MAT137 - Week 4

- The deadline to drop to MAT135 is October 14. (Details on course website.)
- Read on the website the new, updated instructions for problem set submission.
- Notify us as soon as possible if you have a scheduling conflict with Test 1 (Details on the course website.)

- Today's lecture is about limits.
  - More examples of simple $\epsilon - \delta$ proofs.
  - The negating the definition to define a limit not existing.
  - The limit laws.

The formal definition of a limit

Definition

Let f be a function defined at least on an open interval containing a real number a, except possibly at a. Let L be a real number. Then $\lim_{x \to a} f(x) = L$ means

$$\forall \epsilon > 0 \, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

What about one-sided limits

Try to write down the formal definition of $\lim_{x \to a^+} f(x) = L$.

What about one-sided limits

Try to write down the formal definition of $\lim_{x \to a^+} f(x) = L$.

Definition

Let a be a real number, and let f be a function defined at least on an open interval of the form (a, p) for some $p > a$. Let L be a real number. Then $\lim_{x \to a^+} f(x) = L$ means

$$\forall \epsilon > 0 \, \exists \delta > 0 \text{ such that } a < x < a + \delta \implies |f(x) - L| < \epsilon.$$

Now try to do the same for left-hand limits.

What about one-sided limits

We mentioned this before defining limits rigorously, but now we can state it:

Theorem

Let f be a function defined at least on an open interval containing a real number a, except possibly at a. Let L be a real number. Then

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x).$$

Some trickier proofs

Last class we proved a limit of a linear function. As an exercise, prove the more general result about *all* linear functions:

That is, prove that for any slope m and any y-intercept b,

$$\lim_{x \to a}(mx + b) = ma + b.$$

Some trickier proofs

Prove that $\lim_{x \to 4} x^2 - x = 12$.

Negating the definition

What about limits that don't exist? How can we define this?

We can start by negating the definition of the limit we have.

Definition

The statement $\lim_{x \to a} f(x) \neq L$ means

$\exists \epsilon > 0$ such that $\forall \delta > 0$,

$$\text{there is an } x \text{ such that } 0 < |x - a| < \delta \text{ and } |f(x) - L| \geq \epsilon.$$

Using this definition

Prove that $\lim_{x \to 0} \frac{x}{|x|} \neq 0$.

Does this prove that $\lim_{x \to 0} \frac{x}{|x|}$ doesn't exist?

No! It could equal 1. Or π, maybe. Who knows?

Negating the definition

To define the statement

$$\lim_{x \to a} f(x) \text{ does not exist,}$$

we need to make sure that $\lim_{x \to a} f(x)$ doesn't equal *anything*.

That is, we want to define the statement

$$\forall L \in \mathbb{R}, \quad \lim_{x \to a} f(x) \neq L$$

Negating the definition

Definition

The statement

$$\lim_{x \to a} f(x) \text{ does not exist}$$

means

$\forall L \in \mathbb{R}$, $\exists \epsilon > 0$ such that $\forall \delta > 0$,

there is an x such that $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$.

Using this definition

Prove that $\lim_{x \to 0} \frac{x}{|x|}$ does not exist.

Uniqueness

This theorem sort of says that limits are a worthwhile thing to define.

Theorem (Uniqueness of limits)

If $\lim_{x \to a} f(x) = L$ *and* $\lim_{x \to a} f(x) = M$, *then* $L = M$.

You can find the proof on page 73 of your textbook.

Limit laws

Theorem (Limit Laws)

Let f and g be functions defined on an open interval containing a real number a, except possibly at a. Also, suppose that

$$\lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M.$$

Then

- $\lim_{x \to a} [cf(x)] = cL$ *for all* $c \in \mathbb{R}$.
- $\lim_{x \to a} [f(x) + g(x)] = L + M$.
- $\lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M$.
- $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$, *provided that* $M \neq 0$.

Results!

These basic limit laws are already very powerful. For example, they allow us to prove the following very easily:

Corollary

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *be a polynomial, and let* $c \in \mathbb{R}$. *Then*

$$\lim_{x \to c} P(x) = P(c).$$

Proof.

Exercise. □