epsilon_i). \tag{12.7}$$

Using instead the irreducible constituent of $W(\nu)$ in alcove $0''$ we get the following constituents in alcove 0 instead ($1 \le i \le 4$):

$$F(\varpi\mu + p\rho_\varpi - \varpi w \varpi^{-1}(\rho_\varpi + \varepsilon_{(1\ 2\ 3\ 4)}) - \rho - \epsilon_i). \tag{12.8}$$

Finally we need to consider the constituent of $W(\nu)$ in alcove 1. Since we can also write $C_5 = (1\ 2)(3\ 4) \cdot C_0 + p\rho_{(1\ 2)(3\ 4)}$, the diagonal terms in alcove 5 are also given by Weyl modules with highest weights

$$\nu' = (1\ 2)(3\ 4)\varpi(\mu - w\varepsilon_{w_0(1\ 2)(3\ 4)\varpi}) + p\rho_{(1\ 2)(3\ 4)\varpi} - \rho.$$

(Replacing ϖ by $(1\ 2\ 3\ 4)\varpi$ with $(1\ 2\ 3\ 4) \in W_1$, we recover ν.) It can easily be checked that the unique element of W_p sending alcove 5 to alcove 1 is given by

$$\nu' \mapsto (1\ 2\ 4\ 3) \cdot \nu' + p(1, -1, 1, -1).$$

Using that $(1, -1, 1, -1) = \rho_{(1\ 4)} - (1\ 4)\varepsilon_{(1\ 2)(3\ 4)}$ and (12.5), the irreducible constituent of $W(\nu')$ in alcove 1 is the representation in (12.6b).

To finish the proof of the proposition, notice that

$$\begin{aligned} \varepsilon_{(1\ 4\ 3\ 2)} - \epsilon_2 &= \varepsilon_{w_0(1\ 2)(3\ 4)}, & \varepsilon_{(1\ 2\ 3\ 4)} + \epsilon_3 &= \varepsilon_{w_0(1\ 2)(3\ 4)}, \\ \varepsilon_{(1\ 4\ 3\ 2)} - \epsilon_3 &= \varepsilon_{w_0(1\ 2\ 4\ 3)}, & \varepsilon_{(1\ 2\ 3\ 4)} + \epsilon_2 &= \varepsilon_{w_0(1\ 2\ 4\ 3)}. \end{aligned}$$

and compare representations (12.7), (12.8) for the appropriate choice of i with representations (12.6a), (12.6c). $\square$

When $n = 4$ and $\mu = (a, b, c, d)$, define

$$\begin{aligned}
\mu_0 &= (a, b, c, d),\\
\mu_1 &= (a-p, b, c, d+p),\\
\mu_2 &= (a-p, b, c+p, d),\\
\mu_3 &= (a, b-p, c, d+p),\\
\mu_4 &= (a-p, b-p, c+p, d+p),\\
\mu_5 &= (a-2p, b, c, d+2p),\\
\mu_{0'} &= (a-2p, b, c+p, d+p),\\
\mu_{0''} &= (a-p, b-p, c, d+2p).
\end{aligned}$$

Proposition 12.12. *Suppose that (w, μ) lies sufficiently deep in a restricted alcove.*

(i) *The set of diagonal predictions for $F(\mu - \rho)$ is*

$$\{\rho(w, \mu) : w \in W\}.$$

There are $\#W = n!$ diagonal predictions.

(ii) *Suppose that $n = 4$. The set of predictions for $F(\mu - \rho)$ in alcove c is*

$$\{\rho(w, \mu_i) : w \in W,\ i \uparrow c\}.$$

There are $\#W \cdot \#\{i : i \uparrow c\}$ predictions.

Remark 12.13. *Note that the labelling of the μ_i (especially for $i = 0', 0''$) is purely to simplify the statement of the proposition. It is unclear whether even the number of the weights predicted in the generic case will behave in the same way for $n > 4$.*

As an example, let us consider only predictions of niveau 1 for $n = 4$. Then for $\mu = (a, b, c, d)$ sufficiently deep in alcove 2, $F(\mu - \rho)$ predicts the following tame, inertial Galois representations:

$$\begin{pmatrix} \omega^a & & & \\ & \omega^b & & \\ & & \omega^c & \\ & & & \omega^d \end{pmatrix}, \begin{pmatrix} \omega^{a-1} & & & \\ & \omega^b & & \\ & & \omega^c & \\ & & & \omega^{d+1} \end{pmatrix}, \begin{pmatrix} \omega^{a-1} & & & \\ & \omega^b & & \\ & & \omega^{c+1} & \\ & & & \omega^d \end{pmatrix}.$$

Proof. (i) By lemma 12.7(iv), we may assume that $F(\mu - \rho)$ is not predicted by any non-good or non-special $\rho(w, \mu)$. Then we are done immediately by prop. 12.10 and lemma 12.5.

(ii) For any weight ν that lies in a restricted alcove define

$$\begin{aligned}
M_\uparrow(\nu) &= \{\lambda \text{ restricted} : \nu \uparrow \lambda\},\\
M_\downarrow(\nu) &= \{\lambda \text{ restricted} : \lambda \uparrow \nu\}.
\end{aligned}$$

By abuse of notation, also set $M_\uparrow(F(\nu)) = M_\uparrow(\nu)$, $M_\downarrow(F(\nu)) = M_\downarrow(\nu)$.

Let $F = F(\mu - \rho)$. By lemma 12.7(iv) we may assume that F is not predicted by any non-good or non-special $\rho(w, \mu)$. Consider $\nu_d = \mathcal{R}^{-1}(\mu - \rho)$.

If $c \neq 5$ (so that ν_d does not lie in the lowest alcove) we claim that the set of predicted tame inertial representations for F is the set of diagonal predictions for all $F(\nu)$, $\nu \in M_\downarrow(\mu-\rho)$. By lemma 12.11, F will be a predicted weight for some $\rho(w,\mu)$ if and only if F_d is a constituent of a Weyl module appearing as diagonal term in Jantzen's formula for $\overline{R_w(\mu)}$.

Suppose F is predicted for $\rho(w,\mu)$ so, by assumption, $\rho(w,\mu)$ is good and special. As ν_d does not lie in the lowest alcove, the highest weights of the Weyl modules in the restricted region which contain $F(\nu_d)$ as constituent are precisely $M_\uparrow(\nu_d)$ (see prop. 9.3). Take $\nu \in M_\uparrow(\nu_d)$ such that $W(\nu)$ is diagonal term in Jantzen's formula for $\overline{R_w(\mu)}$. Thus $F(\mathcal{R}(\nu)) = \mathcal{R}(F(\nu))$ is a diagonal prediction for $\rho(w,\mu)$, by definition, and note that $\mathcal{R}(\nu) \in M_\downarrow(\mu-\rho)$, as required. The converse follows by reversing the argument.

To find the predictions for F explicitly, we use the reflections in W_p of (9.2) to find $M_\downarrow(\mu-\rho)$ and then apply part (i). Let $\mu=(a,b,c,d)$.

If $c=0$, everything is clear.

If $c=1$, applying the reflection r_{01} to $\mu-\rho$ in alcove 1 we get

$$(d+p,b,c,a-p)-\rho.$$

If $c=2$, applying to $\mu-\rho$ in alcove 2 successively r_{12}, r_{01}, we get

$$(c+p,b,a-p,d)-\rho,\ (d+p,b,a-p,c)-\rho.$$

If $c=3$, analogously we get

$$(a,d+p,c,b-p)-\rho,\ (b,d+p,c,a-p)-\rho.$$

If $c=4$, we get respectively in alcoves 3, 2, 1, 0:

$$(c+p,b,a-p,d)-\rho,\ (a,d+p,c,b-p)-\rho,$$
$$(c+p,d+p,a-p,b-p),\ (b,d+p,a-p,c)-\rho.$$

If $c=5$ (so that ν_d is in the lowest alcove), the argument changes in that $F(\nu_d)$ can occur as constituent only in Weyl modules of alcoves 1, 4 and 5. The way it occurs in the latter is as follows: if ν is in alcove 5, then $W(\nu)$, as representation of GL_4, has no constituent in alcove 0 (see prop. 9.3), but its constituents in alcoves $0'$, $0''$ each decompose into four irreducible constituents in the closure of alcove 0 over $GL_4(\mathbb{F}_p)$ (they might be zero if they land on one of the three non-dominant alcove walls). See the discussion after thm. 10.3.

The highest weights of the Weyl modules in alcove 5 which contain $F(\nu_d)$ as $GL_4(\mathbb{F}_p)$-constituent are the W_p-translates of $\nu_d+p\epsilon_1-\epsilon_i$ in alcove $0'$ and $\nu_d-p\epsilon_4+\epsilon_i$ in alcove $0''$ ($1\le i\le 4$). (We assume here that μ lies sufficiently deep in alcove 5, so that these eight weights do not land on the alcove boundary.) Explicitly,

$$\nu_d=(d+3p,c+2p,b+p,a)-\rho,$$
$$\nu_d+p\epsilon_1-\epsilon_i=(d+4p,c+2p,b+p,a)-\epsilon_i-\rho.$$

Its W_p-translate into alcove 5 is:

$$\nu' = (d + 4p, a + p, b + p, c + p) - \epsilon_j - \rho$$

and

$$\mathcal{R}(\nu') = (c + p, b, a - p, d + p) - \epsilon_k - \rho.$$

In the same way, going via alcove $0''$ gives:

$$\mathcal{R}(\nu'') = (a - p, d + p, c, b - p) + \epsilon_k - \rho.$$

Finally, the W_p-translates of $\mu - \rho$ into alcoves 4 and 1 are

$$(d + 2p, b, c, a - 2p) - \rho, \ (c + p, a - p, d + p, b - p) - \rho.$$

The result follows from (i) by noticing that

$$\begin{aligned} \text{(12.9)} \quad & \rho(w, (c + p, b, a - p, d + p) - \epsilon_k) \\ & \qquad = \rho(w, (c + p, b, a - p, d + p) - p\epsilon_{w^{-1}(k)}) \quad \forall k \end{aligned}$$

and similarly in the other case.

It is an interesting fact that, even though $F(\nu_d)$ does not occur as GL_4-constituent of a Weyl module in alcoves 2, 3, 5, pretending that it does, yields no further ρ. That is, no matter what c, the set of diagonal predictions for all $F(\nu)$, $\nu \in M_{\downarrow}(\mu - \rho)$ is always a subset of the predicted tame inertial representations of $F(\mu - \rho)$.

The claim about the number of predictions follows using lemma 12.5 once we have used the analogue of (12.9) to convert all p's in the μ_i into 1's (depending on w). $\square$

13. Theoretical evidence for the conjecture

In this section we will assume that $n > 1$.

Define $\Gamma_1(N)$ to be the subgroup of those $\gamma \in \Gamma_0(N)$ with $\gamma_{n,n} \equiv 1 \pmod N$, and let $S_1(N)$ be the subgroup of $s \in S_0(N)$ with $s_{n,n} \equiv 1 \pmod N$. Then $(\Gamma_1(N), S_1(N))$ is a Hecke pair, and the corresponding Hecke algebra is denoted by $\mathcal{H}_1(N)$. For $l \nmid N$ let

$$T_{l,i} = [\Gamma_1(N) \begin{pmatrix} l & & \\ & \ddots & \\ & & 1 \end{pmatrix} \widehat{\omega_N(l)} \Gamma_1(N)],$$

with i entries l on the diagonal (the order is irrelevant). Here $\omega_N(l)$ is any element of $\Gamma_0(N)$ such that $l\omega_N(l)_{(n,n)} \equiv 1 \pmod N$, and $\widehat{\omega_N(l)}$ stands for $\omega_N(l)$ if the diagonal matrix has an l as its (n, n)-entry and for 1 otherwise (note that the choice of $\omega_N(l)$ is irrelevant).

Define

$$\begin{aligned} U_1(N) &= \{g \in GL_n(\widehat{\mathbb{Z}}) : \text{last row} \equiv (0, \dots, 0, *) \pmod N\}, \\ \Sigma_1(N) &= \{g \in GL_n(\mathbb{A}^\infty) : g_N \in U_1(N)\}, \end{aligned}$$

where $g_N = \prod_{l|N} g_l$. Then $(U_1(N), \Sigma_1(N))$ is a Hecke pair, and we denote by $\mathcal{H}_1^{\mathbb{A}}(N)$ the Hecke algebra associated to it.

Lemma 13.1. *There is an isomorphism of Hecke algebras*

$$\mathcal{H}_1^{\mathbb{A}}(N) \xrightarrow{\sim} \mathcal{H}_1(N)$$

determined by requiring that

$$[U_1(N)sU_1(N)] \mapsto [\Gamma_1(N)s\Gamma_1(N)]$$

for all $s \in S_1(N)$.

Proof. First we will show that $(\Gamma_1(N), S_1(N)) \subset (U_1(N), \Sigma_1(N))$ are compatible as Hecke pairs (see §2.2). To see that $S_1(N)U_1(N) = \Sigma_1(N)$, note that by strong approximation

$$GL_n(\mathbb{Q})U_1(N) = G(\mathbb{A}^\infty) \supset \Sigma_1(N),$$

so for $\sigma \in \Sigma_1(N)$ write $\sigma = \gamma u$ ($\gamma \in GL_n(\mathbb{Q})$, $u \in U_1(N)$). W.l.o.g., $\det \gamma > 0$. Then it follows immediately that $\gamma \in S_1(N)$. Next, $U_1(N) \cap S_1(N)^{-1}S_1(N) = \Gamma_1(N)$ is obvious. Thus the Hecke pairs are indeed compatible, and we get an injection $\mathcal{H}_1^{\mathbb{A}}(N) \to \mathcal{H}_1(N)$ sending $sU_1(N)$ to $s\Gamma_1(N)$ ($s \in S_1(N)$).

We claim that $U_1(N)sU_1(N) = \Gamma_1(N)sU_1(N)$ for all $s \in S_1(N)$. It suffices to show that $U_1(N) = \Gamma_1(N)(U_1(N) \cap {}^sU_1(N))$. As $s_N \in U_1(N)$ and $U_1(N)$ is compact open, $U_1(N) \cap {}^sU_1(N) \supset U_1(N) \cap U(M)$ for some $(M, Np) = 1$, where $U(M) = \{g \in GL_n(\widehat{\mathbb{Z}}) : g \equiv 1 \pmod{M}\}$. Since $\Gamma_1(N) \twoheadrightarrow SL_n(\mathbb{Z}/M)$, it follows that

$$\{u \in U_1(N) : \det u \equiv 1 \pmod{M}\} \subset \Gamma_1(N)(U_1(N) \cap {}^sU_1(N)).$$

The desired equality follows by noting that the determinant of the right-hand side is $\widehat{\mathbb{Z}}^\times$, which can be seen by using the theorem on elementary divisors for all $l|M$.

