- Problem Set 8 is due tomorrow, 3 March, at 3pm.
- Today we will:
  - Remind ourselves about improper integrals a bit.
  - Talk about series.

Remember improper integrals?

Almost everything we will learn about series this week and next week will be analogous to things we learned about improper integrals before reading break.

So we'll start by reminding ourselves about some of these things.

First, recall that at the beginning of this term, we defined <u>definite integrals</u>, which look like this:

$$\int_a^b f(x)\,dx.$$

These things computed the area underneath the graph of a function on an interval of the form $[a, b]$.

Remember improper integrals?

We then took the definition of

$$\int_a^b f(x)\,dx$$

and asked what happens when b gets bigger and bigger.

More formally, we took a limit as b goes to ∞.

This defined a new concept, which didn't make sense before we defined it:

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty}\left[\int_a^b f(x)\,dx\right].$$

(We defined several other sorts of improper integrals as well, but we don't need those to understand series.)

Remember improper integrals?

Recall that the following two limits are very different things:

$$\lim_{x \to \infty} f(x) \quad \text{and} \quad \lim_{b \to \infty} \left[\int_a^b f(x)\, dx \right].$$

They shouldn't seem very similar, but in the context of series people confuse them all the time.

For example, it *feels* like in order for the improper integral on the right to converge, we need $f(x)$ to "get smaller and smaller" as x increases.

This isn't quite true, but it's good intuition.

Remember improper integrals?

But note that the opposite thing is not true.

We studied improper integrals of the form

$$\int_1^\infty \frac{1}{x^p}\,dx,$$

and saw that some of them converged and some of them diverged, despite the fact that

$$\lim_{x\to\infty} \frac{1}{x^p} = 0 \quad \text{for all } p > 0.$$

Remember improper integrals?

Some other results about improper integrals to remember:

Proposition

Let f be a positive function defined on $[a, \infty)$ and integrable everywhere necessary.

Then $\int_a^\infty f(x)\,dx$ either converges, or diverges to infinity.

Recall that this is true because if f is always positive, then

$$\int_a^b f(x)\,dx$$

must increase as b increases.

Remember improper integrals?

This is the first of two very important comparison tests we had for improper integrals:

Theorem (Basic Comparison Test (BCT))

Let $a \in \mathbb{R}$, and let f, g be positive functions that are integrable on $[a, b]$ for every $b > a$.

Suppose also that $f(x) \leq g(x)$ for all $x \in [a, \infty)$. Then

1. *If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges as well.*
2. *If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges as well.*

Remember improper integrals?

This is the second (and much more important) comparison test we had:

Theorem (Limit Comparison Test (LCT))

Let $a \in \mathbb{R}$, and let f, g be positive functions that are integrable on $[a, b]$ for every $b > a$.

Suppose also that $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ exists and equals a positive constant.

Then:

$$\int_a^{\infty} f(x)\,dx \text{ converges} \iff \int_a^{\infty} g(x)\,dx \text{ converges} .$$

Now that we've reminded ourselves about improper integrals, we can move on to series.

What we're going to do is define a way to add up infinitely many numbers.

This is not something that makes sense to do at this point.

We know very well how to add up finitely many numbers, because we've been doing that since elementary school. We must first define a way to add up infinitely many numbers.

The way we're going to do this is analogous to the way we defined integrals like $\int_a^\infty f(x)\,dx$, which is by taking a limit of things we *do* know how to evaluate.

Series

Definition

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence (in this context, our sequences will often begin with an $n = 0$ term). For each n, define the n^{th} partial sum of the sequence by:

$$s_n = \sum_{k=0}^{n} a_k.$$

That is, $s_n = a_0 + a_1 + a_2 + \cdots + a_n$.

If $\lim_{n\to\infty} s_n$ exists and equals L, we say that the series $\sum_{n=0}^{\infty} a_n$ converges to L, and refer to L as the sum of the series.

If the limit of the sequence above does not exist, we say that $\sum_{n=0}^{\infty} a_n$ diverges.

