MAT137 - Term 2, Week 11

- Test 4 is tomorrow, 24 March, at 4pm. See the course website for details.

- Course evaluations are now available. You should have been notified about them over email. **Please fill one out. It's important to me.**

- Today we will:
    - Remind ourselves a bit about power series and Taylor polynomials
    - Define Taylor series.
    - Talk about analytic functions.
    - Talk about the Lagrange Remainder Theorem
    - Compute some Taylor series of functions we know.

Some reminders from last class

A power series centred at a is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

where $\{a_n\}_{n=1}^{\infty}$ is a sequence.

The x should be treated as a variable, because the goal here is to use the series to define a function of x.

Some reminders from last class

We learned that associated to every power series is a radius of convergence.

If the radius of convergence of $\sum a_n (x-a)^n$ is 0, the series only converges at $x = a$.

If the radius of convergence of $\sum a_n (x-a)^n$ is ∞, the series converges at all real numbers.

If the radius of convergence of $\sum a_n (x-a)^n$ is some value positive real number R, the series converges absolutely on $(a-R, a+R)$, diverges everywhere outside $[a-R, a+R]$, and can have any behaviour at the two endpoints.

Some reminders from last class

The main reason (so far) that we want to define functions with power series is because they "act like polynomials" on the intervals where they converge absolutely.

For example this allows us to differentiate and integrate them easily:

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n (x-a)^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(a_n (x-a)^n\right) = \sum_{n=1}^{\infty} n\, a_n (x-a)^{n-1}$$

This is true so long as the power series converges absolutely at x, or in other words that $|x-a| < R$, where R is the radius of convergence of the power series.

Some reminders from last class

We then moved on to Taylor polynomials.

We developed a way of quantifying how well one function approximates another function:

Definition

Let f and g be two functions such that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ for some real number a.

We say <u>g is a good approximation of order n for f at a</u> if

$$\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n} = 0.$$

Some reminders from last class

Then we found out that if the first n derivatives of f and g have the same values at a, then g is a good approximation of order n for f at a.

This led us to define Taylor polynomials.

Definition

Let f be a function that has all of its derivatives, let a be a real number, and n a positive integer.

The n^{th} Taylor polynomial of f at a is the unique polynomial P_n of smallest degree such that

$$P_n(a) = f(a), P_n'(a) = f(a), \ldots, P_n^{(n-1)}(a) = f^{(n-1)}(a), P_n^{(n)}(a) = f^{(n)}(a)$$

Some reminders from last class

So the n^{th} Taylor polynomial of a function f at a is a polynomial that is a good approximation for f of order n at a.

We also found a formula for what they look like, which is easy to verify by computation:

The n^{th} Taylor polynomial of f at a is:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Examples

Example: Compute the 7^{th} Taylor polynomial of e^x at $x = 0$.

$$P_7(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{7!}x^7.$$

Example: Compute the 5^{th} Taylor polynomial of $g(x) = \cos(x)$ at $x = 0$.

$$P_5(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4$$

Note that the degree of this polynomial is less than 5.

Taylor series

Now we are ready to define Taylor series.

Definition

Suppose f is a function that has all of its derivatives, and a is a real number.

The Taylor series of f and a is the power series S, centred at a, such that

$$S^{(k)}(a) = f^{(k)}(a) \quad \text{for all } k = 0, 1, 2, \ldots$$

In other words, the Taylor series of f at a is the power series S such that all the derivatives of S and f agree at a.

Taylor series

It is a simple exercise in induction to prove that the terms in the Taylor series of f at a look just like the terms in the Taylor polynomials of f at a:

Proposition

Suppose f is a function that has all of its derivatives, and a is a real number.

Then the Taylor series of f at a is the following power series:

$$S(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Examples

Example: At this point it should not be surprising that the Taylor series of e^x at $x = 0$ is:

$$S(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Example: Compute the Taylor series of $\cos(x)$ at $x = 0$.

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Example: Compute the Taylor series of $\sin(x)$ at $x = 0$.

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

There's more going on here than you think.

CAUTION: I have never claimed that a function f equals its Taylor series!

For example, above we found that the Taylor series of $\cos(x)$ at $x = 0$ is

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

but I never claimed that $S(x) = \cos(x)$.