The lemma now follows by noting that if $s \in S_1(N)$ and $U_1(N)sU_1(N) = \coprod s_iU_1(N)$, then $\Gamma_1(N)s\Gamma_1(N) = \coprod s_i\Gamma_1(N)$ (intersect with $S_1(N)$ and use the properties established above). □

Remark 13.2. *In a completely analogous way it follows that $\mathcal{H}_0(N) \xrightarrow{\sim} \mathcal{H}_1(N)$, sending $[\Gamma_0(N)s\Gamma_0(N)]$ to $[\Gamma_1(N)s\Gamma_1(N)]$ for $s \in S_1(N)$. In particular, $T_{l,i}$ is sent to $T_{l,i}$ for $l \nmid N$.*

Proposition 13.3. *Suppose that π is a cuspidal automorphic representation of $GL_n(\mathbb{A}_\mathbb{Q})$ of conductor N. Suppose moreover that for some integers*

$$c_1 > c_2 > \cdots > c_n,$$

π_∞ corresponds, under the Local Langlands Correspondence, to a representation of $W_\mathbb{R}$ sending $z \in \mathbb{C}^\times$ to

$$\operatorname{diag}(z^{-c_1}\bar{z}^{-c_n}, z^{-c_n}\bar{z}^{-c_1}, z^{-c_2}\bar{z}^{-c_{n-1}}, \dots) \otimes (z\bar{z})^{(n-1)/2} \in GL_n(\mathbb{C})$$

and j to an element of determinant $(-1)^{\sum c_i+\lfloor n/2\rfloor}$ (in particular, π is regular algebraic; c.f. [Clo90]*, def. 1.8 and def. 3.12). Let r be the irreducible representation of $GL_{n/\mathbb{C}}$ with highest weight $(c_1-(n-1), c_2-(n-2), \dots, c_n)$. Then there is an $\mathcal{H}_1(N)$-equivariant injection*

$$(\pi^\infty)^{U_1(N)} \hookrightarrow H^e(\Gamma_1(N), r)$$

for any e in the range

$$\frac{n^2-1}{4} \le e \le \left(\frac{n+1}{2}\right)^2 - 1.$$

Remark 13.4.

(i) *As N is the conductor of π, $(\pi^\infty)^{U_1(N)}$ is one-dimensional. Thus we get a Hecke eigenclass in group cohomology.*

(ii) *It is known that $\Gamma_1(N)$ has virtual cohomological dimension $n(n-1)/2$. In particular, $H^e(\Gamma_1(N), r) = 0$ for $e > n(n-1)/2$. (see* [Ser71]*, p. 132 and the remark on p. 101).*

Proof. Let $G = GL_{n/\mathbb{Q}}$, $\mathbb{A} = \mathbb{A}_\mathbb{Q}$. For any open compact subgroup $U \subset G(\mathbb{A}^\infty)$, define

$$\tilde{X}_U = G(\mathbb{R})/O(n) \times G(\mathbb{A}^\infty)/U$$

and

$$X_U = G(\mathbb{Q})\backslash\big(G(\mathbb{R})/O(n) \times G(\mathbb{A}^\infty)/U\big).$$

and denote by $\pi_U : \tilde{X}_U \to X_U$ the natural projection. Then $\tilde{X}_U$ and X_U are real manifolds of dimension $\binom{n+1}{2}$ (X_U is not necessarily connected). If U is sufficiently small, $G(\mathbb{Q})$ will act properly discontinuously on the total space in brackets and the constant sheaf on $\tilde{X}_U$ with fibre r gives rise to a local system on the quotient X_U, which will be denoted by $\mathcal{L}_r$: for any open subset $Z \subset X_U$, $\mathcal{L}_r(Z)$ is the set of locally constant functions

$$\{f : \pi_U^{-1}(Z) \to r : f(\gamma x) = \gamma f(x)\ \forall \gamma \in G(\mathbb{Q}),\ x \in Z\}. \tag{13.1}$$

Notice that $r^\vee$ is the representation of G associated to π_∞ defined in [Clo90], pp. 112–113 (where it is denoted by τ). By [Clo90], lemma 3.15 there is a $G(\mathbb{A}^\infty)$-equivariant injection

$$\bigoplus_\Pi H^e(\mathfrak{sl}_n, O(n); \Pi_\infty \otimes r) \otimes \Pi^\infty \hookrightarrow \varinjlim_V H^e(X_V, \mathcal{L}_r)$$

where Π runs through all cuspidal automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ whose central character agrees with that of $r^\vee$ on $\mathbb{R}_+^\times$, and the limit is over all (sufficiently small) compact open subgroups $V \subset G(\mathbb{A}^\infty)$. The $G(\mathbb{A}^\infty)$-action on the right-hand side is as in sublemma 13.5(ii) below. Here $\mathfrak{sl}_n$ denotes the complexified Lie algebra of $SL_n(\mathbb{R})$.

When n is even, lemma 3.14 in [Clo90] shows that

$$H^e(\mathfrak{sl}_n, O(n); \pi_\infty \otimes r) \cong \wedge^{e-n^2/4}\, \mathbb{C}^{n/2-1},$$

(by the remark on p. 120 in the same reference, there is no quadratic character appearing on the left-hand side).

When n is odd, the condition on the determinant of j made above implies that π_∞ is the induction, using a parabolic subgroup of type $(2, 2, \ldots, 2, 1)$, of $\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_{(n-1)/2} \otimes \chi$ (keeping Clozel's notation), where σ_i is the discrete series representation of central character $|.|^{-c_i - c_{n+1-i}+n-1} \operatorname{sgn}^{c_i + c_{n+1-i}+1}$ and lowest weight $c_i - c_{n+1-i}+1$ and $\chi = |.|^{-c_{(n+1)/2}+(n-1)/2} \operatorname{sgn}^{c_{(n+1)/2}}$. This has the consequence that the character considered in [Clo90], p. 120 is *even* and again we get (without quadratic character on the left-hand side):

$$H^e(\mathfrak{sl}_n, O(n); \pi_\infty \otimes r) \cong \wedge^{e-(n^2-1)/4} \mathbb{C}^{(n-1)/2}.$$

Thus we get an $\mathcal{H}_1(N)$-equivariant homomorphism

$$(\pi^\infty)^{U_1(N)} \hookrightarrow \Big(\varinjlim_{U} H^e(X_U, \mathcal{L}_r) \Big)^{U_1(N)}$$

for any e in the range claimed above. It remains to identify the right-hand side with the appropriate group cohomology group.

Let $H^e(X, \mathcal{L}_r) = \varinjlim_V H^e(X_V, \mathcal{L}_r)$ to simplify notations (X itself will not have any meaning). The following sublemma, whose simple proof we omit, will be useful.

Sublemma 13.5. *Suppose that U, V are sufficiently small compact open subgroups of $G(\mathbb{A}^\infty)$ and $e \geq 0$ arbitrary.*

(i) *If $U \subset V$ consider the natural projection map $f : X_U \to X_V$. Then $f^*\mathcal{L}_r \cong \mathcal{L}_r$ (canonically) and the induced map $f^* : H^e(X_V, \mathcal{L}_r) \to H^e(X_U, \mathcal{L}_r)$ is an injection.*

(ii) *If $g \in G(\mathbb{A}^\infty)$ and $U \subset gVg^{-1}$, denote by $[g]$ the natural map $X_U \to X_V$ given by multiplication on the right by g. Again there is a canonical isomorphism $[g]^*\mathcal{L}_r \cong \mathcal{L}_r$ and an induced map $[g]^* : H^e(X_V, \mathcal{L}_r) \to H^e(X_U, \mathcal{L}_r)$. It is compatible with the maps defined in (i) and yields a smooth left action of $G(\mathbb{A}^\infty)$ on the direct limit $H^e(X, \mathcal{L}_r)$.*

(iii) *The image of the natural map $H^e(X_U, \mathcal{L}_r) \to H^e(X, \mathcal{L}_r)$, which is an injection by (i), is precisely the subspace of U-invariants.*

Choose an auxiliary prime $q \nmid 2N$, and let

$$U = \{g \in U_1(N) : g \equiv 1 \pmod{q}\} \lhd U_1(N).$$

The projection of U to $G(\mathbb{Q}_q)$ contains no elements of finite order, which implies that U is sufficiently small in the above sense, so that $\mathcal{L}_r$ is defined on X_U. By the sublemma, $H^e(X, \mathcal{L}_r)^{U_1(N)} = H^e(X_U, \mathcal{L}_r)^{U_1(N)/U}$.

For now, we allow r to be any $\mathbb{C}[G(\mathbb{Q})]$-module. Let $\Gamma = G(\mathbb{Q}) \cap U_1(N)$, an arithmetic subgroup of G.

Claim: $H^e(X_U, \mathcal{L}_r)^{U_1(N)/U}$ and $H^e(\Gamma, r)$ are universal δ-functors, and they are canonically isomorphic.

First note that if $H \leq K$ are two groups and V is an injective K-module (over $\mathbb{C}$, say), then $V|_H$ is an injective H-module. The reason is that the left adjoint of the forgetful functor K**-mod** $\to H$**-mod** is $\mathbb{C}K \otimes_{\mathbb{C}H} -$, which is exact. By putting $H = \Gamma$, $K = G(\mathbb{Q})$, we see that $H^e(\Gamma, r)$ is a universal δ-functor.

As for the first sequence of functors, note that it is at least a δ-functor: $U_1(N)/U$ is a finite group so that taking $U_1(N)/U$-invariants is an exact functor (we are in characteristic zero!). To demonstrate universality, it suffices to show that $H^e(X_U, \mathcal{L}_r) = 0$ if $e > 0$ and r is an injective $\mathbb{C}[G(\mathbb{Q})]$-module. By the strong approximation theorem,

$$G(\mathbb{A}) = \coprod_{i=1}^{t} G(\mathbb{Q}) g_i U G(\mathbb{R})$$

for some $g_i \in G(\mathbb{A}^\infty)$. This implies that

$$X_U \cong \coprod_{i=1}^{t} (G(\mathbb{Q}) \cap {}^{g_i}U) \backslash G(\mathbb{R})/O(n).$$

Under this isomorphism, $\mathcal{L}_r$ corresponds to a local system on each space in the disjoint union. It is easy to see that on the i-th piece it is the one induced by the constant sheaf on $G(\mathbb{R})/O(n)$ with fibre r under the $(G(\mathbb{Q}) \cap {}^{g_i}U)$-action (as in (13.1)). It will be denoted by $\mathcal{L}_r$ as well. By [Mum70], appendix to §I.2, for example, $H^e((G(\mathbb{Q}) \cap {}^{g_i}U) \backslash G(\mathbb{R})/O(n), \mathcal{L}_r) = 0$ if $e > 0$ and r injective as $(G(\mathbb{Q}) \cap {}^{g_i}U)$-module; in particular if r is injective as $G(\mathbb{Q})$-module. (Note that for the constant sheaf $\underline{r}$, $H^i(G(\mathbb{R})/O(n), \underline{r}) = 0$ for $i > 0$ since $G(\mathbb{R})/O(n)$ is contractible; see [Bre97], thm. III.1.1 for the comparison of sheaf cohomology with singular cohomology).

To check that the two universal δ-functors above are canonically isomorphic, it is enough to identify them in degree 0. By (13.1), $H^0(X_U, \mathcal{L}_r)^{U_1(N)/U}$ is the set of locally constant, $G(\mathbb{Q})$-invariant functions $f : G(\mathbb{A})/U_1(N)O(n) \to r$. By the strong approximation theorem, using that $\det U_1(N) = \widehat{\mathbb{Z}}^\times$, such a function is determined by its values on $G(\mathbb{R})$; by local constancy it is even determined by $f(1) \in r$. It follows easily that the set of possibly values of $f(1)$ is precisely $r^\Gamma = H^0(\Gamma, r)$. This establishes the claim.

Claim: The map of δ-functors $H^e(\Gamma, r) \xrightarrow{\text{res}} H^e(\Gamma_1(N), r)$ is a (canonically split) injection.

As $(\Gamma : \Gamma_1(N)) = 2$, this is clear: $\frac{1}{2}$ cores provides the splitting, where cores is the corestriction map.

Claim: The above canonical injection

$$H^e(X, \mathcal{L}_r)^{U_1(N)} \hookrightarrow H^e(\Gamma_1(N), r)$$

of δ-functors is $\mathcal{H}_1^{\mathbb{A}}(N) \cong \mathcal{H}_1(N)$-equivariant.

Note that the Hecke action on the left is defined through the $G(\mathbb{A}^\infty)$-action of sublemma 13.5, whereas the one on the right is the usual one on group cohomology (see §2.2). Both Hecke actions are clearly maps of δ-functors, so again it suffices to check the claim in degree 0. Given $s \in S_1(N)$, we know by lemma 13.1 that the Hecke operator $T_s = [\Gamma_1(N)s\Gamma_1(N)] \in \mathcal{H}_1(N)$ corresponds to $T_s = [U_1(N)sU_1(N)] \in \mathcal{H}_1^{\mathbb{A}}(N)$. Moreover, the same lemma shows that if $s_i \in S_1(N)$ $(1 \leq i \leq n)$ are chosen such that

$$\Gamma_1(N)s\Gamma_1(N) = \coprod s_i\Gamma_1(N),$$

then also

$$U_1(N)sU_1(N) = \coprod s_iU_1(N).$$

An element of $H^0(X, \mathcal{L}_r)^{U_1(N)}$ is a locally constant, $G(\mathbb{Q})$-invariant function

$$f : G(\mathbb{A})/U_1(N)O(n) \to r$$

which is determined by $f(1) \in r^\Gamma \subset r^{\Gamma_1(N)}$. By the sublemma, $T_s(f)$ is the function sending $g \in G(\mathbb{A})$ to $\sum f(gs_i)$; in particular, the image of 1 is $\sum f(s_i) = \sum s_i f(1) = T_s(f(1))$ (we used that f is locally constant). This verifies the Hecke equivariance. $\square$

Fix an isomorphism $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_p$.

