MAT137 - Week 6

- Your first midterm is tomorrow, 4-6pm. Details are on the course website.
  - Make sure you go to the correct room.
  - Know your UTorID.
  - Know your **@mail.utoronto.ca** email address.
  - Know your student number.

- Today's lecture is about derivatives.
  - Some intuition.
  - The definition of the derivative and some examples.
  - Differentiability vs. continuity.
  - Some derivative rules.

Derivatives

Definition

Let a be a real number, and let f be a function defined at least on an open interval containing a. f is said to be differentiable at a if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists.

In this case its value is denoted by $f'(a)$, and called the derivative of f at a.

Derivatives

The following two limits are equal, and so they are used interchangeably in the definition above:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Derivatives as functions

When we calculate $f'(a)$ for some different values of a, we often find that the calculations feel the same.

So we just define the derivative *function* in general as:

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Examples

Compute the derivatives of the following functions:

$f(x) = x^2$.

$g(x) = \sqrt{x}$.

$h(x) = \frac{1}{x}$.

Any constant function.

Tangent lines

We know that $f'(a)$ is the slope of the tangent line to the graph of f at the point a. What is the equation of this tangent line?

It's the line with slope $f'(a)$ that passes through the point $(a, f(a))$.

Therefore, its equation is

$$y - f(a) = f'(a)(x - a).$$

Example: Find the equation of the tangent line to $f(x) = \frac{1}{x}$ at $x = 2$.

Differentiability vs. continuity

We know that $f(x) = |x|$ is continuous everywhere. Show that it is not differentiable at 0.

This means that continuity at a point **does not imply** differentiability at that point.

The converse is true, however:

Theorem

If f is differentiable at a, then f is continuous at a.

Notation

There are two major ways of notating derivatives and differentiation.

- **Lagrange notation**: What we've been using so far. Given a function f, its derivative is called f'
- **Leibniz notation**: Given a function f, suppose $y = f(x)$, and denote its derivative by $\frac{dy}{dx}$.

Leibniz notation is often more useful, because you can represent differentiation as an operation. For example, you can write:

$$\frac{d}{dx} f(x) \quad \text{or} \quad \frac{d}{dx}\left(\frac{x^2 + 7x + 1}{2x + 9}\right).$$

Notation

The downside of Leibniz notation: It's harder to express evaluation of derivatives at a point.

Lagrange notation: $f'(a)$

Leibniz notation: $\left.\frac{d}{dx}f(x)\right|_{x=a}$

New derivatives from old ones

This should remind you of the limit laws.

Theorem

Let f and g be differentiable at a real number a. Let c be any real number.

then $f+g$ and cf are differentiable at a, and:

1. $(cf)'(a) = \frac{d}{dx}\left[cf(x)\right]\Big|_{x=a} = cf'(a).$
2. $(f+g)'(a) = \frac{d}{dx}\left[f(x)+g(x)\right]\Big|_{x=a} = f'(a)+g'(a).$

The proof is an easy exercise in using the limit laws.

Actual derivatives of actual functions

Enough abstraction. Teach me how to take derivatives of actual functions!

Theorem

For any positive integer n, $\dfrac{d}{dx}\left[x^n\right] = nx^{n-1}$.

Key fact for the proof, which is a generalization of "difference of squares" and "difference of cubes": for any positive n, and real numbers a and b,

$$(a^n - b^n) = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}).$$

Example

Compute the derivative of:

$$f(x) = 3x^{723} + 2x^{42} - 17x^3 + 1,000,000$$

What about products?

We saw how to differentiate sums of differentiable functions. For products, it's trickier.

Theorem (Product Rule)

If f and g are differentiable at a real number x, then so is fg, and

$$(fg)'(x) = \frac{d}{dx}\big[f(x)g(x)\big] = f'(x)g(x) + f(x)g'(x).$$

Example

Compute the derivative of the following function at $x = 1$:

$$f(x) = (3x^2 + 2x + 1)(x^7 - 2).$$

What about quotients?

Quotients are even more tricky.

Theorem (Quotient Rule)

Let f and g be differentiable at a real number x, and suppose $g'(x) \neq 0$. Then $\frac{f}{g}$ is differentiable at x, and

$$\left(\frac{f}{g}\right)'(x) = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$