- The deadline to drop to MAT135 is **tomorrow**. (Details on course website.)
- The deadline to let us know you have a scheduling conflict with Test 1 is also **tomorrow**. (Details on the course website.)

- Today's lecture is about continuity.
  - The definition, and some "continuity laws".
  - Types of discontinuities.
  - The Squeeze Theorem.
  - Two "special limits".
  - The IVT and EVT.

Continuity

Earlier, we talked about predictions. We said that if

$$\lim_{x \to a} f(x) = L,$$

then we're making a prediction for how f should act near a. Specifically, that $f(a)$ "should equal" L.

We saw that sometimes our predictions are wrong, but that was no concern to us.

Continuous functions are functions which *actually do what we predict they will do*.

Continuity

Definition

Let $a \in \mathbb{R}$, and let f be defined on an open interval containing a.

(Note that we *do* require the function to be defined at a.)

We say $\underline{f \text{ is continuous at } a}$ if

$$\lim_{x \to a} f(x) = f(a).$$

Continuity

We can also translate this into an $\epsilon - \delta$ definition as follows:

Definition

Let $a \in \mathbb{R}$, and let f be defined on an open interval containing a.

We say <u>f is continuous at a</u> if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \epsilon.$$

The benefit of knowing f is continuous at a: you can evaluate $\lim_{x \to a} f(x)$ simply by plugging in a.

Some trickier proofs

We have already shown that:

- $f(x) = 2x + 1$ is continuous at 2.
- In fact, we've shown that all lines are continuous at all real numbers:

$$\lim_{x \to a} (mx + b) = ma + b.$$

- *In fact*, we've shown that all polynomials are continuous at all real numbers. (Last class, as a corollary of the limit laws.)

Continuity

Other functions:

- $f(x) = \sqrt{x}$ is continuous at all positive real numbers.

  (Proof that it's continuous at 4 is in the textbook, section 2.2. The general proof is more or less the same.)
  Question: Is it continuous at 0?

- $g(x) = |x|$ is continuous at all real numbers.

  (Proof is in the textbook, section 2.2.)

- The sine and cosine functions are both continuous at all real numbers.

  (Proof is in the textbook, section 2.5.)

Combining continuous functions.

Corollary (...of the Limit Laws)

Suppose f and g are functions, $a \in \mathbb{R}$, and both f and g are continuous at a.

Then:

1. *cf is continuous at a for any real number c.*
2. *$f + g$ is continuous at a.*
3. *fg is continuous at a.*
4. *$\frac{f}{g}$ is continuous at a, provided that $g(a) \neq 0$.*

All of these results follow *immediately* from the Limit Laws. Try to prove them yourself.

Notable uses of this corollary.

The previous result gets us some useful things:

- $f(x) = \tan(x)$ is continuous at every point where it is defined.
- The same is true for the other three trigonometric functions:

$$\sec(x), \quad \csc(x), \quad \cot(x)$$

What about compositions?

Suppose you know that

- $\lim_{x \to a} f(x) = L$
- $\lim_{x \to b} g(x) = M$

Can you say anything about $\lim_{x \to a} g(f(x))$?

What would you need to know in order to be able to say something?

What about compositions?

Suppose we know a little more.

- $\lim_{x \to a} f(x) = L$
- $\lim_{x \to L} g(x) = M$

Now can you say anything about $\lim_{x \to a} g(f(x))$?

Compositions of continuous functions.

Theorem

Suppose that:

- *a is a real number,*
- *f is continuous at a,*
- *g is continuous at $f(a)$.*

Then $g \circ f$ is continuous at a.

In other words, under these hypotheses:

$$\lim_{x \to a} g(f(x)) = g(f(a)).$$

This is Theorem 2.4.4 in the book, and the proof is there.

Example

These are very powerful tools. As an example, consider:

$$f(x) = \sqrt{\frac{3|x|^3 + \pi|x|}{(1 - x^2)^3}}.$$

At what points is this function continuous?

Two more simple definitions

We'll need these two definitions shortly. They're not tricky.

Definition

Let a be a real number.

- Suppose f is a function defined on an open interval to the left of a.

  We say <u>f is continuous from the left at a</u> if

  $$\lim_{x \to a^-} f(x) = f(a).$$

- Suppose f is a function defined on an open interval to the right of a.

  We say <u>f is continuous from the right at a</u> if

  $$\lim_{x \to a^+} f(x) = f(a).$$

Discontinuities

We saw that <u>f is continuous at a</u> means $\lim_{x \to a} f(x) = f(a)$.

We can break this down into three parts:

- f is defined at a. In other words, $f(a)$ exists.
- $\lim_{x \to a} f(x)$ exists.
- The previous two things are equal.

The Squeeze theorem

Problem 5 on your PS2 is sort of like a version of the following theorem.

This theorem is easy to understand with a picture, so draw one.

Theorem

Let a be a real number, and suppose f, g, and h are all functions defined on an open interval containing a, except possibly at a.

Moreover, suppose they have the following relationship on that interval:

$$f(x) \leq g(x) \leq h(x).$$

Finally, suppose that $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$.

Then $\lim_{x \to a} g(x) = L$ *as well.*

Example

Show that $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Two useful limits

Your textbook uses the Squeeze Theorem to do two important things:

- Prove that sine and cosine are continuous.
- Prove the following two limits.

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0$$

A small, useful result

The following result is intuitively clear, and useful.

Theorem

If $\lim_{x\to 0} f(x) = L$ *and* $c \neq 0$*, then* $\lim_{x\to 0} f(cx) = L$*.*

The proof is left as an easy exercise.

Examples

Evaluate $\lim_{x \to 0} \frac{\sin(7x)}{x}$.

Evaluate $\lim_{x \to 0} \frac{x^3 \sin(x)}{\sin^4(7x)}$.

Two important theorems

Theorem (Intermediate Value Theorem)

Suppose f is continuous on an interval of the form $[a, b]$, and let V be any number strictly between $f(a)$ and $f(b)$.

Then there exists a $c \in (a, b)$ such that $f(c) = V$.

With this theorem again, the picture tells the whole story.

Example: Show that there are at least two solutions to the equation

$$x^2 - 1 = \sin(x).$$

Two important theorems

Theorem (Extreme Value Theorem)

Suppose f is continuous on an interval of the form $[a, b]$.

Then f attains both a maximum value M and a minimum value m.

There are not many interesting applications of this theorem alone, but it will be very important for us later.