Proposition 13.6. *Suppose $p > 2$. Given integers $a > b > c > d$ with $a + d = b + c$. Let τ be any of the following tame inertial Galois representations $\tau : I_p \to GL_4(\overline{\mathbb{F}}_p)$:*

$$\begin{pmatrix} \omega^a &&& \\ & \omega^b && \\ && \omega^c & \\ &&& \omega^d \end{pmatrix}, \begin{pmatrix} \omega_2^{a+pd} &&& \\ & ** && \\ && \omega_2^{b+pc} & \\ &&& ** \end{pmatrix}, \begin{pmatrix} \omega_2^{a+pb} &&& \\ & ** && \\ && \omega_2^{c+pd} & \\ &&& ** \end{pmatrix},$$

$$\begin{pmatrix} \omega_2^{a+pc} &&& \\ & ** && \\ && \omega_2^{b+pd} & \\ &&& ** \end{pmatrix}, \begin{pmatrix} \omega_2^{a+pd} &&& \\ & ** && \\ && \omega^{b} & \\ &&& \omega^c \end{pmatrix}, \begin{pmatrix} \omega_2^{b+pc} &&& \\ & ** && \\ && \omega^{a} & \\ &&& \omega^d \end{pmatrix},$$

$$\begin{pmatrix} \omega_4^{a+pb+p^2d+p^3c} &&& \\ & ** && \\ && ** & \\ &&& ** \end{pmatrix}, \begin{pmatrix} \omega_4^{a+pc+p^2d+p^3b} &&& \\ & ** && \\ && ** & \\ &&& ** \end{pmatrix}.$$

Then there is an integer N prime to p, a continuous, irreducible, odd Galois representation $\rho : G_{\mathbb{Q}} \to GL_4(\overline{\mathbb{F}}_p)$ with $\rho|_{I_p} \cong \tau$ and a Hecke eigenclass in

$$H^e(\Gamma_1(N), F)$$

with attached Galois representation ρ for some e and for some Jordan-Hölder constituent F of $W(a-3, b-2, c-1, d)$.

Note that the symbol "$**$" stands for the p-th power of the previous diagonal matrix entry. Part (iii) of the following lemma will be needed in the proof, whose proof is given on p. 64.

Lemma 13.7. *Given a prime p.*

(i) *The Galois group of a quartic (totally complex) CM field can be either of $\mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathbb{Z}/4$ or D_8.*
(ii) *There is a quartic (totally complex) CM field K with Galois group $\mathbb{Z}/2 \times \mathbb{Z}/2$, unramified at p such that* Frob_p *is (a) trivial, (b) the complex conjugation, or (c) another element of order 2.*

(iii) *There is a cyclic quartic (totally complex) CM field K, unramified at p such that* Frob_p *is (a) trivial, (b) the complex conjugation, or (c) an element of order 4.*
(iv) *There is a quartic (totally complex) CM field K with Galois group $G \cong D_8$, unramified at p such that* $\mathrm{Frob}_p \in G$ *is (a) trivial, (b) the complex conjugation, (c) a (non-central) element of order 2 not fixing K^+, (d) a non-central element of order 2 fixing K^+, or (e) an element of order 4.*

Note that if K is a CM field, we have denoted its totally real subfield by K^+, so that $[K : K^+] \leq 2$. Also note that the above cases (ii)(a)–(c) (iii)(a)–(c) and (iv)(a)–(e) exhaust the respective Galois group.

For both the proof of the proposition and the lemma it will be very useful to keep a diagram of the subgroup lattice of D_8 at hand, together with explicit generators of each subgroup.

Proof of prop. 13.6. By lemma 13.7, choose a quartic totally complex CM field $K/\mathbb{Q}$, unramified at p, with Galois group G dihedral of order 8. We will fix the conjugacy class of Frob_p later. Let $\mu(K)$ be the torsion subgroup of $\mathcal{O}_K^\times$ and let $w(K)$ be its order; denote by L the Galois closure of $K/\mathbb{Q}$ and let $c \in G$ denote the complex conjugation (which is central!).

We now want to make a careful choice of a Hecke character χ over K that is unramified outside l. For this recall:

Sublemma 13.8. *There is a bijection between Hecke characters χ over K and 4-tuples $(\mathfrak{f}, \epsilon, \epsilon_\mathfrak{f}, \epsilon_\infty)$, where $\mathfrak{f} \lhd \mathcal{O}_K$, $\epsilon : I_K^\mathfrak{f} \to \mathbb{C}^\times$ ($I_K^\mathfrak{f}$ being the ideals prime to $\mathfrak{f}$), $\epsilon_f : (\mathcal{O}_K/\mathfrak{f})^\times \to \mathbb{C}^\times$* primitive *(i.e., not factoring through $(\mathcal{O}_K/\mathfrak{f}_1)^\times$ for any proper divisor $\mathfrak{f}_1|\mathfrak{f}$), $\epsilon_\infty : K_\infty^\times \to \mathbb{C}^\times$ continuous such that for all $x \in K^\times$, x prime to $\mathfrak{f}$,*

$$\epsilon((x)) = \epsilon_\mathfrak{f}(x)\epsilon_\infty(x). \tag{13.2}$$

(By weak approximation, $\epsilon_\mathfrak{f}$ and ϵ_∞ are determined by $(\mathfrak{f}, \epsilon)$.) The conductor of χ is $\mathfrak{f}$. The bijection is determined by demanding that

$$\chi(x) = \epsilon_\mathfrak{f}(x_\mathfrak{f})^{-1}\epsilon_\infty(x_\infty)^{-1}\epsilon((x))$$

for all $x \in \mathbb{A}_K^\times$ that are prime to $\mathfrak{f}$.

Fix for each $\sigma : K \to \mathbb{C}$ an integer n_σ with the property that $n_\sigma + n_{\sigma c} = w$ for all σ (some w). These will be pinned down further on. Let $\epsilon_\infty : K_\infty^\times \to \mathbb{C}^\times$ be given by $\epsilon_\infty(x) = |x|^{-3/2}\prod_\sigma \sigma(x)^{n_\sigma}$. (Here, for $x \in \mathbb{A}_K^\times$, $|x|$ is the norm on $\mathbb{A}_K^\times$ and $\sigma(x)$ means $\sigma(x_v)$ for the unique place $v|\infty$ which is induced by σ on K.)

Claim: $\epsilon_\infty(\mathcal{O}_K^\times) \subset \mu_{w(L)[L:K]}(\mathbb{C})$.

Fix an embedding $j : L \to \mathbb{C}$ and for $\tau \in G$ let $m_\tau = n_{j\tau|K}$. In particular, $m_\tau + m_{\tau c} = w$ for all τ. It will suffice to show that $\prod_\tau \tau(-)^{m_\tau}$ acts trivially on $(\mathcal{O}_L^\times)_{tor-free}$. For, this group contains $(\mathcal{O}_K^\times)_{tor-free}$ as subgroup and on it, $j\prod_\tau \tau(-)^{m_\tau} = \prod_\sigma \sigma(-)^{n_\sigma[L:K]}$.

By the unit theorem, $(\mathcal{O}_L^\times)_{tor-free} \hookrightarrow \mathrm{Map}(S_\infty, \mathbb{R})_0$ as G-module, where S_∞ is the set of archimedean places of L and the subscript "0" denotes the subspace of $f : S_\infty \to \mathbb{R}$ with $\sum_v f(v) = 0$. As G acts transitively on S_∞ with stabiliser $\langle c\rangle$, $\mathrm{Map}(S_\infty, \mathbb{R})_0 \cong \mathbb{R}[G/\langle c\rangle]_0$ as $\mathbb{R}G$-module, where the subscript "0" now refers to $\sum_{G/\langle c\rangle} \lambda_g g$ with $\sum_{G/\langle c\rangle} \lambda_g = 0$ (i.e., the augmentation ideal). It will suffice to show that for $\bar\nu \in G/\langle c\rangle$, the action of $\sum_G m_\tau \tau(-)$ on $\bar\nu \in \mathbb{R}[G/\langle c\rangle]$ is independent of $\bar\nu$. Indeed,

$$\sum_{\tau\in G} m_\tau \tau\bar\nu = \sum_{\tau\in G/\langle c\rangle} (m_\tau\tau + (w - m_\tau)\tau c)\bar\nu = w \sum_{\tau\in G/\langle c\rangle} \overline{\tau\nu} = w \sum_{\tau\in G/\langle c\rangle} \bar\tau$$

is independent of $\bar\nu$. This proves the claim.

So far, the argument only depended on K being a CM field. Note now that in our situation (K being quartic totally complex CM with dihedral Galois group), L does not have any *abelian* totally complex CM subfields. Thus $w(L) = 2$ and by the claim $\epsilon_\infty(\mathcal{O}_K^\times) \subset \mu_4(\mathbb{C})$.

Choose now distinct rational primes $q_i \nmid 2p$ ($1 \le i \le t$, any $t \ge 3$) that stay inert in K (equivalently, $\mathrm{Frob}_{q_i} \in G$ has order 4). This can be achieved by the Cebotarev density theorem. Denote by $\mathfrak{q}_i$ the prime of K lying above q_i.

If $\alpha \equiv 1 \pmod{\prod \mathfrak{q}_i}$ then in particular $\epsilon_\infty(\alpha) \equiv 1 \pmod{q_1}$ (in the subring $\bar{\mathbb{Z}} \subset \mathbb{C}$). But $\epsilon_\infty(\alpha) \in \mu_4(\mathbb{C})$ by above and hence it is 1 (as q_1 odd). Therefore $\epsilon_\infty|_{\mathcal{O}_K^\times}$ can be written as

$$\epsilon_\infty|_{\mathcal{O}_K^\times} : \mathcal{O}_K^\times \to (\mathcal{O}_K/\prod \mathfrak{q}_i)^\times \xrightarrow{\theta} \mathbb{C}^\times$$

where θ is not uniquely determined! Letting A be the image of $\mathcal{O}_K^\times$ in $(\mathcal{O}_K/\prod \mathfrak{q}_i)^\times$, we see that θ is determined by ϵ_∞ on A but nowhere else (the characters of $(\mathcal{O}_K/\prod \mathfrak{q}_i)^\times/A$ separate points).

Let B_p be the p-Sylow subgroup of $(\mathcal{O}_K/\prod \mathfrak{q}_i)^\times$. Observe that

$$\prod_{i=1}^t ((\mathcal{O}_K/\mathfrak{q}_i)^\times)^{q_i^2-1} \not\subset A \cdot B_p.$$

This is because the size of the 2-torsion on the left-hand side is exactly $2^t \ge 8$, whereas on the right it is bounded above by 4 due to the unit theorem. Therefore we can assume, without loss of generality, that θ is non-trivial on $\prod_{i=1}^t ((\mathcal{O}_K/\mathfrak{q}_i)^\times)^{q_i^2-1}$ while being of order prime to p (simply first extend the given map on A to $A \cdot B_p$ by making it trivial on B_p).

Let $\mathfrak{f} = \prod \mathfrak{q}_i$ and $\epsilon_\mathfrak{f} = \theta^{-1}$ (if necessary, some of the $\mathfrak{q}_i$'s need to be removed from $\mathfrak{f}$ to make $\epsilon_\mathfrak{f}$ primitive). Writing $\epsilon_\mathfrak{f} = \prod \epsilon_{\mathfrak{q}_i}$ (with the obvious meaning), we see that $\epsilon_{\mathfrak{q}_i}$ has order not dividing $q_i^2 - 1$ for some i. By permuting the $\mathfrak{q}_i$, let us assume that this happens when $i = 1$ and set $\mathfrak{q} = \mathfrak{q}_1$, $q = q_1$.

By construction, $\epsilon_{\mathfrak{f}}\epsilon_\infty$ is trivial on $\mathcal{O}_K^\times$. Now ϵ can be defined by (13.2) on the finite index subgroup of $I_K^{\mathfrak{f}}$ generated by (x) with $x \in K^\times$ prime to $\mathfrak{f}$ and extended arbitrarily to $I_K^{\mathfrak{f}}$. The above sublemma yields a Hecke character χ over K; we record here some of its properties:

(13.3)

- $\chi_\infty(x) = |x|^{3/2} \prod_\sigma \sigma(x)^{-n_\sigma}$,
- χ has conductor $\prod \mathfrak{q}_i$ (prime to p),
- $\chi_{\mathfrak{q}}|_{\mathcal{O}_{K_{\mathfrak{q}}}^\times}$ has order dividing $q^4 - 1$ but not dividing $q^2 - 1$,
- $\chi(\prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times)$ has order prime to p.

By [AC89], §III.6 we can consider the automorphic induction $\pi = \operatorname{Ind}_K^{\mathbb{Q}} \chi$, which is obtained in two stages: first inducing along the cyclic extension K/K^+: $\Pi = \operatorname{Ind}_K^{K^+} \chi$; then inducing along the cyclic extension $K^+/\mathbb{Q}$: $\pi = \operatorname{Ind}_{K^+}^{\mathbb{Q}} \Pi$.

Suppose now that the n_σ above are a permutation of $\{a, b, c, d\}$ (note that there are only 8 possible choices due to the restriction $n_\sigma + n_{\sigma c} = w \ \forall \sigma$ made above).

Claim: π is a cuspidal automorphic representation of $GL_4(\mathbb{A}_{\mathbb{Q}})$ of conductor prime to p to which prop. 13.3 applies with $(c_1, c_2, c_3, c_4) = (a, b, c, d)$.

Note the following facts about Arthur-Clozel's cyclic automorphic inductions: (i) they construct them using cyclic base change ([AC89], thm. III.6.2), (ii) global cyclic base change is compatible with local base change at all (finite or infinite) places (see [AC89], thm. III.5.1), (iii) local cyclic base change is compatible with restriction under the Local Langlands Correspondence (see [AC89], p. 71 in the archimedean case and [HT01], thm. VII.2.6 in the non-archimedean case).

As $\chi \not\cong \chi^c$ (look at the either of the infinite components), Π is cuspidal and is determined by

$$\mathrm{BC}_K^{K^+} \Pi \cong \chi \oplus \chi^c,$$

where BC denotes base change ([AC89], bottom of p. 216). In particular, under the Local Langlands Correspondence the infinite components of Π correspond to the representations sending

$$z \mapsto |z|^3 \operatorname{diag}(z^{-a}\bar{z}^{-d}, z^{-d}\bar{z}^{-a}), \text{ resp.}$$
$$z \mapsto |z|^3 \operatorname{diag}(z^{-b}\bar{z}^{-c}, z^{-c}\bar{z}^{-b}),$$

for $z \in W_{\mathbb{C}} = \mathbb{C}^\times$. Repeating the argument shows that π is cuspidal and that under the Local Langlands Correspondence π_∞ corresponds to a representation sending

$$z \mapsto |z|^3 \operatorname{diag}(z^{-a}\bar{z}^{-d}, z^{-d}\bar{z}^{-a}, z^{-b}\bar{z}^{-c}, z^{-c}\bar{z}^{-b})$$

for $z \in W_{\mathbb{C}}$. As $a \neq d$ and $b \neq c$, by the classification of representations of $W_{\mathbb{R}}$ (see e.g. [Tat79], (2.2.2)), this representation is the direct sum of

$$z \mapsto |z|^3 \begin{pmatrix} z^{-a}\bar{z}^{-d} & \\ & z^{-d}\bar{z}^{-a} \end{pmatrix}$$
$$j \mapsto \begin{pmatrix} & 1 \\ (-1)^{a+d} & \end{pmatrix}$$

and the same with (a, d) replaced by (b, c). This shows that $(c_1, c_2, c_3, c_4) = (a, b, c, d)$ in the notation of prop. 13.3.