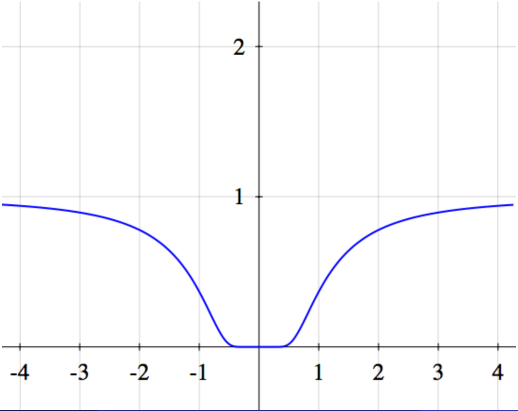
This is a very subtle and important point. First we'll give an example of how this can go wrong, then describe how we ensure it doesn't go wrong.

A very pathalogical example

Example: Consider the following function:

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Here's the relevant part of its graph:



A very pathological example

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Now let's compute the Taylor series of this function.

It is a a tedious (but purely computational) exercise to check that:

$$f^{(k)}(0) = 0 \quad \text{for all } k = 0, 1, 2, \ldots$$

This implies that the Taylor series of this function is:

$$S(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0.$$

So the Taylor series absolutely converges everywhere, but $S(x) \neq f(x)$ for all $x \neq 0$.

Now what?

The whole idea here is to use the Taylor series of a function to understand the function better, or manipulate it more easily.

So our next goal is to find out when and how we can guarantee a function f equals its Taylor series.

Such a function will be called an analytic function, roughly speaking. We'll make this more precise later.

For now just note that we just saw a function that has all of its derivatives everywhere, but which is not analytic.

Minimizing error

Think back to our discussion of "good approximations of order n", error functions, etc.

Provided f has all of its derivatives at a, we can define

- P_n, its n^{th} Taylor polynomial at a, for all n, and
- S, its Taylor series at a.

We know that P_n is a good approximation of order n for f at a, and that this approximation likely has some error for values of x other than a. Call this error $R_n(x)$.

Therefore, we'll have that for all x, $f(x) = P_n(x) + R_n(x)$.

Minimizing error

First of all, note that if x is inside the radius of convergence of the Taylor series S, we have:

$$\lim_{n\to\infty} P_n(x) = S(x)$$

This is simply by definition of series convergence; $P_n(x)$ are the partial sums of the series $S(x)$.

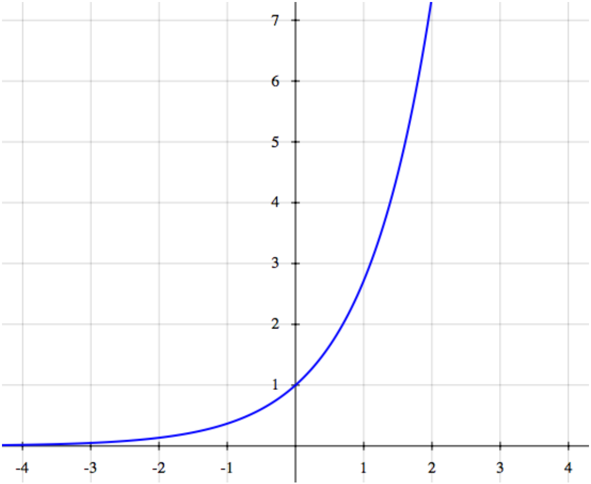
If we also knew that $\lim_{n\to\infty} R_n(x) = 0$, we would have:

$$f(x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty} P_n(x) + R_n(x) = \lim_{n\to\infty} P_n(x) + \lim_{n\to\infty} R_n(x) = S(x).$$

In other words, we have shown that if $R_n(x) \to 0$, then $f(x) = S(x)$ for any x inside the radius of convergence of S.

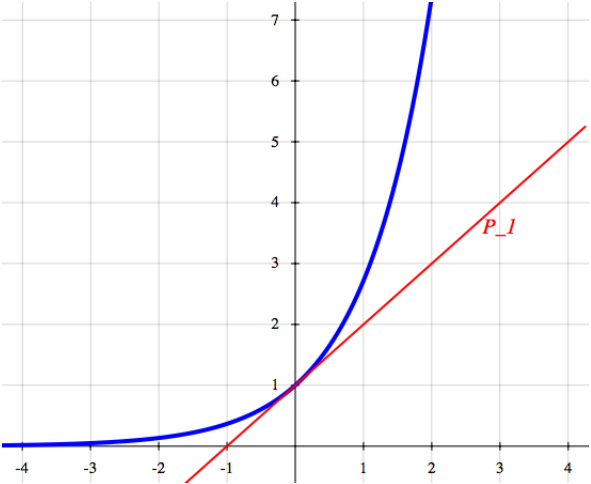
Here's what this looks like

Here's an example of what it looks like when $R_n(x) \to 0$: the exponential function $f(x) = e^x$.