Examples

Example: The series $\sum_{n=0}^{\infty}(-1)^n$ diverges.

The sequence $\{(-1)^n\}_{n=0}^{\infty}$ looks like this:

$$1, -1, 1, -1, 1, -1, \ldots$$

In this case, the partial sums are easy to evaluate:

$$s_n = \sum_{k=1}^{n}(-1)^k = \begin{cases} 1 & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}$$

Clearly then $\lim_{n\to\infty} s_n$ does not exist, so the series diverges.

Example: The series $\sum_{n=1}^{\infty} 7$ diverges.

This series is even easier to work with than the previous one. We can see that:

$$s_n = \sum_{k=1}^{n} 7 = \underbrace{7 + 7 + \cdots + 7}_{n \text{ times}} = 7n$$

Since $\lim_{n\to\infty} 7n$ does not exist, the series diverges.

We'll see some examples of series that converge a bit later.

Facts: It doesn't matter where you start.

Something we know about sequences is that if you care about the limit of a sequence, it doesn't matter where you start.

In other words, $\{a_n\}_{n=1}^{\infty}$ and $\{a_n\}_{n=17}^{\infty}$ have the same limit (if it exists), and the fact that the first 16 terms of the latter sequence are missing doesn't matter.

The same is true of series, in the following sense. Suppose $\{a_n\}_{n=0}^{\infty}$ is a sequence, and $M \in \mathbb{N}$. Then:

$$\sum_{n=0}^{\infty} a_n \text{ converges} \iff \sum_{n=M}^{\infty} a_n \text{ converges.}$$

The partial sums of these two series differ by precisely $\sum_{n=0}^{M} a_n$, which is simply a finite number.

Facts: Linearity

Like every other kind of limit we've defined, there are some "limit law" type results about series:

Proposition

Suppose $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *converge to* L *and* M*, respectively. Then:*

1. $\sum_{n=0}^{\infty} (a_n + b_n)$ *converges to* $L + M$.
2. $\sum_{n=0}^{\infty} (c\, a_n)$ *converges to* cL *for all* $c \in \mathbb{R}$.

These results both follow easily from what we know about finite sums of numbers, and the limits laws for sequences. For example, for the second part:

$$\sum_{n=0}^{\infty} (c\, a_n) = \lim_{n\to\infty} \left[\sum_{k=0}^{n} c\, a_k \right] = \lim_{n\to\infty} \left[c \sum_{k=0}^{n} a_k \right] = c \lim_{n\to\infty} \left[\sum_{k=0}^{n} a_k \right] = cL.$$

The "Necessary Condition Test"

This is an extremely fundamental result to the study of series.

Your intuition should be that if a series is going to converge, it's terms should "get smaller", so that the partial sums get closer and closer together. This theorem formalizes this intuition.

Theorem (Necessary Condition Test (NCT))

Suppose $\{a_n\}_{n=0}^{\infty}$ *is a sequence. If* $\sum_{n=0}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

This theorem is most useful in its contrapositive form:

$$\text{If} \quad \lim_{n\to\infty} a_n \neq 0, \quad \text{then} \sum_{n=0}^{\infty} a_n \quad \text{diverges.}$$

Example

The NCT allows us to easily determine that many series diverge. For example, all of the following series diverge because the limit of their terms is not zero:

- $\sum_{n=0}^{\infty} 7$.
- $\sum_{n=0}^{\infty} n$.
- $\sum_{n=0}^{\infty} \frac{n}{n+1}$.
- $\sum_{n=0}^{\infty} (-1)^n$.
- $\sum_{n=1}^{\infty} \log(n)$.
- $\sum_{n=0}^{\infty} \sin(n)$.

In the last case, it takes a bit of work to show that $\lim_{n\to\infty} \sin(n) \neq 0$, but it shouldn't be surprising.

WARNING

The converse of the NCT is not true.

That is:

$$\text{If } \lim_{n\to\infty} a_n = 0, \text{ it does not follow that } \sum_{n=0}^{\infty} a_n \text{ converges.}$$

Repeat this to yourself five times every day, until you begin saying it in your sleep.