Let S be the set of primes l that either ramify in K or divide a prime where χ is ramified. For $l \notin S$, π_l is an unramified principal series which corresponds to

$$\sigma_l = \bigoplus_{\lambda | l} \operatorname{Ind}_{W_\lambda}^{W_l} \chi_\lambda \tag{13.4}$$

under the Local Langlands Correspondence (see [AC89], p. 214f). In particular, the conductor N of π is prime to p.

Let $\Sigma = \operatorname{Ind}_{W_K}^{W_{\mathbb{Q}}} \chi$. Since

$$\Sigma|_{I_q} \cong \bigoplus_{i \bmod 4} (\chi_{\mathfrak{q}}|_{I_{\mathfrak{q}}})^{q^i},$$

Σ is irreducible (this uses (13.3)). The previous paragraph shows that Σ_v and π_v correspond to each other under the unramified Langlands Correspondence for almost all places v. Therefore we can use corollary 4.5 of [Hen86] to see that at *all* finite places v, the L-factors (and even the ϵ-factors) of Σ_v and π_v agree. In particular, Σ and π are ramified at the same set of finite places (namely those finite primes at which the L-factor has degree less than 4; for π this characterisation follows from [Jac79], §3). It follows that S is precisely the set of prime divisors of N.

This establishes the claim. We get, for e as in the sublemma, an $\mathcal{H}_1(N)$-equivariant injection

$$(\pi^\infty)^{U_1(N)} \hookrightarrow H^e(\Gamma_1(N), r) \tag{13.5}$$

with r of highest weight $(a-3, b-2, c-1, d)$.

For $l \nmid N$, let $t_{l,j}$ $(1 \leq j \leq 4)$ denote the eigenvalues of $\sigma_l(\mathrm{Frob}_l)$. It is a standard (and easy) fact that $[G(\mathbb{Z}_l)\begin{pmatrix} l & & \\ & \ddots & \\ & & 1 \end{pmatrix}G(\mathbb{Z}_l)]$ (i times l) has eigenvalue $s_i(t_{l,1}, \dots)l^{i(4-i)/2}$ on $\pi_l^{G(\mathbb{Z}_l)}$, where s_i denotes the i-th elementary symmetric function. Therefore $[U_1(N)\begin{pmatrix} l & & \\ & \ddots & \\ & & 1 \end{pmatrix}_l U_1(N)]$ has the same eigenvalue on $(\pi^\infty)^{U_1(N)}$. Since this Hecke operator corresponds to $T_{l,i} \in \mathcal{H}_1(N)$, equation (13.5) gives a Hecke eigenclass in $H^e(\Gamma_1(N), r)$ whose $T_{l,i}$-eigenvalue is $s_i(t_{l,1}, \dots) \cdot l^{i(4-i)/2}$ $(\forall l \nmid N, \forall i)$. Equivalently, there is an eigenclass in $H^e(\Gamma_1(N), r \otimes_{\mathbb{C},\iota} \overline{\mathbb{Q}}_p)$ with $T_{l,i}$-eigenvalue $\iota(s_i(t_{l,1}, \dots) \cdot l^{i(4-i)/2})$ $(\forall l \nmid N, \forall i)$.

Claim: There is a Hecke eigenclass in $H^e(\Gamma_1(N), F(\lambda))$ with $T_{l,i}$-eigenvalue
$$\overline{\iota(s_i(t_{l,1},\dots)\cdot l^{i(4-i)/2})}$$
($\forall l \nmid Np$, $\forall i$) for some Jordan-Hölder constituent $F(\lambda)$ of $W(a-3, b-2, c-1, d)$ ($\lambda \in X(T)_+$).

It is a classical fact that every representation r of the algebraic group GL_n over $\mathbb{C}$ can be defined over $\mathbb{Q}$ (e.g., it follows from [FH91], §15.3; or see [Jan03] II.2.7, II.5.6, II.5.10 in the much more general context of connected, split reductive groups over a field of characteristic zero). By [Jan03] I.10.4 it has a model M even over $\mathbb{Z}_{(p)}$ (a representation of the reductive group scheme $GL_{n/\mathbb{Z}_{(p)}}$). Let $\overline{M}$ denote its reduction modulo p, a representation of GL_n over $\mathbb{F}_p$. By [Jan03] I.2.11(3) (or II.1.1(2) over $\mathbb{Z}$), M has a weight space decomposition which has to be compatible with that of r and $\overline{M}$ so that each weight occurs with the same multiplicity in each of r, M, $\overline{M}$.

By [Ser71], §2.4, thm. 4, $\Gamma_1(N)$ is of type (WFL). In particular, for any noetherian ring A, $H^e(\Gamma_1(N), P)$ is a finite A-module whenever P is a finite A-module with commuting $\Gamma_1(N)$-action (see [Ser71], rk. on p. 101).

Consider now only the Hecke operators $T_{l,i}$ with $l \nmid Np$. For any $\mathbb{Z}_{(p)}$-algebra R, let $r_R = M \otimes_{\mathbb{Z}_{(p)}} R$. Note that $r_{\overline{\mathbb{Z}}_p}$ is a $GL_n(\mathbb{Z}_{(p)})$-invariant $\overline{\mathbb{Z}}_p$-lattice in $r_{\overline{\mathbb{Q}}_p} \cong r \otimes_{\mathbb{C},\iota} \overline{\mathbb{Q}}_p$. Since
$$H^e(\Gamma_1(N), r_{\overline{\mathbb{Q}}_p}) \cong H^e(\Gamma_1(N), r_{\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$$
(Hecke equivariantly) and this space is finite-dimensional over $\overline{\mathbb{Q}}_p$, the simultaneous generalised eigenspaces for $T_{l,i}$ with $l \nmid Np$ can be defined over some finite extension $E/\mathbb{Q}_p$. Thus the above set of Hecke eigenvalues also occurs in $H^e(\Gamma_1(N), r_E)$. Consider the following Hecke equivariant map:
$$H^e(\Gamma_1(N), r_{\mathcal{O}_E})_{tor-free} \hookrightarrow H^e(\Gamma_1(N), r_E).$$
The image of the map is a lattice in $H^e(\Gamma_1(N), r_E)$; this follows by looking at the long exact sequence associated to $0 \to r_{\mathcal{O}_E} \to r_E \to r_E/r_{\mathcal{O}_E} \to 0$. By scaling the Hecke eigenclass in $H^e(\Gamma_1(N), r_E)$, we may assume it lies in this sublattice and has non-zero reduction in $H^e(\Gamma_1(N), r_{\mathcal{O}_E})_{tor-free} \otimes_{\mathcal{O}_E} k_E$. Consider the Hecke equivariant map:
$$H^e(\Gamma_1(N), r_{\mathcal{O}_E}) \otimes_{\mathcal{O}_E} k_E \twoheadrightarrow H^e(\Gamma_1(N), r_{\mathcal{O}_E})_{tor-free} \otimes_{\mathcal{O}_E} k_E.$$
By the Ash-Stevens lifting lemma ([AS86], prop. 1.2.2), by enlarging E if necessary, we can lift the system of Hecke eigenvalues to $H^e(\Gamma_1(N), r_{\mathcal{O}_E}) \otimes_{\mathcal{O}_E} k_E$. That is, there is an eigenclass in it with the same system of Hecke eigenvalues, although this eigenclass does not necessarily lift the eigenclass we had in $H^e(\Gamma_1(N), r_{\mathcal{O}_E})_{tor-free} \otimes_{\mathcal{O}_E} k_E$.

Finally, the long exact sequence associated to $0 \to r_{\mathcal{O}_E} \to r_{\mathcal{O}_E} \to r_{k_E} \to 0$ yields a Hecke equivariant injection
$$H^e(\Gamma_1(N), r_{\mathcal{O}_E}) \otimes_{\mathcal{O}_E} k_E \hookrightarrow H^e(\Gamma_1(N), r_{k_E}) \hookrightarrow H^e(\Gamma_1(N), r_{\overline{\mathbb{F}}_p}).$$

Thus there is an eigenclass in $H^e(\Gamma_1(N), r_{\overline{\mathbb{F}}_p})$ with $T_{l,i}$-eigenvalue $\overline{\iota(s_i(t_{l,1},\dots)\cdot l^{i(4-i)/2})}$ ($\forall l\nmid N$, $\forall i$).

Note that the GL_4-representations $r_{\overline{\mathbb{F}}_p} = \overline{M}\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p$ and $W(a-3,b-2,c-1,d)$ have the same formal character: by above, $\overline{M}$ has the same formal character as r, and both r and $W(a-3,b-2,c-1,d)$ have formal character given by the Weyl character formula for the highest weight $(a-3,b-2,c-1,d)$ ([Jan03] II.5.10). Now by [Jan03] II.2.7 and II.5.8, $r_{\overline{\mathbb{F}}_p}$ and $W(a-3,b-2,c-1,d)$ have isomorphic semisimplifications (as GL_4-modules). Using the Ash-Stevens lifting lemma ([AS86], prop. 1.2.2), the same system of Hecke eigenvalues obtained in $H^e(\Gamma_1(N), r_{\overline{\mathbb{F}}_p})$ also occurs in $H^e(\Gamma_1(N), F(\lambda))$ for some Jordan-Hölder constituent $F(\lambda)$ of $W(a-3,b-2,c-1,d)$ ($\lambda\in X(T)_+$). This establishes the claim.

The Hecke character $\eta=\chi^{-1}|.|^{3/2}$ is algebraic with algebraic infinity type $\eta_\infty(x)=\prod\sigma(x)^{n_\sigma}$. Recall the definition of the associated p-adic Galois character $\eta^{(p)}$ (using the global Artin map; see e.g. [HT01], p. 20f):

$$\begin{aligned} \eta^{(p)}: G_K^{ab}\cong \overline{K^\times K^\times_{\infty,+}}\backslash\mathbb{A}_K^\times &\to \overline{\mathbb{Q}}_p^\times \\ x &\mapsto \iota\eta(x^\infty)\prod_{\tau:K\to\overline{\mathbb{Q}}_p}\tau(x_p)^{n_{\iota^{-1}(\tau)}}. \end{aligned} \tag{13.6}$$

Here, the convention is that $\sigma(x_p)$ means $\sigma(x_v)$ for the unique $v|p$ induced by σ on K. In particular, $\eta^{(p)}|_{G_{K_\lambda}}=\iota\chi_\lambda^{-1}$ under the local Artin map for all $\lambda|l$, $l\nmid Np$.

Claim: The Galois representation

$$\rho=\mathrm{Ind}_{G_K}^{G_\mathbb{Q}}(\overline{\eta^{(p)}})$$

is attached to the eigenclass in $H^e(\Gamma_1(N), F(\lambda))$ constructed above. It it continuous, irreducible, odd and its ramification outside p occurs precisely at all $l|N$.

Clearly, ρ is continuous. By Mackey's formula, using the local Artin map, for any prime l,

$$\rho|_{I_l}\cong\bigoplus_{\lambda|l}\ \bigoplus_{g\in\mathrm{Gal}(K_\lambda^0/\mathbb{Q}_l)}\mathrm{Ind}_{I_\lambda}^{I_l}(\overline{\iota\chi_\lambda}^{-1}|_{I_\lambda}^g),$$

where K_λ^0 is the maximal unramified subextension of $K_\lambda/\mathbb{Q}_l$. By Frobenius reciprocity, I_l acts trivially on the direct summand corresponding to the index (λ,g) if and only if $I_\lambda=I_l$ and $\overline{\iota\chi}|_{I_\lambda}=1$. Thus the claim about ramification outside p follows from (13.3) and the fact that S is the set of primes dividing N. Specialising now to $l=q$ we even get:

$$\rho|_{I_q}\cong\bigoplus_{i \bmod 4}(\overline{\iota\chi}_{\mathfrak{q}}|_{I_\mathfrak{q}})^{-q^i}.$$

Note that even $\overline{\iota\chi}_\mathfrak{q}|_{I_\mathfrak{q}}$ has order not dividing q^2-1, by (13.3). Hence $\rho|_{G_{\mathbb{Q}_q}}$ is irreducible; a fortiori, so is ρ.

For $l \nmid Np$,

$$\rho|_{G_{\mathbb{Q}_l}} \cong \bigoplus_{\lambda|l} \mathrm{Ind}_{G_{K_\lambda}}^{G_{\mathbb{Q}_l}}(\overline{\eta^{(p)}}|_{G_{K_\lambda}}).$$

We know by above that this is unramified. Using an explicit basis, we see that $\rho(\mathrm{Frob}_l)$ has characteristic polynomial

$$X^{[K_\lambda:\mathbb{Q}_l]} - \overline{\eta^{(p)}(\mathrm{Frob}_\lambda)}$$

on the λ-direct summand. A similar consideration applied to σ_l in (13.4) shows that the eigenvalues of $\rho(\mathrm{Frob}_l)$ are $\overline{\iota(t_{l,j}^{-1} l^{-3/2})}$ (recall that the $t_{l,j}$ are the eigenvalues of $\sigma_l(\mathrm{Frob}_l)$ and that $\overline{\eta^{(p)}}|_{G_{K_\lambda}} = \overline{\iota(\chi_\lambda^{-1}|.|_\lambda^{3/2})}$). Incidentally, $\rho(\mathrm{Frob}_l)$ is not semisimple in case $p|[K_\lambda : \mathbb{Q}_l]$.

By the following simple computation, and the fact that S is the set of prime divisors of N, we see that ρ is attached to the eigenclass constructed above: for all $l \nmid Np$,

$$\sum_{i=0}^{4} (-1)^i l^{i(i-1)/2} \overline{s_i(\iota t_{l,1}, \dots) \cdot \iota l^{i(4-i)/2}} X^i = \prod_{j=1}^{4} (1 - \overline{\iota(t_{l,j} l^{3/2})} \cdot X).$$

Finally, note that

$$\rho|_{G_{\mathbb{R}}} \cong \left(\mathrm{Ind}_{G_{\mathbb{C}}}^{G_{\mathbb{R}}}(1)\right)^{\oplus 2},$$

which has eigenvalues 1 and -1 twice each on complex conjugation. Thus ρ is odd and the claim is established.