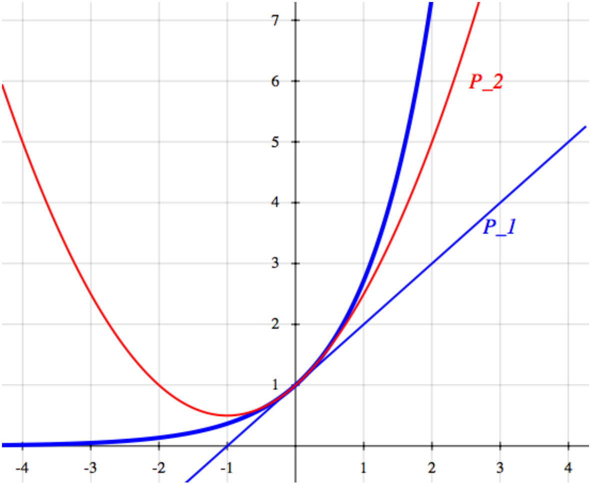
Here's what this looks like

The exponential function $f(x) = e^x$, and its first Taylor polynomial P_1



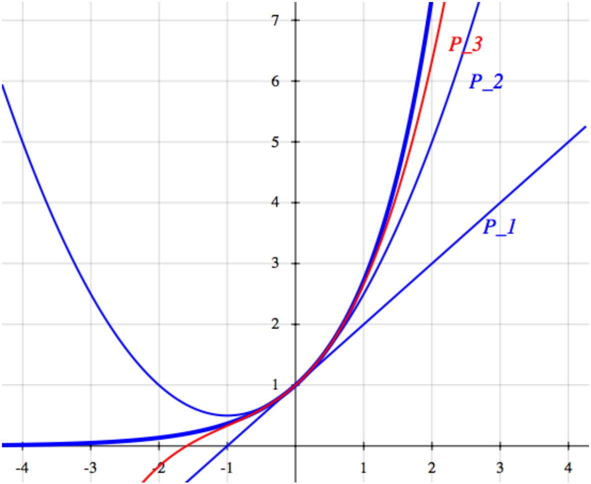
Here's what this looks like

The exponential function $f(x) = e^x$, and its second Taylor polynomial P_2



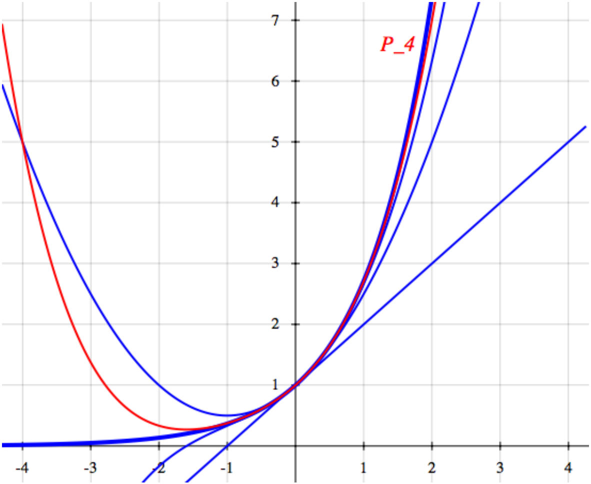
Here's what this looks like

The exponential function $f(x) = e^x$, and its third Taylor polynomial P_3



Here's what this looks like

The exponential function $f(x) = e^x$, and its fourth Taylor polynomial P_4



Analytic functions

Roughly speaking, functions for which $R_n(x) \to 0$, so that they equal their Taylor series on some interval, are called analytic.

We'll state this definition more precisely in a moment, but for now you can have this fact for free:

Theorem

For all $a \in \mathbb{R}$, e^x, $\sin(x)$, and $\cos(x)$ equal their Taylor series centred at a for all x.

In particular, the following are true for all x:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Some definitions

Now let's be a bit more precise.

Definition

Let f be a function defined at least on an open interval I.

- f is $\underline{C^1 \text{ on } I}$ of f' exists and is continuous on I.
- More generally, f is $\underline{C^n \text{ on } I}$ if $f', f'', f''', \dots, f^{(n)}$ all exist and are all continuous on I.
- f is $\underline{C^\infty \text{ on } I}$ (or sometimes $\underline{\text{smooth on } I}$ if f has all of its derivatives at every point of I.
- f is $\underline{\text{analytic on } I}$ if for every $a \in I$, the Taylor series of f centred at a converges to $f(x)$ for all x near a.

