MAT137 - Week 3

- Problem Set 1 is due next week, 28 September, by 11:59pm.
  - You will soon receive an email with a personal link to submit it. You *must* use that link, and should not share it.
  - If you don't get such an email, check your spam folder.
  - If you still can't find it, contact Alfonso. All of this is on the course website.
  - **Do not leave the submission process to the last minute.**
- If you need to change tutorials, you'll get info on how to do this very soon.
- Today's lecture is about limits. I will assume you have watched videos 2.1 through 2.7.
- For next week's lecture, watch vidoes 2.8 through 2.14.

Some review of absolute values and inequalities

Problem. For what values of x is the following inequality true?

$$|x-7| < 3$$

In other words, what values of x are within a distance 3 from 7?

Notice that thinking about the expression $|x-7|$ as "the distance between x and 7" makes this problem *much* easier.

What about the following inequality?

$$0 < |x-7| < 3$$

Some review of absolute values and inequalities

Problem. Suppose x is a real number that satisfies the inequality

$$|x-2| < 1.$$

What bounds, if any, can you put on $|x-7|$?

Here's how you should read this question:

If x is within a distance 1 from 2, how far can x be from 7?

Again, I hope you agree that when phrased in terms of distances, this is a straightforward problem.

You have just proved the following conditional:

$$|x-2| < 1 \implies 4 < |x-7| < 6$$

Some review of absolute values and inequalities

Now we know how to bound values of x with inqualities and absolute values. Let's use bounds on x to find bounds on the values of functions of x.

Problem. Suppose x is a real number that satisfies the inequality

$$|x+7| < 2.$$

How big can $|3x+21|$ be?

In words:

If x is within a distance 2 from -7, how far can $3x$ be from -21?

You have just proved the following conditional:

$$|x+7| < 2 \implies |3x+21| < 6$$

A Greek letter!

Learn to write this Greek letter:

$$\delta$$

Absolute values and conditionals

Now let's reverse idea in the previous exercise.

Problem 1. Find **one** positive value of δ that makes the following conditional true.

$$\text{If } |x - 3| < \delta, \text{ then } |4x - 12| < 6.$$

Problem 2. Find **all** positive values of δ that make the above conditional true.

Absolute values and conditionals

Now we know that for any $0 < \delta \leq \frac{3}{2}$, the following conditional is true:

$$\text{If } |x - 3| < \delta, \text{ then } |4x - 12| < 6.$$

We'll work with this idea a bit.

Problem 3. Suppose we want a tighter restriction on $|4x - 12|$ in the conditional above. For example, let's say we want the distance between $4x$ and 12 to be less than 1:

$$\text{If } |x - 3| < \delta, \text{ then } |4x - 12| < 1.$$

Will all of the same values of δ that worked before work now?

No! To make $4x$ closer to 12, we must make x closer to 3. Which values of δ will work here?

Absolute values and conditionals

Any good mathematical concept is worth generalizing.

-Michael Spivak

Problem 4. Let ϵ be a fixed positive real number. Is it possible to find a positive δ that **does not** depend on ϵ and that makes the following conditional true?

$$\text{If } |x-3| < \delta, \text{ then } |4x-12| < \epsilon.$$

No! The smaller ϵ is, the smaller δ should have to be! Just like before.

Problem 5. Let ϵ be a fixed positive real number. Find a value of δ, in terms of ϵ, that makes the following conditional true:

$$\text{If } |x-3| < \delta, \text{ then } |4x-12| < \epsilon.$$

Absolute values and conditionals

Now we know that for a fixed positive real number ϵ, the following conditional is true:

$$\text{If } |x-3| < \frac{\epsilon}{4}, \text{ then } |4x-12| < \epsilon.$$

Congratulations! You've just done the "hard part" of proving that $\lim_{x \to 3} 4x = 12$.

Not so bad, right?

Limits, intuitively

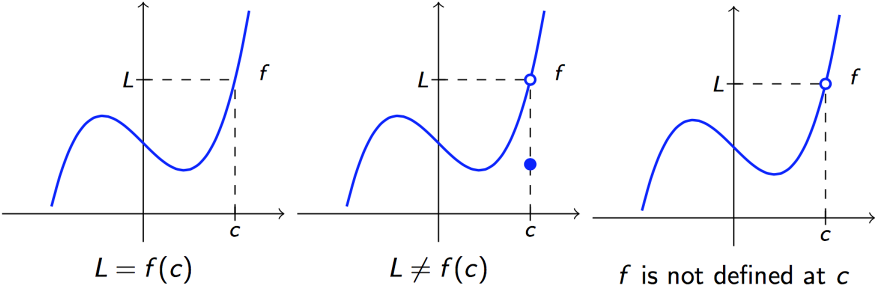
Recall the intuitive definition of a limit given in the videos:

$$\lim_{x \to c} f(x) = L \qquad \text{means}$$

If x is close to c (but not equal to c), then $f(x)$ is close to L.