To determine $\rho|_{I_p}$, note that

$$\rho|_{I_p} \cong \bigoplus_{\mathfrak{p}|p} \bigoplus_{i \bmod f_{\mathfrak{p}}} \overline{\eta^{(p)}}|_{I_{\mathfrak{p}}}^{p^i}$$

where $f_{\mathfrak{p}}$ is the inertial degree. Also, as χ is unramified at all $\mathfrak{p}|p$ we get from (13.6),

$$\overline{\eta^{(p)}} : x_p \mapsto \prod_{\tau: K \to \overline{\mathbb{Q}}_p} \overline{\tau(x_p)}^{\,n_{\iota^{-1}(\tau)}}$$

for $x_p \in \prod_{\mathfrak{p}|p} \mathcal{O}_{K_{\mathfrak{p}}}^\times$. Fix for each $\mathfrak{p}$ an embedding $\tau_{\mathfrak{p}} : K \to \overline{\mathbb{Q}}_p$ which induces the place $\mathfrak{p}$ on K and denote by $\phi : \mathbb{Q}_p^{nr} \to \mathbb{Q}_p^{nr}$ the arithmetic Frobenius. Recall that the composite $I_{K_{\mathfrak{p}}} \twoheadrightarrow \mathcal{O}_{K_{\mathfrak{p}}}^\times \to k_{\mathfrak{p}}^\times$, where the first map is induced by local class field theory and the second is $x_{\mathfrak{p}} \mapsto \bar{x}_{\mathfrak{p}}$, is the fundamental tame character $\theta_{\mathfrak{p}}$ of level $f_{\mathfrak{p}}$ (see [Ser72], prop. 3 with $L = K_d$ in Serre's notation; notice the different sign convention for the local Artin map). We get

$$\overline{\eta^{(p)}} : x_{\mathfrak{p}} \mapsto \overline{\tau_{\mathfrak{p}}} \theta_{\mathfrak{p}}^{\sum_{i \bmod f_{\mathfrak{p}}} p^i n_{\iota^{-1}(\phi^i \tau_{\mathfrak{p}})}}$$

for $x_{\mathfrak{p}} \in \mathcal{O}_{K_{\mathfrak{p}}}^\times$.

Now we let the n_σ vary through the 8 allowed permutations of $\{a, b, c, d\}$ (recall that $n_\sigma + n_{\sigma c} = w \; \forall \sigma$ has to hold). To see which $\rho|_{I_p}$ can be obtained for a fixed conjugacy class of $\mathrm{Frob}_p \in G$, it thus only matters how complex conjugation acts on the set of $\mathfrak{p}|p$, and what $f_\mathfrak{p}$ is in each case. In the notation of lemma 13.7(iv) we can obtain all of the following:

$$\begin{pmatrix} \omega^a & & & \\ & \omega^b & & \\ & & \omega^c & \\ & & & \omega^d \end{pmatrix} \qquad \text{in case (a),}$$

$$\begin{pmatrix} \omega_2^{a+pd} & & & \\ & ** & & \\ & & \omega_2^{b+pc} & \\ & & & ** \end{pmatrix} \qquad \text{in case (b),}$$

$$\begin{pmatrix} \omega_2^{a+pb} & & & \\ & ** & & \\ & & \omega_2^{c+pd} & \\ & & & ** \end{pmatrix}, \begin{pmatrix} \omega_2^{a+pc} & & & \\ & ** & & \\ & & \omega_2^{b+pd} & \\ & & & ** \end{pmatrix} \qquad \text{in case (c),}$$

$$\begin{pmatrix} \omega_2^{a+pd} & & & \\ & ** & & \\ & & \omega^b & \\ & & & \omega^c \end{pmatrix}, \begin{pmatrix} \omega_2^{b+pc} & & & \\ & ** & & \\ & & \omega^a & \\ & & & \omega^d \end{pmatrix} \qquad \text{in case (d),}$$

$$\begin{pmatrix} \omega_4^{a+pb+p^2d+p^3c} & & & \\ & ** & & \\ & & ** & \\ & & & ** \end{pmatrix}, \begin{pmatrix} \omega_4^{a+pc+p^2d+p^3b} & & & \\ & ** & & \\ & & ** & \\ & & & ** \end{pmatrix} \qquad \text{in case (e).}$$

as claimed.

$\square$

Theorem 13.9. *Generically, proposition 13.6 is compatible with the conjecture. That is, suppose given $\nu = (a,b,c,d) - \rho$ with $a+d = b+c$ sufficiently deep in a restricted alcove C for GL_4 (in particular, $a > b > c > d$). Then whenever prop. 13.6 produces a Hecke eigenclass in*

$$\bigoplus_{\substack{e \geq 0 \\ F \in JH(W(\nu))}} H^e(\Gamma_1(N), F)$$

with attached Galois representation ρ, $\rho|_{I_p}$ is actually predicted by the conjecture for some $F \in JH(W(\nu))$.

Assume moreover that the conjecture correctly predicts tame $\rho|_{I_p}$ for all weights in all alcoves strictly below C, at least for all δ-deep weights, some $\delta > 0$. Say

$$JH(W(\nu)) = \{F(\nu)\} \sqcup \{F_\alpha : \alpha \in I\},$$

where all F_α lie in alcoves strictly below C. Then in the above situation if F_α does not predict $\rho|_{I_p}$ for any α and ν lies sufficiently deep in C, we obtain evidence for $F(\nu)$ predicting $\rho|_{I_p}$. This happens for a certain proportion of the predicted tame $\rho|_{I_p}$; see table 1.

Remark 13.10. *We hope to verify that prop. 13.6 is compatible with the conjecture in every case. We have so far done this in the case of niveau 1 and assuming that all constituents of $W(\mu - \rho)$ are regular Serre weights.*

Proof. Note first that, in the notation of prop. 9.3, all tame inertial Galois representations produced by prop. 13.6 are of the form $\rho(w, \mu_0)$.

Suppose first $C \neq C_5$. Let $W = W(\mu - \rho)$ and $F = F(\mu - \rho)$. Then by the proof of prop. 12.12(ii), the predictions for F are the union of the

	alcove					
	0	1^*	2^*	3^*	4^*	5^*
niveau 1	1	$\frac{1}{2}$	0	0	$\frac{1}{5}$	$\frac{1}{8}$
niveau (2,2)	1	$\frac{1}{2}$	0	0	$\frac{1}{5}$	$\frac{1}{8}$
niveau (2,1,1)	$\frac{1}{3}$	$\frac{1}{6}$	0	0	$\frac{1}{15}$	$\frac{1}{24}$
niveau 4	$\frac{1}{3}$	$\frac{1}{6}$	0	0	$\frac{1}{15}$	$\frac{1}{24}$
niveau (3,1)	0	0	0	0	0	0

TABLE 1. The proportion of predicted tame $\rho|_{I_p}$ of a given niveau produced by prop. 13.6 for all sufficiently deep weights in a given alcove. The * indicates that this is conditional on the conjecture being correct in all lower alcoves.

diagonal predictions for all $F(\nu)$, $\nu \in M_\downarrow(\mu - \rho)$. Thus, taking into account prop. 9.3, we see that, if all predictions in lower alcoves are correct, F has to predict $\rho(w, \mu_0)$ for all $w \in W$.

Now suppose $C = C_5$. As $\mu - \rho$ is sufficiently deep inside C_5, W has 13 irreducible constituents over $GL_4(\mathbb{F}_p)$, 8 of which are in alcove 0 (use prop. 9.3 and the comments after thm. 10.3 to decompose irreducible GL_4-representations in alcoves $0'$, $0''$ over $GL_4(\mathbb{F}_p)$). Let us determine all tame inertial representations predicted by F but not by any of the other constituents of W.

The W_p-translates of $\mu - \rho$ in alcoves 0–4 have respective highest weights

$$\begin{gathered}(b, a-p, d+p, c) - \rho,\\ (c+p, a-p, d+p, b-p) - \rho,\ (d+2p, a-p, c, b-p) - \rho,\\ (c+p, b, d+p, a-2p) - \rho,\ (d+2p, b, c, a-2p) - \rho.\end{gathered}$$

Combining this with prop. 12.12, we see that the constituents of W in alcoves 1–4 predict precisely all $\rho(w, \mu_i)$ for $w \in W$, $i \in \{1, 4, 0', 0'', 5\}$.

By (9.2), (10.3), the GL_4-constituent of W in alcove $0'$ decomposes into the following four weights in alcove 0:

$$F((a-p, d+p, c, b-p) + \epsilon_i - \rho),\ 1 \le i \le 4,$$

Prop. 12.12 together with the observation of (12.9) shows that these constituents of W predict precisely all $\rho(w, \mu_i)$ for $w \in W$, $i \in \{1, 3, 4, 0''\}$.

Similarly (or by dualising), the constituents of W that arise through alcove $0''$ predict precisely all $\rho(w, \mu_i)$ for $w \in W$, $i \in \{1, 2, 4, 0'\}$. The upshot is, once again, that if all predictions in lower alcoves are correct, F has to predict $\rho(w, \mu_0)$ for all $w \in W$. $\square$

Proof of lemma 13.7. (i) Choose $\sigma \in G_{\mathbb{Q}}$ inducing the non-trivial automorphism of $K^+/\mathbb{Q}$. Then the compositum $L = K \cdot \sigma(K)$ is the normal closure over $K/\mathbb{Q}$. We are done if $K = \sigma(K)$, so assume the contrary. Then the Galois group is a transitive permutation group on four letters which has a central element of order 2 (as L is CM). The result follows by considering the centralisers of a 2-cycle (it is the Klein 4-group) and of a permutation of cycle type $(2,2)$ (it is dihedral of order 8).

(ii) Such a K contains three quadratic subextensions that are unramified at p, precisely one of which is totally real; conversely any such K is obtained as the compositum of two quadratic fields that are unramified at p, precisely one of which is totally real. It is easy to see that the three cases correspond respectively to: (a) all three quadratic fields are split at p, (b) only the totally real quadratic field is split it p, (c) only one of the imaginary quadratic fields is split at p. The claim is now obvious.

(iii) Here is one way of doing it, obtaining a K which is ramified at at most two rational primes. A proof as in part (iv) would also work. Suppose first that $p > 2$. Using Dirichlet's theorem on primes in arithmetic progressions, we find primes l, l' such that $l \equiv 5 \pmod 8$, $l' \equiv 1 \pmod 8$ and both are quadratic non-residues mod p. Choose generators ω of $\mathbb{F}_l^\times$ and ω' of $\mathbb{F}_{l'}^\times$. For any integer a consider the following map

$$\begin{aligned} \phi : \widehat{\mathbb{Z}}^\times \cong \prod \mathbb{Z}_\ell^\times &\twoheadrightarrow \mathbb{Z}/4 \\ (n_\ell) &\mapsto \log_\omega(\bar{n}_l) + a \log_{\omega'}(\bar{n}_{l'}) \bmod 4. \end{aligned}$$

Note that $\phi(-1) \neq 0$. Thus by "class field theory", $\ker \phi$ corresponds to a cyclic degree 4 CM field. The choice of a controls whether it is of type (a), (b) or (c). For $p = 2$, it is not hard to see that there is unique such K with (a) $K \subset \mathbb{Q}(\mu_{3 \cdot 17})$, (b) $K \subset \mathbb{Q}(\mu_{7 \cdot 17})$, (c) $K = \mathbb{Q}(\mu_5)$. (Explicitly, in (a), $K = \mathbb{Q}(\sqrt{3(-17+4\sqrt{17})}$, and in (b), $K = \mathbb{Q}(\sqrt{7(-17+4\sqrt{17})}$.)

(iv) Consider $K = \mathbb{Q}(\sqrt{a+b\sqrt{d}})$ with integers a, b, d, with normal closure (over $\mathbb{Q}$) denoted by L. Suppose that $a < 0$ and that $d > 0$ and $a^2 - b^2 d > 0$ lie in different (non-trivial) square classes of $\mathbb{Q}^\times$. In that case K is a quartic CM field with dihedral Galois group. For, K is a totally complex quadratic extension of $\mathbb{Q}(\sqrt{d})$, a totally real quadratic field. Moreover, $K/\mathbb{Q}$ is not Galois, as it would otherwise contain a square root of $(a+b\sqrt{d})(a-b\sqrt{d}) = a^2 - b^2 d > 0$, which is ruled out by the assumptions.

Note that cases (c) and (d) are equivalent, upon interchanging K and $\tau(K)$, where τ is any element in the Galois group of order 4.

First suppose that p is odd. In addition to requiring $a < 0$, $a^2 - b^2 d > 0$ and $d > 1$ with d square-free, also impose that:

- $a \equiv d \equiv 1$, $b \equiv 0 \pmod p$ and $d \nmid a$ in case (a),
- $\left(\frac{a}{p}\right) = -1$, $d \equiv 1$, $b \equiv 0 \pmod p$ and $d \nmid a$ in case (b),
- $\left(\frac{2a-1}{p}\right) = -1$, $d \equiv 1$, $b \equiv a - 1 \pmod p$ in case (c),

- $\circ$ $\left(\frac{d}{p}\right) = \left(\frac{a^2-b^2d}{p}\right) = -1$ and $d \nmid a$ in case (e).

In the first three cases it is easy to choose a, b, d (d could always taken to be prime, for example). In the fourth case, choose d first with $\left(\frac{d}{p}\right) = -1$, $\left(\frac{d-1}{p}\right) = 1$. Then $a \equiv d$, $b \equiv 1 \pmod p$ will work.

Clearly, the conditions ensure that $a^2 - b^2d$ and d lie in different non-trivial square classes. The corresponding CM field K is unramified at p, as d and $a^2 - b^2d$ are prime to p. In the first two cases, L^+ is split at p, as $\left(\frac{d}{p}\right) = \left(\frac{a^2-b^2d}{p}\right) = 1$. Moreover $\mathbb{Q}_p(\sqrt{a+b\sqrt{d}}) = \mathbb{Q}_p$ in the first, but not the second, case as the reduction mod p of $a + b\sqrt{d}$ is a square, resp. a non-square, in $\mathbb{F}_p^\times$. Thus K is as required in the first two cases. In the third case, K^+ is split at p whereas the other two quadratic subfields of L are inert at p, establishing that K is as in (c). The fourth case is similar with $\mathbb{Q}(\sqrt{d(a^2-b^2d)})$ split at p and the other two quadratic subfields of L inert at p, once we see that the quadratic field just mentioned is indeed the one fixed by the elements of order 4 in G. As $L = \mathbb{Q}(\alpha, \alpha')$ with $\alpha = \sqrt{a+b\sqrt{d}}$, $\alpha' = \sqrt{a-b\sqrt{d}}$, any element of G is determined by its action on α and α'. The conjugates of α are $S = \{\pm\alpha, \pm\alpha'\}$. Given s_1, $s_2 \in S$, $s_1 \neq \pm s_2$ there is a $\tau \in G$ such that $\tau(\alpha) = s_1$ and $\tau(\alpha') = s_2$ (as $\#G = 8$). Thus an element of order 4 in G is given by τ with $\tau(\alpha) = \alpha'$, $\tau(\alpha') = -\alpha$. In particular, $\tau(\sqrt{d}) = -\sqrt{d}$ and hence τ fixes $\alpha\alpha'\sqrt{d}$, as required.

