MAT137 - Week 7

- Today we're still talking about derivatives.
  - Shapes of graphs.
  - Higher derivatives.
  - Derivatives of trig functions.
  - Chain Rule
  - Implicit differentiation (maybe, if we have time)

Shapes of graphs

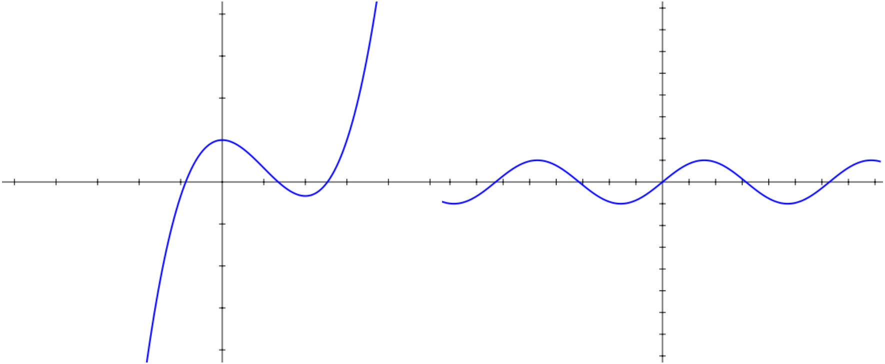
Remember that if f is differentiable (on all of $\mathbb{R}$, let's say), then f' is itself a function.

Its value at a point x is the slope of the tangent line to the graph of f at the same point x.

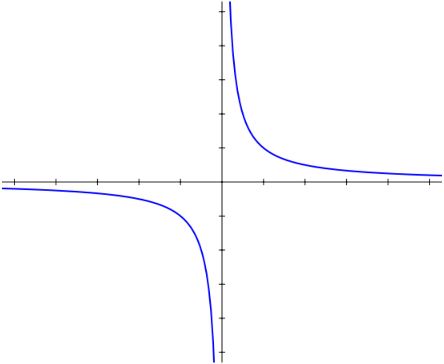
Just knowing this (ie. without knowing any ways of computing derivatives) we can sketch lots of graphs.

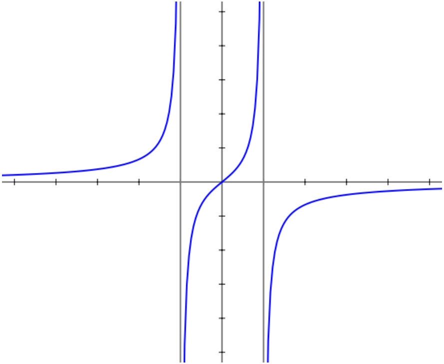
Shapes of graphs

Sketch the graphs of the derivatives of these functions.



Sketch the graphs functions whose derivatives look like these functions.





Reminder

Here are the derivatives we know how to compute thusfar:

- The derivative of any constant function is 0.

- For **any nonzero integer** n, we know that $\frac{d}{dx}\left(x^n\right) = n\,x^{n-1}$.

- We also know $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$, which is true for all positive x.

- We also know how to differentiate sums, products, and quotients of functions whose derivatives we already know.

Higher derivatives

As we mentioned, the derivative of a differentiable function f is another function, which we call f'.

...

Well why not try to differentiate f'!?

Notation: Given a function f, we call...

- ...its *first derivative* f'.
- ...its *second derivative* $f'' := (f')'$.
- ...its *third derivative* $f''' := (f'')'$.
- ...it's n^{th} *derivative* $f^{(n)}$.

Higher derivatives in Leibniz notation

Here's how these are written in Leibniz notation. If $y = f(x)$, then we call

- ...its *first derivative* $\frac{dy}{dx}$.
- ...its *second derivative* $\frac{d^2y}{dx^2} := \frac{d}{dx}\left(\frac{dy}{dx}\right)$.
- ...its *third derivative* $\frac{d^3y}{dx^3} := \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$.
- ...it's n^{th} *derivative* $\frac{d^ny}{dx^n}$.

Idea: $\frac{d^ny}{dx^n}$ looks sort of like $\left(\frac{d}{dx}\right)^n (y)$

Example

There's no trick to computing higher derivatives.