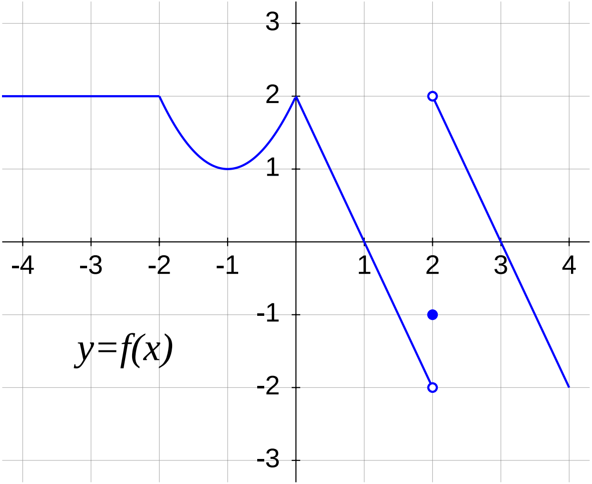
Note that a limit ***never*** cares about what's happening at c. Only near c.

All of the following functions have the same limit L at c:



In particular, this means that in principle you can ***never*** evaluate a limit simply by plugging $x = c$ into the function. More about this next class.

Limits from a graph



Find the value of

1. $\lim_{x \to 2} f(x)$
2. $\lim_{x \to 0} f(f(x))$
3. $\lim_{x \to -3} f(f(x))$

Designing functions to get certain limits

Problem. Construct two functions f and g such that the following three things are true:

- $\lim_{x \to 1} f(x) = 2$.
- $\lim_{u \to 2} g(u) = 3$.
- $\lim_{x \to 1} g(f(x)) = 100$.

The precise definition of a limit

Here's the precise definition of a limit given in the videos.

Definition

Let f be a function defined on an open interval containing a real number c, except possibly at c. Let L be a real number. Then $\lim_{x \to c} f(x) = L$ means

$$\forall \epsilon > 0 \, \exists \delta > 0 \text{ such that } 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

The precise definition of a limit

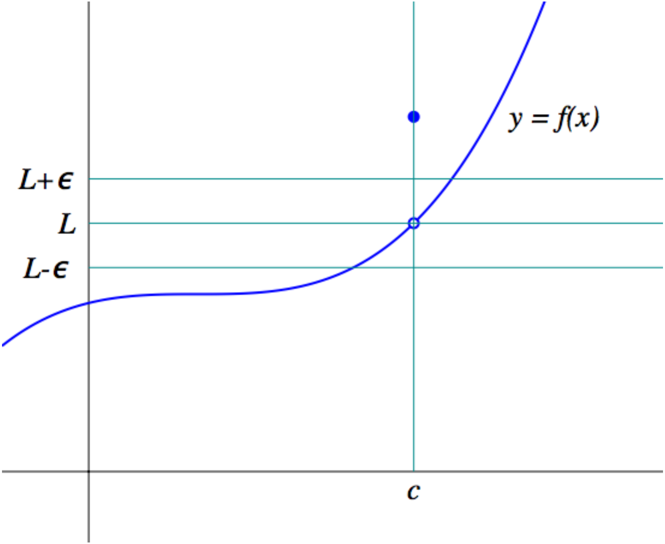
$$\forall \epsilon > 0 \, \exists \delta > 0 \text{ such that } 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

Translation into normal English:

$\forall \epsilon > 0$	"No matter how close you require f to get, ..."
$\exists \delta > 0$ such that	"...there is a distance δ such that..."
$0 < \lvert x - c \rvert < \delta \implies$	"...if x is within δ of (but not equal to) c, then..."
$\lvert f(x) - L \rvert < \epsilon$	"...$f(x)$ is closer to L than you required."

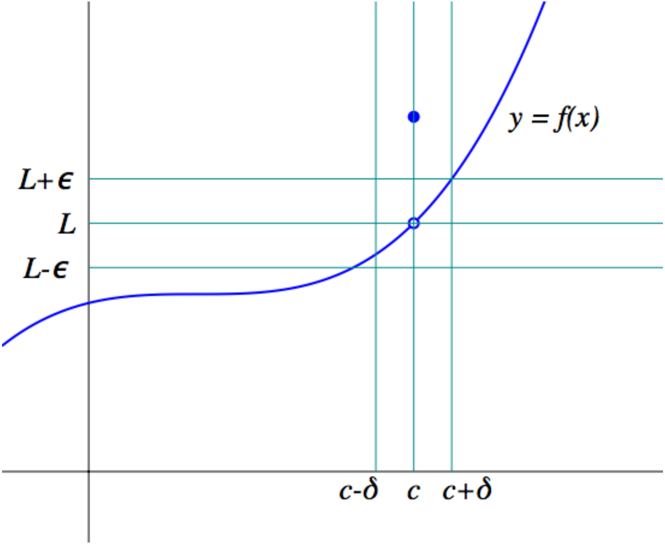
In pictures

For any positive distance ϵ from L...



In pictures

...there is some positive distance δ from c, which may depend on ϵ...



In pictures

...such that $0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$.