If $p = 2$, it is not hard to check that the following work instead:

- $\circ$ $a \equiv d \equiv 1$, $b \equiv 0 \pmod 8$, $d \nmid a$ in case (a),
- $\circ$ $a \equiv 5$, $d \equiv 1$, $b \equiv 0 \pmod 8$, $d \nmid a$ in case (b),
- $\circ$ a odd, $b \equiv 2 \pmod 4$, $d \equiv 1 \pmod 8$ in case (c),
- $\circ$ a odd, $b \equiv 2 \pmod 4$, $d \equiv 5 \pmod 8$, $d \nmid a$ in case (e),

in addition to requiring $a < 0$, $a^2 - b^2d > 0$ and $d > 1$ with d square-free. $\square$

14. Weights in Serre's Conjecture for Hilbert modular forms

In [BDJ], Buzzard, Diamond and Jarvis formulate a version of Serre's conjecture for Hilbert modular forms. Theorem 14.2 below will show that their weight conjecture in the tame case is related, via an operation on the Serre weights analogous to $\mathcal{R}$ in section 6, to the decomposition of irreducible representations of $GL_2(\mathbb{F})$ over $\overline{\mathbb{Q}}_p$ when reduced mod p (where $\mathbb{F}$ is a finite field of characteristic p).

Suppose that K is a totally real number field *unramified at* p and $\rho : G_K \to GL_2(\overline{\mathbb{F}}_p)$ is a (continuous) irreducible, totally odd representation. A *Serre weight* in this context is an isomorphism class of irreducible representations of $GL_2(\mathcal{O}_K/p) \cong \prod_{\mathfrak{p}|p} GL_2(k_\mathfrak{p})$ over $\overline{\mathbb{F}}_p$ where $k_\mathfrak{p}$ is the residue field of K at $\mathfrak{p}$. Any such representation is isomorphic to $\bigotimes_{\mathfrak{p}|p} W_\mathfrak{p}$ with $W_\mathfrak{p}$ an irreducible representation of $GL_2(k_\mathfrak{p})$. The weight conjecture in [BDJ] defines these components independently of one another in terms of $\rho|_{I_\mathfrak{p}}$. Let us therefore restrict our attention to a single prime dividing p.

Fix an embedding $\overline{K} \to \overline{\mathbb{Q}}_p$, determining a place $\mathfrak{p}|p$ of K. Let $I_\mathfrak{p}$ denote the corresponding inertia subgroup. Let $k'_\mathfrak{p}$ be the quadratic intermediate field of $k_\mathfrak{p} \to \overline{\mathbb{F}}_p$. Then there are canonical fundamental tame characters $\psi : I_\mathfrak{p} \twoheadrightarrow k_\mathfrak{p}^\times$ and $\psi' : I_\mathfrak{p} \twoheadrightarrow (k'_\mathfrak{p})^\times$.

Let $f = [k_\mathfrak{p} : \mathbb{F}_p]$. For $i \in \mathbb{Z}/f$, let λ_i be the p^i-th power of $k_\mathfrak{p}^\times \to \overline{\mathbb{F}}_p^\times$ induced by the embedding above and for $i \in \mathbb{Z}/2f$ let $\lambda_{i'}$ be the p^i-th power of $(k'_\mathfrak{p})^\times \to \overline{\mathbb{F}}_p^\times$ induced by the above embedding.

Also let $\psi_i = \lambda_i \circ \psi$ (resp. $\psi_{i'} = \lambda_{i'} \circ \psi'$) be the fundamental tame level f (resp. $2f$) characters of $I_\mathfrak{p}$.

To describe the set $W_{Ser,\mathfrak{p}}$ of isomorphism classes of irreducible representations of $GL_2(k_\mathfrak{p})$ over $k_\mathfrak{p}$ (*Serre weights at* $\mathfrak{p}$), note first that theorem 4.6 together with remark 4.7 show that

$$W_{Ser,\mathfrak{p}} = \{F(a,b) : 0 \le a-b \le p^f-1, 0 \le b < p^f-1\}.$$

If we write $a-b = \sum_{i=0}^{f-1} m_i p^i$, $b = \sum_{i=0}^{f-1} b_i p^i$ with $0 \le m_i,\ b_i \le p-1$ then by the Steinberg tensor product theorem (thm. 4.5),

$$F(a,b) \cong \bigotimes_{i=0}^{f-1} F(b_i+m_i, b_i)^{(p^i)}.$$

Since $F(b_i+m_i,b_i) \cong \mathrm{Sym}^{m_i} \otimes \det^{b_i}$,

$$F(a,b) \cong \bigotimes_{i=0}^{f-1} (\mathrm{Sym}^{m_i} k_\mathfrak{p}^2 \otimes \det{}^{b_i}) \otimes_{k_\mathfrak{p},\phi^i} k_\mathfrak{p}$$

where ϕ is the Frobenius element. This representation will also be denoted by $F_{\vec{m},\vec{b}}$.

Suppose that $\rho : G_K \to GL_2(\overline{\mathbb{F}}_p)$ is (continuous) irreducible, totally odd and assume that it is *tame at* $\mathfrak{p}$. Then we can write $\rho|_{I_\mathfrak{p}} \cong \chi_1 \oplus \chi_2$. We say that $\rho|_{I_\mathfrak{p}}$ is of *niveau 1* if $\chi_i^{p^f-1} = 1$ $(i = 1, 2)$ and of *niveau 2* otherwise. Let us recall the definition of the conjectured set of weights from [BDJ] in the tame case. If $\rho|_{I_\mathfrak{p}}$ is of niveau 1 then $W_{conj,\mathfrak{p}}(\rho)$ consists of all $F_{\vec{m},\vec{b}}$ such that

$$\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \prod_J \psi_i^{m_i+1} & \\ & \prod_{J^c} \psi_i^{m_i+1} \end{pmatrix} \prod \psi_i^{b_i} \tag{14.1}$$

for some $J \subset \mathbb{Z}/f$. If $\rho|_{I_\mathfrak{p}}$ is of niveau 2 then $W_{conj,\mathfrak{p}}(\rho)$ consists of all $F_{\vec{m},\vec{b}}$ such that

$$\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \prod_J \psi_{i'}^{m_i+1} & \\ & \prod_{J^c} \psi_{i'}^{m_i+1} \end{pmatrix} \prod \psi_i^{b_i}$$

for some $J \subset \mathbb{Z}/2f$ projecting *bijectively* onto $\mathbb{Z}/f$ (under the natural map). Here we are abusing notation: $m_i = m_{i \bmod f}$ and $b_i = b_{i \bmod f}$.

Associated to each $\rho|_{I_\mathfrak{p}}$ define a representation $V_\mathfrak{p}(\rho|_{I_\mathfrak{p}})$ of $GL_2(k_\mathfrak{p})$ over $\overline{\mathbb{Q}}_p$. The Teichmüller lift will again be denoted by $\tilde{}$. For characters $\chi_i : k_\mathfrak{p}^\times \to \overline{\mathbb{Q}}_p^\times$, $I(\chi_1, \chi_2)$ will denote the induction from the Borel subgroup of upper-triangular matrices to $GL_2(k_\mathfrak{p})$ of $\chi_1 \otimes \chi_2$, whereas for a character $\chi : (k'_\mathfrak{p})^\times \to \overline{\mathbb{Q}}_p^\times$ which does not factor through the norm $(k'_\mathfrak{p})^\times \to k_\mathfrak{p}^\times$, the cuspidal representation $\Theta(\chi)$ of $GL_2(k_\mathfrak{p})$ was defined in §2.1:

Definition 14.1.

(i) *If* $\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \prod \psi_i^{m_i} & \\ & \prod \psi_i^{n_i} \end{pmatrix} \prod \psi_i^{c_i}$ *is of niveau 1,*

$$V_\mathfrak{p}(\rho|_{I_\mathfrak{p}}) = I(\textstyle\prod \tilde{\lambda}_i^{m_i}, \prod \tilde{\lambda}_i^{n_i}) \otimes \prod \tilde{\lambda}_i^{c_i}.$$

(ii) *If* $\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \prod \psi_{i'}^{m_i} & \\ & \prod \psi_{i'}^{p^f m_i} \end{pmatrix} \prod \psi_i^{c_i}$ *is of niveau 2,*

$$V_\mathfrak{p}(\rho|_{I_\mathfrak{p}}) = \Theta(\textstyle\prod \tilde{\lambda}_{i'}^{m_i}) \otimes \prod \tilde{\lambda}_i^{c_i}.$$

Note that in (ii), in the products involving $\psi_{i'}$ or $\lambda_{i'}$, i runs over $\mathbb{Z}/2f$, whereas in the other products i runs through $\mathbb{Z}/f$. In particular, $V_\mathfrak{p}(\rho|_{I_\mathfrak{p}} \otimes (\chi \circ \psi)) \cong V_\mathfrak{p}(\rho|_{I_\mathfrak{p}}) \otimes \tilde{\chi}$ for any character $\chi : k_\mathfrak{p}^\times \to \overline{\mathbb{F}}_p^\times$.

A *regular Serre weight at* $\mathfrak{p}$ is any Serre weight $F_{\vec{m},\vec{b}}$ with $0 \le m_i < p-1$ for all i. The set of regular Serre weights at $\mathfrak{p}$ is denoted by $W_{reg,\mathfrak{p}}$. Define $\mathcal{R}_\mathfrak{p} : W_{reg,\mathfrak{p}} \to W_{reg,\mathfrak{p}}$ by

$$\mathcal{R}_\mathfrak{p}(F(a,b)) = F(b + (p-2)\textstyle\sum_{i=0}^{f-1} p^i, a),$$

(compare this with (6.1) on page 24 in the GL_3-case).

Theorem 14.2. *Suppose that* $\rho : G_K \to GL_2(\overline{\mathbb{F}}_p)$ *is continuous, irreducible, totally odd, and tame at* $\mathfrak{p}$.

(i) $W_{conj,\mathfrak{p}}(\rho) \cap W_{reg,\mathfrak{p}} = \mathcal{R}_\mathfrak{p}(JH(\overline{V_\mathfrak{p}(\rho|_{I_\mathfrak{p}})}) \cap W_{reg,\mathfrak{p}})$.

(ii) *There is a* multi-valued *function* $\mathcal{R}_{ext,\mathfrak{p}} : W_{Ser,\mathfrak{p}} \to W_{Ser,\mathfrak{p}}$ *that extends* $\mathcal{R}_\mathfrak{p}$ *such that*

$$W_{conj,\mathfrak{p}}(\rho) = \mathcal{R}_{ext,\mathfrak{p}}(JH(\overline{V_\mathfrak{p}(\rho|_{I_\mathfrak{p}})})).$$

The following definition of $\mathcal{R}_{ext,\mathfrak{p}}$ will be shown to satisfy part (ii) of the theorem. Suppose $F = F(a,b)$, $0 \le a - b \le p^f - 1$. Express $a - b$ as $\sum_{i=0}^{f-1} m_i p^i$ $(0 \le m_i \le p-1)$. Define a collection $\mathcal{S}(F)$ of subsets of $\mathbb{Z}/f$ by: $S \in \mathcal{S}(F)$ if and only if for all $s \in S$, $m_s \ne 0$ and there is an i such that $m_i = p-1$, $m_{i+1} = \cdots = m_{s-1} = p-2$ and $\{i, i+1, \dots, s-1\} \cap S = \varnothing$. Then $\mathcal{R}_{ext,\mathfrak{p}}(F)$ is defined to be

$$\{F(a',b') : \exists S \in \mathcal{S}(F), a' \equiv b - \sum_{i \notin S} p^i,\ b' \equiv a - \sum_{i \in S} p^i \pmod{p^f - 1}\}.$$

In particular, for this choice of $\mathcal{R}_{ext,\mathfrak{p}}$, if F is a regular Serre weight then $\mathcal{S}(F)=\{\varnothing\}$ and $\mathcal{R}_{\mathfrak{p}}(F)=\mathcal{R}_{ext,\mathfrak{p}}(F)$ unless F is a twist of $F((p-2)\sum p^i,0)$ in which case $\mathcal{R}_{ext,\mathfrak{p}}(F)$ contains one more weight.

The proof will require several lemmas, proved below.

Lemma 14.3. *Suppose that $0\le m_i\le p-1$ $(i\in\mathbb{Z}/f)$.*

(i) *Suppose that $\rho|_{I_\mathfrak{p}}$ is of niveau 1. Then $F_{\vec{m},\vec{b}}$ is a constituent of $\overline{V_\mathfrak{p}(\rho|_{I_\mathfrak{p}})}$ if and only if*

$$\rho|_{I_\mathfrak{p}}\sim\begin{pmatrix}\prod_{J^c}\psi_i^{p-1-m_i} & \\ & \prod_J\psi_i^{p-1-m_i}\end{pmatrix}\prod\psi_i^{m_i+b_i}$$

for some $J\subset\mathbb{Z}/f$.

(ii) *Suppose that $\rho|_{I_\mathfrak{p}}$ is of niveau 2. Then $F_{\vec{m},\vec{b}}$ is a constituent of $\overline{V_\mathfrak{p}(\rho|_{I_\mathfrak{p}})}$ if and only if*

$$\rho|_{I_\mathfrak{p}}\sim\begin{pmatrix}\prod_{J^c}\psi_{i'}^{p-1-m_i} & \\ & \prod_J\psi_{i'}^{p-1-m_i}\end{pmatrix}\prod\psi_i^{m_i+b_i}$$

for some $J\subset\mathbb{Z}/2f$ projecting bijectively onto $\mathbb{Z}/f$.