These properties are listed in increasing order of strength.

Some definitions

We know now that e^x, $\sin(x)$, and $\cos(x)$ are analytic on $\mathbb{R}$.

We showed that

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is C^∞ on $\mathbb{R}$, but not analytic on any interval containing 0.

The function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable everywhere, but its derivative is not continuous, so it is not C^1.

For each k, the function $f(x) = |x|^{k+1}$ is C^k but not C^{k+1}.

How can we tell a function is analytic?

So far, I've just asked you to believe that e^x, $\sin(x)$, and $\cos(x)$ are analytic.

Previous work we've done about geometric series leads us to believe that $f(x) = \dfrac{1}{1-x}$ is analytic on $(-1, 1)$.

In general, if we want to prove a function is analytic, we have to prove that $\lim_{n\to\infty} R_n(x) = 0$ for all x in some interval, as we discussed earlier.

We can do this with the help of theorems that gives us a way to control $R_n(x)$. We'll see the most famous one here.

Taylor's Theorem

The famous theorem below gives an explicit expression for the remainder, usually called the "integral form":

Theorem

Let n be a positive integer, and suppose f is C^{n+1} on an interval I that contains a point a. Let P_n be its n^{th} Taylor polynomial at a.

Then for all $x \in I$, we have $f(x) = P_n(x) + R_n(x)$, where

$$R_n(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n \, dt.$$

This integral is a mess though, so instead of working with it we use theorems that estimate its value

Lagrange's Remainder Theorem

Theorem (LRT)

Let n be a positive integer, and suppose f is C^{n+1} on an interval I containing a point a.

Then for any $x \in I$, we have:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some number c in between a and x.

This is a consequence of the MVT. Note that the value of c depends on n and x.

We can use this theorem to prove that e^x is analytic on $\mathbb{R}$, without much difficulty.

Proving the exponential is analytic

We'll start by doing the proof for $a = 0$ in detail.

Proof: Fix an arbitrary real number x. We'd like to show that $R_n(x) \to 0$.

By the LRT, we have an expression for this remainder:

$$R_n(x) = \frac{e^c}{(n+1)!}\, x^{n+1},$$

where c is some number between 0 and x. Here we've used the fact for any n, the $(n+1)^{\text{th}}$ derivative of e^x is itself.

Proving the exponential is analytic

$$R_n(x) = \frac{e^c}{(n+1)!}\, x^{n+1},$$

It's tough to control this sequence as it is, since the value of c can change depending on n.

So instead of proving $R_n(x) \to 0$ directly, we'll show that $|R_n(x)| \to 0$, and then apply the Squeeze Theorem.

We can check:

$$|R_n(x)| \quad = \quad \frac{e^c}{(n+1)!}\, |x|^{n+1} \quad < \quad \frac{M}{(n+1)!}\, |x|^{n+1}$$

where M is any upper bound for the value of e^c on the interval joining 0 to x. We can use M equalling either 1 or e^x depending on the sign of x.

Proving the exponential is analytic

$$|R_n(x)| \quad = \quad \frac{e^c}{(n+1)!}\,|x|^{n+1} \quad < \quad \frac{M}{(n+1)!}\,|x|^{n+1}$$

The sequence on the right goes to zero by the Big Theoren, since the $(n+1)!$ term in the denominator grows much faster than the $|x|^{n+1}$ term in the numerator.

It follows that $|R_n(x)| \to 0$, and in turn that $R_n(x) \to 0$, each by the Squeeze theorem, since:

$$0 \le |R_n(x)| \le \frac{M}{(n+1)!}\,|x|^{n+1} \quad \text{and} \quad -|R_n(x)| \le R_n(x) \le |R_n(x)|$$

Therefore, we have shown that:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}\,x^n.$$

Proving the exponential is analytic

To prove that e^x is analytic, we actually have to show that it equals its Taylor series centred at any $a \in \mathbb{R}$. In other words, that:

$$e^x = \sum_{n=0}^{\infty} \frac{e^a}{n!} (x - a)^n \quad \text{for all } a \in \mathbb{R}.$$

This is almost as easy as the $a = 0$ case though. The only difference is that in choosing M, we have to find an upper bound for e^x on the interval joining a to x.

Similarly to before, we can choose M equalling either e^x or e^a, depending on whether x is larger or smaller than a.