Example: Let $f(x) = x^7 + 4x + \frac{1}{x}$.

Then:

$$f'(x) = 7x^6 + 4 - \frac{1}{x^2}.$$

$$f''(x) = 42x^5 + 0 + \frac{2}{x^3}.$$

$$f'''(x) = 210x^4 - \frac{6}{x^4}.$$

And so on...

Exercises

Exercise: What is the 7^{th} derivative of x^7?
(Try to figure out what the answer is without explicitly calculating it on paper.)

Exercise: Let p be a polynomial of degree 7. What is its 8^{th} derivative?

Exercise: Let $f(x) = \frac{1}{x}$. Try to find a formula for the n^{th} derivative of f.

Derivatives of trigonometric functions

We have not yet explained how to differentiate $\sin(x)$ and $\cos(x)$.

For the computation, recall the following two trig identities:

1. $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.
2. $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$.

Using these and our two special limits, we derive:

$$\frac{d}{dx}\sin(x) = \cos(x) \quad \text{and} \quad \frac{d}{dx}\cos(x) = -\sin(x).$$

REMEMBER: x is measured in radians. These formulas are *different* if x is measured in degrees!

Derivatives of trigonometric functions

Using the quotient rule, we can obtain the derivatives of the other four trig functions:

- $\dfrac{d}{dx}\tan(x) = \dfrac{d}{dx}\left(\dfrac{\sin(x)}{\cos(x)}\right) = \sec^2(x).$
- $\dfrac{d}{dx}\sec(x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos(x)}\right) = \sec(x)\tan(x).$
- $\dfrac{d}{dx}\csc(x) = \dfrac{d}{dx}\left(\dfrac{1}{\csc(x)}\right) = -\csc(x)\cot(x).$
- $\dfrac{d}{dx}\cot(x) = \dfrac{d}{dx}\left(\dfrac{\cos(x)}{\sin(x)}\right) = -\csc^2(x).$

YOU DON'T NEED TO MEMORIZE THESE!

What about compositions?

Imagine you're driving up a mountain.

You can look at an altimeter and derive a function h whose value at time t is your height. This describes your height as $h(t)$.

Suppose you also know that the temperature T outside varies with height h. This describes the temperature outside as $T(h)$.

We can now calculate, in principle, $h'(t)$ and $T'(h)$. What do these represent?

What if we want to know how quickly the temperature is changing? That is, what is $\frac{dT}{dt} = (T \circ h)'$?

If the functions are lines, this is easy to do:

Example: Suppose $T(h) = -5h + 10$ and $h(t) = 2t + 3$. Compute $\frac{dT}{dt} = (T \circ h)'(t)$.

Answer: $(T \circ h)'(t) = -10$.

In general...

In general, at a time t_0, we expect to get

$$(T \circ h)'(t_0) = T'(h(t_0))\, h'(t_0).$$

Generalizing this to the composition of two general functions f and g, we expect to get

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$

In general...

In Leibniz notation, this is very convincing.

Let $w = f(y)$ and $y = g(x)$. The derivative of their composition is then $\frac{dw}{dx}$, and the statement above translates to:

$$\frac{dw}{dx} = \frac{dw}{dy}\frac{dy}{dx}.$$

Example

Let $f(x) = 3x^2 + 1$ and $g(x) = \sqrt{x} + 1$.

Compute $(f \circ g)'(x)$ by explicitly composing the functions and differentiating the result.

The Chain Rule

We are now able to state the general theorem here.

Theorem (Chain Rule)

Suppose f, g are functions such that g is differentiable at x and f is differentiable at $g(x)$.
Then $f \circ g$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$

We'll give a "wrong" proof that *feels* right.

Uses of the Chain Rule

Here are some things we can do with the Chain Rule:

1. Prove the quotient rule. To do this, note that $\dfrac{f(x)}{g(x)} = f(x)\big(g(x)\big)^{-1}$.
2. Calculate the derivative of $f(x) = (2x^3 + 7x + 10)^{1{,}000{,}000}$.
3. Calculate the derivative of $g(x) = \sin^2(2x + 1)$. Three functions here!