Let me explain the idea of the proof of the theorem. The above lemma is the key tool that lets us relate the conjectured weight set $W_{conj,\mathfrak{p}}(\rho)$ with the decomposition of $\overline{V_\mathfrak{p}(\rho|_{I_\mathfrak{p}})}$. This works perfectly for regular Serre weights. In general the problem is that the number of constituents of $\overline{V_\mathfrak{p}(\rho|_{I_\mathfrak{p}})}$ might be a lot smaller than $\#W_{conj,\mathfrak{p}}(\rho)$. This suggests looking for a multi-valued function extending $\mathcal{R}$. In view of lemma 14.3, we have to find rules to convert an expression of the form

$$\rho|_{I_\mathfrak{p}}\sim\begin{pmatrix}\prod_J\psi_i^{\alpha(i)} & \\ & \prod_{J^c}\psi_i^{\alpha(i)}\end{pmatrix}\chi$$

for some $J\subset\mathbb{Z}/f$, $0\le\alpha(i)\le p-1$ and some character χ into an expression of the form

$$\rho|_{I_\mathfrak{p}}\sim\begin{pmatrix}\prod_L\psi_i^{\beta(i)} & \\ & \prod_{L^c}\psi_i^{\beta(i)}\end{pmatrix}\chi'$$

for some $L\subset\mathbb{Z}/f$, $1\le\beta(i)\le p$ and some character χ' in such a way that the map

$$(\alpha,\chi)\mapsto(\beta,\chi')$$

does not depend on J and works equally well for the analogous expressions of niveau 2. The theorem shows, roughly speaking, that there are enough such rules to explain all of $W_{conj,\mathfrak{p}}(\rho)$.

To make this principle concrete, consider $f=3$ and $\vec{\alpha}=(0,1,p-1)$ and $\chi=1$. It is very instructive to check that there are such rules giving rise to the following pairs (β,χ'):

$$((p,p,p-2),1),\ ((p,2,p-1),\psi_1^{-1}),\ ((p,p,p),\psi_2^{-1}).$$

For example, here are two instances of the second rule:

$$\begin{pmatrix} \psi_1 & \\ & \psi_2^{p-1} \end{pmatrix} \sim \begin{pmatrix} \psi_1^2 & \\ & \psi_0^p \psi_2^{p-1} \end{pmatrix} \psi_1^{-1}$$

and

$$\begin{pmatrix} \psi_{1'} \psi_{2'}^{p-1} & \\ & \psi_{4'} \psi_{5'}^{p-1} \end{pmatrix} \sim \begin{pmatrix} \psi_{3'}^p \psi_{1'}^2 \psi_{2'}^{p-1} & \\ & \psi_{0'}^p \psi_{4'}^2 \psi_{5'}^{p-1} \end{pmatrix} \psi_1^{-1}.$$

In the end, these rules consist of multiple uses of the identity

$$\psi_{j+1} = \psi_i^p \psi_{i+1}^{p-1} \cdots \psi_j^{p-1}$$

when $\alpha(i) = \cdots = \alpha(j) = 0$ ($\alpha(i) = 1$ is allowed if ψ_i is itself to be expanded in this manner!). Of course this works equally well for $\psi_{(j+1)'}$. To compare with the formalism below, let us indicate in each case the corresponding choice of $\mathcal{I}$:

$$\underline{0},_{-}\underline{1},_{-}p-1, \quad \underline{0},_{+}1,\ p-1, \quad \underline{0},_{-}\underline{1},_{+}p-1.$$

Note that the last of these is not covered by the $\mathcal{R}_{ext,\mathfrak{p}}$ we defined above. In fact, it is not hard to see that axiom A4 below could be weakened to:

- A4′ If an $\mathcal{I}$-interval is positive, its successor does not lie in any $\mathcal{I}$-interval.

This corresponds to removing the condition $m_s \neq 0$ in the definition of $\mathcal{R}_{ext,\mathfrak{p}}$ above. If we denote this modified version of $\mathcal{R}_{ext,\mathfrak{p}}$ by $\mathcal{R}'_{ext,\mathfrak{p}}$ then it is clear that *any* multi-valued function between $\mathcal{R}_{ext,\mathfrak{p}}$ and $\mathcal{R}'_{ext,\mathfrak{p}}$ (i.e., such that there is a containment pointwise) satisfies thm. 14.2(ii).

* * *

For our purposes, an *interval* in $\mathbb{Z}/f$ is any stretch of numbers $[\![i,j]\!] = \{i, i+1, \ldots, j\}$ in $\mathbb{Z}/f$. That is, the start and end points are remembered. For example, $[\![0, p-1]\!] \neq [\![1, 0]\!]$, even though the underlying sets are the same. The *successor* of an interval $[\![i,j]\!]$ is $j+1$.

Suppose that α is a function $\mathbb{Z}/f \to \{0, 1, \ldots, p-1\}$, and suppose that $\mathcal{I}$ a collection of disjoint intervals I in $\mathbb{Z}/f$, each labelled with a sign (thought of as pertaining to the entry following that interval). Define the set $\mathcal{L}_{[0,p-1]}$ to consist of all $(\alpha, \mathcal{I})$ which satisfy the following rules:

- A1 For each interval $I \in \mathcal{I}$, $\alpha(I) \subset \{0, 1\}$.
- A2 If $i \in \bigcup \mathcal{I}$ then $\alpha(i) = 1$ if and only if i is start point of an $\mathcal{I}$-interval and $i - 1 \in \bigcup \mathcal{I}$.
- A3 If $i \notin \bigcup \mathcal{I}$ and $\alpha(i) = 0$, then $i - 1 \in \bigcup \mathcal{I}$.
- A4 If an $\mathcal{I}$-interval is positive, its successor does not lie in any $\mathcal{I}$-interval and has α-value in $[0, p-2]$.
- A5 If an $\mathcal{I}$-interval is negative, its successor lies in another $\mathcal{I}$-interval or has α-value in $[2, p-1]$.

Note that every function $\alpha : \mathbb{Z}/f \to \{0, 1, \ldots, p-1\}$ can be equipped with intervals and signs satisfying these rules.

Similarly, suppose that β is a function $\mathbb{Z}/f \to \{1,2,\dots,p\}$, and suppose that $\mathcal{I}$ a collection of disjoint intervals in $\mathbb{Z}/f$, each labelled with a sign (thought of as pertaining to the entry following that interval). Define the set $\mathcal{L}_{[1,p]}$ to consist of all $(\beta,\mathcal{I})$ which satisfy the following rules:

B1 For each interval $I \in \mathcal{I}$, $\beta(I) \subset \{p-1,p\}$.

B2 The set of start points of $\mathcal{I}$-intervals is $\beta^{-1}(p)$.

B3 If an $\mathcal{I}$-interval is positive, its successor does not lie in any $\mathcal{I}$-interval and has β-value in $[1,p-1]$.

B4 If an $\mathcal{I}$-interval is negative, its successor lies in another $\mathcal{I}$-interval or has β-value in $[1,p-2]$.

Note that every function $\beta : \mathbb{Z}/f \to \{1,2,\dots,p\}$ can be equipped with intervals and signs satisfying these rules.

To define a map $\phi : \mathcal{L}_{[0,p-1]} \to \mathcal{L}_{[1,p]}$, represent α as the string of numbers $\alpha(0), \alpha(1), \dots, \alpha(f-1)$; underline each $\mathcal{I}$-interval and put the corresponding sign just after the last entry of the interval. In this way the function ϕ has the following effect on each interval and its successor (it leaves all other entries unchanged):

$$\underline{(1),0,\dots,0,}_{\pm} a,\dots \mapsto \underline{p,p-1,\dots,p-1,}_{\pm} a \pm 1,\dots$$
$$\underline{\dots,0,0,}_{-}\underline{1,0,\dots} \mapsto \underline{\dots,p-1,p-1,}_{-}\underline{p,p-1,\dots}$$

Lemma 14.4. *The map ϕ is well defined and in fact a bijection.*

Lemma 14.5. *Suppose that $\alpha : \mathbb{Z}/f \to \{0,1,\dots,p-1\}$. Then the following are equivalent for a subset $S \subset \mathbb{Z}/f$:*

(i) *$S \in \mathcal{S}(F_{\vec{p}-\vec{1}-\vec{\alpha},\vec{x}})$ for some $\vec{x}$.*

(ii) *$S \in \mathcal{S}(F_{\vec{p}-\vec{1}-\vec{\alpha},\vec{x}})$ for all $\vec{x}$.*

(iii) *S is the set of successors of positive intervals in $\mathcal{I}$ for some $\mathcal{I}$ with $(\alpha,\mathcal{I}) \in \mathcal{L}_{[0,p-1]}$.*

Proof of the theorem. (i) This is a straightforward application of lemma 14.3. First consider the niveau 1 case. Suppose $F \in W_{conj,\mathfrak{p}}(\rho)$ and F regular. By twisting, we can assume without loss of generality that $F = F_{\vec{b}-\vec{1},\vec{0}}$ $(1 \le b_i \le p-1)$ and

$$\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \prod_J \psi_i^{b_i} & \\ & \prod_{J^c} \psi_i^{b_i} \end{pmatrix}$$

for some $J \subset \mathbb{Z}/f$. By lemma 14.3, the regular Serre weight $F_{\vec{p}-\vec{1}-\vec{b},\vec{b}}$ is a constituent of $\overline{V_\mathfrak{p}(\rho|_{I_\mathfrak{p}})}$. Applying $\mathcal{R}_\mathfrak{p}$ produces $F_{\vec{b}-\vec{1},\vec{0}}$. Reversing the argument yields the other inclusion.

The niveau 2 case works exactly the same way.

(ii) *Step 1: Show that $\mathcal{R}_{ext,\mathfrak{p}}(F) \subset W_{conj,\mathfrak{p}}(\rho)$ if F is a constituent of $\overline{V_\mathfrak{p}(\rho|_{I_\mathfrak{p}})}$.*

Without loss of generality (twisting ρ and F) we may assume that $F = F_{\vec{m},\vec{0}}$ $(0 \le m_i \le p-1)$. If $\rho|_{I_\mathfrak{p}}$ *has niveau 1*, then by lemma 14.3 we can write

$$\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \prod_J \psi_i^{p-1-m_i} & \\ & \prod_{J^c} \psi_i^{p-1-m_i} \end{pmatrix} \prod \psi_i^{m_i} \tag{14.2}$$

for some subset $J \subset \mathbb{Z}/f$. Define $\alpha : \mathbb{Z}/f \to \{0, 1, \dots, p-1\}$, $i \mapsto p-1-m_i$. Given $S \in \mathcal{S}(F)$, we can by lemma 14.5 choose a collection $\mathcal{I}$ of signed intervals such that $(\alpha, \mathcal{I}) \in \mathcal{L}_{[0,p-1]}$ and S is the set of successors of positive $\mathcal{I}$-intervals. Let J_+ (resp. J_-) denote those elements of J that succeed positive (resp. negative) intervals of $\mathcal{I}$. Similarly define J_+^c and J_-^c. Let J_0 (resp. J_0^c) denote those elements of J (resp. J^c) that do not lie in any interval of $\mathcal{I}$. Note that $S = J_+ \cup J_+^c$. Then

$$\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix} \prod_S \psi_i^{-1} \prod_i \psi_i^{m_i} \tag{14.3}$$

where

$$\chi_1 = \prod_{J_+} \psi_i^{\alpha(i)+1} \prod_{J_0 \setminus (J_+ \cup J_-)} \psi_i^{\alpha(i)} \prod_{J_0 \cap J_-} \psi_i^{\alpha(i)-1} \prod_{\substack{j+1 \in J_- \cup J_+^c \\ [\![i,j]\!] \in \mathcal{I}}} (\psi_i^p \psi_{i+1}^{p-1} \cdots \psi_j^{p-1})$$

and χ_2 is obtained by interchanging the roles of J and J^c. Note that each ψ_i appears with non-zero exponent in precisely one of χ_1, χ_2 (the way they are expressed here); call this non-zero exponent $\beta(i)$. It is not hard to see that $\phi(\alpha, \mathcal{I}) = (\beta, \mathcal{I})$. Thus

$$\chi_1 = \prod_L \psi_i^{\beta(i)}, \ \chi_2 = \prod_{L^c} \psi_i^{\beta(i)}$$

for some $L \subset \mathbb{Z}/f$ and all exponents $\beta(i)$ are in $[1, p]$, so (14.3) gives rise to a Serre weight $F(A, B) \in W_{conj,\mathfrak{p}}(\rho)$ (by (14.1)). Combining equations (14.2) and (14.3) we find that

$$\det(\rho|_{I_\mathfrak{p}} \cdot \prod \psi_i^{-m_i}) = \psi_0^{-\sum m_i p^i} = \psi_0^{\sum (\beta(i) - 2 \cdot 1_S(i)) p^i}.$$

Using this, we easily see that $F(A, B)$ satisfies

$$A \equiv -\sum_{S^c} p^i, \ B \equiv \sum m_i p^i - \sum_S p^i \pmod{p^f - 1}.$$

We are done except for showing that any other weight $F(A', B')$ satisfying these congruences is in the conjectured weight set. But these congruences determine $F(A, B)$ except for the pairs $\{F(x, x), F(x + p^f - 1, x)\}$ and for all x, $F(x, x) \in W_{conj,\mathfrak{p}}(\rho)$ if and only if $F(x + p^f - 1, x) \in W_{conj,\mathfrak{p}}(\rho)$ (this follows directly from the definition). Therefore $\mathcal{R}_{ext,\mathfrak{p}}(F) \subset W_{conj,\mathfrak{p}}(\rho)$.

If $\rho|_{I_\mathfrak{p}}$ *has niveau 2*, then

$$\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \prod_J \psi_{i'}^{p-1-m_i} & \\ & \prod_{J^c} \psi_{i'}^{p-1-m_i} \end{pmatrix} \prod \psi_i^{m_i}$$

for some $J \subset \mathbb{Z}/2f$ projecting bijectively onto $\mathbb{Z}/f$. The argument is now formally identical to the niveau 1 case provided we replace each ψ_i by $\psi_{i'}$

and "$[\![i,j]\!] \in \mathcal{I}$" in the subscript of the right-most product in the expression for χ_1 by "$[\![i,j]\!] \in \widetilde{\mathcal{I}}$", where $\widetilde{\mathcal{I}}$ is the set of intervals in $\mathbb{Z}/2f$ which project bijectively onto the $\mathcal{I}$-intervals in $\mathbb{Z}/f$.

Step 2: Show that all weights F in $W_{conj,\mathfrak{p}}(\rho)$ are obtained in this way.

If $\rho|_{I_\mathfrak{p}}$ *has niveau 1*, then we can twist by characters and assume without loss of generality that $F = F_{\vec{\beta}-\vec{1},\vec{0}}$ $(1 \le \beta(i) \le p)$ and

$$\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \prod_L \psi_i^{\beta(i)} & \\ & \prod_{L^c} \psi_i^{\beta(i)} \end{pmatrix} \tag{14.4}$$

for some $L \subset \mathbb{Z}/f$. Define a collection $\mathcal{I}$ of disjoint signed intervals in $\mathbb{Z}/f$ which is in bijection with $\beta^{-1}(p)$, as follows. Whenever $\beta(i) = p$ and $i \in L$ (resp. L^c) choose j such that all numbers in $\beta([\![i,j]\!]-\{i\}) \subset \{p-1\}$, $[\![i,j]\!] \subset L$ (resp. L^c) and j is maximal with respect to these properties (i.e., j cannot be replaced by $j+1$). In that case $[\![i,j]\!]$ is the $\mathcal{I}$-interval corresponding to $i \in \beta^{-1}(p)$. We let it be negative if and only if $\beta(j+1) = p$ or $j+1 \in L$ (resp. L^c). Observe that $(\beta, \mathcal{I}) \in \mathcal{L}_{[1,p]}$.