The Harmonic Series

This is probably the single most important series we will see, and it demonstrates why the warning is necessary.

Example: The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the harmonic series. It diverges.

I've actually shown you this series before, in disguise.

One of the first examples of recursively defined sequences I gave you was the sequence of Harmonic numbers:

$$H_1 = 1, \quad H_{n+1} = H_n + \frac{1}{n+1}.$$

(You can find this on slide 24, from Term 2 Lecture 6.)

The Harmonic Series

Thinking about this a bit, we can see that for every n:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}.$$

In other words, the Harmonic numbers are exactly the partial sums of the harmonic series.

When we first saw the Harmonic numbers, we proved that the sequence $\{H_n\}_{n=1}^{\infty}$ is unbounded, and therefore $\lim\limits_{n\to\infty} H_n$ does not exist.

That means that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{n\to\infty} H_n$ diverges.

Another result that caries over from integrals

Proposition

If $\{a_n\}_{n=0}^{\infty}$ is a sequence of positive numbers, then

$$\sum_{n=0}^{\infty} a_n$$

either converges, or diverges to infinity.

To see this, note that if the terms of the sequence are positive, then the sequence of partial sums must be increasing, since:

$$s_{n+1} - s_n = \sum_{k=0}^{n+1} a_k - \sum_{k=0}^{n} a_k = a_{n+1} > 0.$$

Okay, how can we evaluate series?

We have now defined series convergence, seen some series that diverge, and even have a theorem that tells us when some series diverge.

How can we actually evaluate some convergent series?

In general, this is very difficult. Given some sequence $\{a_n\}_{n=0}^{\infty}$ it's usually hard to say anything meaningful about

$$s_n = \sum_{k=0}^{n} a_k.$$

Computing this is like the "series version" of computing indefinite integrals.

We'll proceed slowly, by looking at some special cases and eventually devising some tests for convergence of series.

Telescoping series

The easiest series to work with are so-called "telescoping series".

Example: Compute $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

First, note the following very convenient fact:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

We can use this to evaluate the partial sums:

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = 1 - \frac{1}{k+1}$$

And so we can see that $\lim_{n \to \infty} s_n = 1$.

Geometric series

Given a real number r (r stands for "ratio"), the series

$$\sum_{n=0}^{\infty} r^n$$

is called a geometric series. These are series we can evaluate explicitly.

Theorem

The geometric series $\sum_{n=0}^{\infty} r^n$ converges if and only if $|r| < 1$.

In this case, $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

Caution with geometric series

Just a quick warning about the previous result.

Earlier we said that where you start a series doesn't matter from the point of view of convergence.

It <u>does</u> matter for computing the actual value of a series though.

For example, the previous result says that $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \left(\frac{1}{2}\right)} = 2$.

However, if we start the series from $n = 1$, we have:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \left[\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n\right] - 1 = 2 - 1 = 1.$$

The Integral Test

Theorem

Suppose f is a continuous, positive, decreasing function defined on $[1, \infty)$.

Then

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges.}$$

NOTE: This doesn't say the series *equals* the integral.

The Basic Comparison Test

Theorem (Basic Comparison Test (BCT))

Suppose that $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *are sequences with* *positive* *terms.*

Suppose also that $a_n \leq b_n$ *for all* n*.*

1. *If* $\sum_{n=0}^{\infty} b_n$ *converges, then* $\sum_{n=0}^{\infty} a_n$ *converges.*
2. *If* $\sum_{n=0}^{\infty} a_n$ *diverges, then* $\sum_{n=0}^{\infty} b_n$ *diverges.*

The Limit Comparison Test

Theorem (Limit Comparison Test (LCT))

Suppose that $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *are sequences with* *positive* *terms.*

Suppose also that $\lim_{n\to\infty} \frac{a_n}{b_n}$ *exists and equals a positive constant.*

Then

$$\sum_{n=0}^{\infty} a_n \text{ converges} \iff \sum_{n=0}^{\infty} b_n \text{ converges.}$$