MAT137 - Week 10

- Your fourth problem set is due **today, at 11:59pm**. Don't leave the submission process until the last minute.
- Today's lecture is about the Mean Value Theorem and optimization.
- NOTE: You have homework from this lecture. See the last slide.

MVT

Theorem (Mean Value Theorem)

Let f be continuous on an interval $[a, b]$ and differentiable on the corresponding open interval (a, b).

Then there is a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Observations about the MVT

- The theorem is not the equation at the end. It's the entire statement. Both hypotheses are critical.
- This theorem is an *existence theorem*, like the IVT. It doesn't tell you what the value of c is. It just tells you that such a c exists. Broadly speaking, this theorem says something like:

  > *If ⟨some hypotheses are true⟩, then an equation has a solution.*

- If $f(a) = f(b)$, the MVT reduces to Rolle's Theorem. We say that Rolle's Theorem is a *special case* of the MVT.

MVT

The “moral” of the story is that because of the MVT, your intuition about derivatives as slopes of tangent lines usually leads you to correct conclusions.

Derivatives are “locally defined”. That means that in order to define and compute $f'(a)$, you only need to look at points *very close* to a. The MVT allows you to take this local data about a function and draw conclusions about the large-scale behaviour of the function.

We'll start with two of the most important examples of this, the first of which was proved in one of the videos:

Corollary

Let f be differentiable on an open interval I.

Then f is a constant function if and only if $f'(x) = 0$ for all $x \in I$.

Increasing functions

The second important application is to increasing and decreasing functions. We'll talk about increasing functions here.

Definition

Let f be a function and let I be an interval on which it is defined. We say <u>f is increasing on I</u> if whenever $x_1, x_2 \in I$ and $x_1 < x_2$, then $f(x_1) < f(x_2)$.

Note that this definition doesn't have anything to do with derivatives.

Corollary

Let f be differentiable on an open interval I, and suppose that $f'(x) > 0$ for all $x \in I$.

Then f is increasing on I.

The analogous theorem about negative derivatives and decreasing functions is also true, of course.

Increasing functions

Here's the result again.

Corollary

Let f be differentiable on an open interval I, and suppose that $f'(x) > 0$ for all $x \in I$.

Then f is increasing on I.

Exercise: Prove this theorem.

- Start by clearly stating what you have to prove.
- If you want to apply the MVT, make sure to check the hypotheses of the theorem are satisfied.
- After you've got an idea of how to do it, take a moment to think about why you couldn't have done this without the MVT.

What's wrong with this proof?

"Proof".

Let f be differentiable on (a, b), and suppose $f'(x) > 0$ for all $x \in (a, b)$.

By the MVT, we know that $\dfrac{f(b) - f(a)}{b - a} = f'(c) > 0$.

Therefore $f(b) - f(a) > 0$, so $f(b) > f(a)$.

This means f is increasing on (a, b), as required. $\square$

Exercise

Because of the result you proved, we now know we can determine whether a differentiable function is increasing or decreasing by finding the sign of its derivative.

Problem 1. On what intervals is the function $f(x) = \dfrac{3x^2}{x^3 - 4}$ increasing or decreasing?

After you've done that, try to sketch the graph as best you can.

To save you the time, here's the derivative:

$$f'(x) = \frac{6x(x^3 - 4) - 3x^2(3x^2)}{(x^3 - 4)^2} = \frac{-3x(x^3 + 8)}{(x^3 - 4)^2}.$$

Another similar exercise

Here's another similar exercise.

Problem 2. On what intervals is the function $f(x) = x\sqrt[3]{x^2 - 4}$ increasing or decreasing?

After you've done that, try to sketch the graph as best you can (you don't have to be super precise).

To save you the time, here's the derivative:

$$f'(x) = \frac{5x^2 - 12}{3(x^2 - 4)^{2/3}}.$$

Optimization

Optimization problems are one of the most important applications of calculus.

The idea is to look at a situation, model it with a function, and use techniques from calculus to find where the function achieves a maximum or minimum value.

Here's a problem you'll do in a minute, to give you an idea of what these sorts of problems look like:

Problem. A farmer has 700 metres of fencing and wants to fence off a rectangular field, along with extra fencing that divides the area into two equal parts down the middle. What dimensions should his field have?

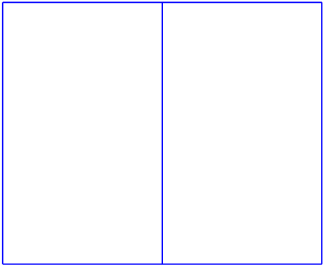
Optimization

General strategy for optimization problems:

1. Draw a picture of the situation, if possible.
2. Label relevant quantities you know and assign variables to quantities you don't know.
3. Define a function involving those variables, whose maximal or minimal value is the answer to the question.
4. Make sure to note the domain of your function in the context of the problem.
5. Find all local critical points of the function on its domain.
6. Check whether these critical points represent minima or maxima of the function.
7. Check the endpoints of the domain, if any.

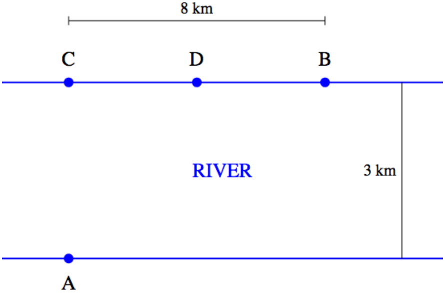
Here's that problem again, which you will now do.

Problem 1. A farmer has 700 metres of fencing and wants to fence off a rectangular field, along with extra fencing that divides the area into two equal parts down the middle. What dimensions should his field have?



We didn't see this during class, but it's a great practise problem.

Problem 2. You launch a rowboat from point A on a bank of a river, 3km wide, and want to reach point B, 8 km downstream on the opposite bank, as quickly as possible. You can row from A to the point C directly across the river and then run to B, or you can row directly from A to B, or you can row to some point D between C and B and then run to B. If you row at 6 km/h and run at 8 km/h, where should you land?



Do this problem as homework for the next lecture. Come prepared with a solution.

Problem 3. Find the area of the smallest circle centred at the point $(1, 4)$ which intersects the parabola $y^2 = 2x$.