Let Σ_L (resp. Σ_{L^c}) be the set of successors of $\mathcal{I}$-intervals contained in L (resp. L^c). The notations L_0, L_0^c have the same meaning as in the previous part. Note that $S = \Sigma_L \cap L_0^c \cup \Sigma_{L^c} \cap L_0$ is the set of successors of positive $\mathcal{I}$-intervals. We see that

$$\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix} \prod_S \psi_i \tag{14.5}$$

where

$$\chi_1 = \prod_{L_0 \cap \Sigma_{L^c}} \psi_i^{\beta(i)-1} \prod_{L_0 \setminus (\Sigma_L \cup \Sigma_{L^c})} \psi_i^{\beta(i)} \prod_{L_0 \cap \Sigma_L} \psi_i^{\beta(i)+1} \prod_{\Sigma_L \setminus (L_0 \cup L_0^c)} \psi_i$$

and χ_2 is obtained by interchanging the roles of L and L^c. Every ψ_i occurs with a non-zero exponent in at most one of χ_1, χ_2 (the way they are expressed here); call this exponent $\alpha(i) \in \{0, 1, \dots, p-1\}$. By lemma 14.3, taking into account the twist, this decomposition shows that $F' = F_{\vec{p}-\vec{1}-\vec{\alpha},\vec{\alpha}+\vec{1_S}}$ is a constituent of $\overline{V_\mathfrak{p}(\rho|_{I_\mathfrak{p}})}$ (here 1_S is the characteristic function of S).

It is not hard to see that $\phi^{-1}(\beta, \mathcal{I}) = (\alpha, \mathcal{I})$. In particular, by lemma 14.5 $S \in \mathcal{S}(F')$. Equations (14.4), (14.5) yield

$$\det(\rho|_{I_\mathfrak{p}}) = \psi_0^{\sum(\alpha(i)+2\cdot 1_S(i))p^i} = \psi_0^{\sum \beta(i)p^i}.$$

We see that the weight in $\mathcal{R}_{ext,\mathfrak{p}}(F')$ corresponding to $S \in \mathcal{S}(F')$ is $F_{\vec{\beta}-\vec{1},\vec{0}} = F$, and we are done.

If $\rho|_{I_\mathfrak{p}}$ *has niveau 2*, the argument is completely analogous (as in Step 1). □

Proof of lemma 14.3. (i) First let us show the implication "⇒". Without loss of generality,

$$\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \prod \psi_i^{n_i} & \\ & 1 \end{pmatrix}$$

for some $0 \le n_i \le p-1$. By [Dia], prop. 1.1, the constituents of $\overline{V_\mathfrak{p}(\rho|_{I_\mathfrak{p}})}$ are the $F_{\vec{c}_J,\vec{d}_J}$ where $J \subset \mathbb{Z}/f$ and

$$c_{J,i} = \begin{cases} n_i + \delta_J(i) - 1 & \text{if } i \in J \\ p - 1 - n_i - \delta_J(i) & \text{if } i \notin J \end{cases}$$

$$d_{J,i} = \begin{cases} 0 & \text{if } i \in J \\ n_i + \delta_J(i) & \text{if } i \notin J \end{cases}$$

where δ_J is the characteristic function of $\{i+1 : i \in J\}$. Also, the convention is that $F_{\vec{c}_J,\vec{d}_J} = (0)$ if $c_{J,i} = -1$ for some i. Now note that

$$\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \prod_{J^c} \psi_i^{n_i+\delta_J(i)} & \\ & \prod_J \psi_i^{p-n_i-\delta_J(i)} \end{pmatrix} \prod_J \psi_i^{n_i+\delta_J(i)-1} \prod_{J^c} \psi_i^{p-1}.$$

Conversely, suppose without loss of generality that $\rho|_{I_\mathfrak{p}}$ is as in the statement of the lemma with $\vec{b} = 0$. Note that whenever $m_i = p-1$ it is irrelevant whether $i \in J$ or not. Thus for all such i we can prescribe whether or not $i \in J$. There is a unique way to alter J in this manner such that for all i with $m_i = p-1$, $i \in J \Leftrightarrow i-1 \in J$ (the latter is equivalent to $\delta_J(i) = 1$). Note that

$$\begin{aligned} V_\mathfrak{p}(\rho|_{I_\mathfrak{p}}) &\cong I(\prod_{J^c} \lambda_i^{p-1-m_i} \prod_J \lambda_i^{m_i+1-p}, 1) \otimes \prod \lambda_i^{m_i} \prod_J \lambda_i^{p-1-m_i} \\ &\cong I(\prod_{J^c} \lambda_i^{p-1-m_i-\delta_J(i)} \prod_J \lambda_i^{m_i+1-\delta_J(i)}, 1) \otimes \prod \lambda_i^{m_i} \prod_J \lambda_i^{p-1-m_i}. \end{aligned}$$

By our choice of J, all exponents of the first character in the induction are contained in $\{0, 1, \dots, p-1\}$. It follows from [Dia], prop. 1.1 (using the same subset J) that $F_{\vec{m},\vec{0}}$ is a constituent of $\overline{V_\mathfrak{p}(\rho|_{I_\mathfrak{p}})}$, as required.

(ii) This works completely analogously, it is only more cumbersome to write out. Note that we can assume $\vec{m} \ne \vec{p} - \vec{1}$ as on the one hand

$$\dim F_{\vec{p}-\vec{1},\vec{b}} = p^f > p^f - 1 = \dim V_\mathfrak{p}(\rho|_{I_\mathfrak{p}})$$

and on the other hand $\rho|_{I_\mathfrak{p}}$ cannot be unramified up to twist (being of niveau 2). □

Proof of lemma 14.4. This is straightforward. □

Proof of lemma 14.5. Note that the first two statements are equivalent, by the definition of $\mathcal{S}(F)$, to

(i′) For all $s \in S$,

(a) $\alpha(s) \ne p - 1$.

(b) There is an $i \in \mathbb{Z}/f$ such that $[\![i, s-1]\!] \cap S = \varnothing$ and $\alpha(i) = 0$, $\alpha(i+1) = \cdots = \alpha(s-1) = 1$.

We will now show that (i′) $\Leftrightarrow$ (iii).

First suppose that $(\alpha, \mathcal{I}) \in \mathcal{L}_{[0,p-1]}$ and let S be the set of successors of positive intervals. Then by property A4 (see p. 70), $\alpha(s) \neq p-1$ if $s \in S$. Moreover, $\alpha(s-1) \in \{0,1\}$ and $s-1 \notin S$ (as $s-1$ is in an interval). If it is 1, by property A2 the preceding entry lies in a different (negative) interval and iterating this process gives the desired interval $[\![i, s-1]\!]$. Note that the process has to stop (i.e., eventually we hit a 0) because $s \in S$ cannot itself lie in an interval (by A4).

Conversely, suppose given S satisfying (i) and (ii). Here is a way to define $\mathcal{I}$ having S as set of successors of positive intervals and such that $(\alpha, \mathcal{I}) \in \mathcal{L}_{[0,p-1]}$ (in fact it is the unique way). It is easier to define $\bigcup \mathcal{I}$ first: we let $i \in \bigcup \mathcal{I}$ if and only if there is a j such that $[\![j, i]\!] \subset S^c$ and $\alpha(j) = 0$, $\alpha(j+1) = \cdots = \alpha(i) = 1$ (in particular, this whole interval will be contained in $\bigcup \mathcal{I}$). We let $i \in \bigcup \mathcal{I}$ be start point of an $\mathcal{I}$-interval if and only if $i-1 \notin \bigcup \mathcal{I}$ or $i-1 \in \bigcup \mathcal{I}$ and $\alpha(i) = 1$. We let an $\mathcal{I}$-interval be positive if and only if its successor is in S.

It is straightforward to see that $(\alpha, \mathcal{I}) \in \mathcal{L}_{[0,p-1]}$; by definition S is the set of successors of positive intervals. $\square$

References

[AC89] James Arthur and Laurent Clozel. *Simple algebras, base change, and the advanced theory of the trace formula*, volume 120 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[ADP02] Avner Ash, Darrin Doud, and David Pollack. Galois representations with conjectural connections to arithmetic cohomology. *Duke Math. J.*, 112(3):521–579, 2002.

[AS86] Avner Ash and Glenn Stevens. Cohomology of arithmetic groups and congruences between systems of Hecke eigenvalues. *J. Reine Angew. Math.*, 365:192–220, 1986.

[AS00] Avner Ash and Warren Sinnott. An analogue of Serre's conjecture for Galois representations and Hecke eigenclasses in the mod p cohomology of $\mathrm{GL}(n, \mathbf{Z})$. *Duke Math. J.*, 105(1):1–24, 2000.

[Ash92] Avner Ash. Galois representations attached to mod p cohomology of $\mathrm{GL}(n, \mathbf{Z})$. *Duke Math. J.*, 65(2):235–255, 1992.

[BDJ] Kevin Buzzard, Fred Diamond, and Frazer Jarvis. On Serre's conjecture for mod l Galois representations over totally real fields. Preliminary version.

[Bre97] Glen E. Bredon. *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[Bum97] Daniel Bump. *Automorphic forms and representations*, volume 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[Car85] Roger W. Carter. *Finite groups of Lie type*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1985. Conjugacy classes and complex characters, A Wiley-Interscience Publication.

[Clo90] Laurent Clozel. Motifs et formes automorphes: applications du principe de fonctorialité. In *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann*

Arbor, MI, 1988), volume 10 of *Perspect. Math.*, pages 77–159. Academic Press, Boston, MA, 1990.

[Dem65] Michel Demazure. Schémas en groupes réductifs. *Bull. Soc. Math. France*, 93:369–413, 1965.

[Dia] Fred Diamond. A correspondence between representations of local Galois groups and Lie-type groups. Preprint.

[DL76] P. Deligne and G. Lusztig. Representations of reductive groups over finite fields. *Ann. of Math. (2)*, 103(1):103–161, 1976.

[Dou] Darrin Doud. Supersingular Galois representations and a generalization of a conjecture of Serre. To appear in Experimental Mathematics.

[Dou02] Darrin Doud. Three-dimensional Galois representations with conjectural connections to arithmetic cohomology. In *Number theory for the millennium, I (Urbana, IL, 2000)*, pages 365–375. A K Peters, Natick, MA, 2002.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[Gee] Toby Gee. On the weights of mod p Hilbert modular forms (with an appendix by David Savitt), arXiv:math.NT/0601516.

[Gre55] J. A. Green. The characters of the finite general linear groups. *Trans. Amer. Math. Soc.*, 80:402–447, 1955.

[Hen86] Guy Henniart. On the local Langlands conjecture for $\mathrm{GL}(n)$: the cyclic case. *Ann. of Math. (2)*, 123(1):145–203, 1986.

[HT01] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[Jac79] Hervé Jacquet. Principal L-functions of the linear group. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 63–86. Amer. Math. Soc., Providence, R.I., 1979.

[Jan74] Jens C. Jantzen. Zur Charakterformel gewisser Darstellungen halbeinfacher Gruppen und Lie-Algebren. *Math. Z.*, 140:127–149, 1974.

[Jan77] Jens C. Jantzen. Über das Dekompositionsverhalten gewisser modularer Darstellungen halbeinfacher Gruppen und ihrer Lie-Algebren. *J. Algebra*, 49(2):441–469, 1977.

[Jan81] Jens Carsten Jantzen. Zur Reduktion modulo p der Charaktere von Deligne und Lusztig. *J. Algebra*, 70(2):452–474, 1981.

[Jan87] J. C. Jantzen. Representations of Chevalley groups in their own characteristic. In *The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)*, volume 47 of *Proc. Sympos. Pure Math.*, pages 127–146. Amer. Math. Soc., Providence, RI, 1987.

[Jan03] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[Jan05] Jens Carsten Jantzen. Deligne-Lusztig modulo p: revisited. Personal correspondence, 2005.

[Kha] Chandrashekhar Khare. On Serre's modularity conjecture for 2-dimensional mod p representations of $G_{\mathbf{Q}}$ unramified outside p, arXiv:math.NT/0504080.

[KWa] Chandrashekhar Khare and Jean-Pierre Wintenberger. On Serre's reciprocity conjecture for 2-dimensional mod p representations of the Galois group of $\mathbf{Q}$, arXiv:math.NT/0412076.

[KWb] Chandrashekhar Khare and Jean-Pierre Wintenberger. Serre's modularity conjecture: the odd conductor case (I). Preprint.

[Mum70] David Mumford. *Abelian varieties.* Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay, 1970.

[Ser71] Jean-Pierre Serre. Cohomologie des groupes discrets. In *Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970)*, pages 77–169. Ann. of Math. Studies, No. 70. Princeton Univ. Press, Princeton, N.J., 1971.

[Ser72] Jean-Pierre Serre. Propriétés galoisiennes des points d'ordre fini des courbes elliptiques. *Invent. Math.*, 15(4):259–331, 1972.

[Ser75] Jean-Pierre Serre. Valeurs propres des opérateurs de Hecke modulo l. In *Journées Arithmétiques de Bordeaux (Conf., Univ. Bordeaux, 1974)*, pages 109–117. Astérisque, Nos. 24–25. Soc. Math. France, Paris, 1975.

[Ser87] Jean-Pierre Serre. Sur les représentations modulaires de degré 2 de $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. *Duke Math. J.*, 54(1):179–230, 1987.

[Spr70a] T. A. Springer. Characters of special groups. In *Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)*, Lecture Notes in Mathematics, Vol. 131, pages 121–166. Springer, Berlin, 1970.

[Spr70b] T. A. Springer. Cusp forms for finite groups. In *Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)*, Lecture Notes in Mathematics, Vol. 131, pages 97–120. Springer, Berlin, 1970.

[Ste51] Robert Steinberg. The representations of $\mathrm{GL}(3,q), \mathrm{GL}(4,q), \mathrm{PGL}(3,q)$, and $\mathrm{PGL}(4,q)$. *Canadian J. Math.*, 3:225–235, 1951.

[Tat79] J. Tate. Number theoretic background. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 3–26. Amer. Math. Soc., Providence, R.I., 1979.

Dept. of Mathematics, Harvard University, Cambridge MA 02138, USA
E-mail address: `herzig@math.harvard.edu`