MAT137 - Week 11

- Your second test is on **Friday, 1 December, 4-6pm**. See the course website for details.
- Today's lecture is primarily about limits at infinity, and l'Hôpital's rule.
- You have homework from this lecture. See slide 17.

A problem from last class

Last class I left you with this problem for homework.

Problem 3. Find the area of the smallest circle centred at the point $(1, 4)$ which intersects the parabola $y^2 = 2x$.

Limits at infinity

So far we've defined limits at a point, which are written like

$$\lim_{x \to a} f(x) = L$$

(if they exist). We've also defined

$$\lim_{x \to a} f(x) = \infty \quad \text{and} \quad \lim_{x \to a} f(x) = -\infty.$$

Today we'll talk about limits at infinity:

$$\lim_{x \to \infty} f(x) = L \quad \text{and} \lim_{x \to -\infty} f(x) = L.$$

Motivation

When discussing limits at a point, our intuition was that

$$\lim_{x \to a} f(x) = L$$

means something like

> *$f(x)$ can be made arbitrarily close to L by making x sufficiently close to a.*

Now, consider the function $g(x) = \frac{1}{x}$.

Exercise: Convince yourself that you can make $g(x)$ as close as you want to 0 by making x sufficiently large.

Limits at infinity

Intuitively, the notation

$$\lim_{x \to \infty} f(x) = L$$

should mean something like

> *$f(x)$ can be made arbitrarily close to L by making x sufficiently large.*

Exercise: Suppose f is a function defined on an interval of the form (p, ∞) for some real number p. Write down a definition for the statement

$$\lim_{x \to \infty} f(x) = L.$$

Limits at infinity

Definition

Suppose f is a function defined on an interval of the form (p, ∞) for some real number p. Then

$$\lim_{x \to \infty} f(x) = L$$

means

$$\forall \epsilon > 0 \ \exists M \in \mathbb{R} \text{ such that } x > M \implies |f(x) - L| < \epsilon.$$

Exercise: Write down a similar definition for the statement $\lim_{x \to -\infty} f(x) = L$.

Examples

1. Prove that $\lim_{x \to \infty} \frac{1}{x^2} = 0$.

2. Prove that $\lim_{x \to \infty} \frac{\sin(x)}{x^2} = 0$.

(This reminds us of the Squeeze Theorem, which does also apply to limits at infinity.)

3. Prove that $\lim_{x \to \infty} \cos(x)$ does not exist.

Examples

1. Evaluate $\lim_{x \to \infty} x - \sqrt{x^2 + 7}$.

2. Evaluate $\lim_{x \to \infty} \dfrac{3x^2 + 7x + 1}{8x^2 + 4}$.

The moral of Problem 2 is that we found the fastest-growing term, and divided by them to "cancel out" the growth.

Exercise. Evaluate $\lim_{x \to \infty} \dfrac{7e^{7x} + \sin(x)}{e^{7x} + 7}$.

Be careful

What's wrong with this solution? Compute: $\lim_{x\to-\infty} x - \sqrt{x^2+x}$.

Proof.

$$\lim_{x\to-\infty} x - \sqrt{x^2+x} = \lim_{x\to-\infty} x - \sqrt{x^2+x} \cdot \frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}}$$

$$= \lim_{x\to-\infty} \frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}}$$

$$= \lim_{x\to-\infty} \frac{-x}{x\left(1+\sqrt{1+\frac{1}{x}}\right)}$$

$$= \lim_{x\to-\infty} \frac{-1}{1+\sqrt{1+\frac{1}{x}}} = -\frac{1}{2}$$

□

Indeterminate forms

Recall that if we know

$$\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M,$$

(and $M \neq 0$), then the limit law for quotients tells us that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

In other words, by knowing the limits of f and g, we can determine the limit of $\frac{f}{g}$ from the *form* of the function alone.

Indeterminate forms

The same is not true if $L = M = 0$.

Exercise. For each part, find a pair of functions f and g such that

$$\lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} g(x),$$

but such that...

1. ... $\lim_{x \to 0} \frac{f(x)}{g(x)} = 7$.
2. ... $\lim_{x \to 0} \frac{f(x)}{g(x)} = 0$.
3. ... $\lim_{x \to 0} \frac{f(x)}{g(x)} = \infty$.
4. ... $\lim_{x \to 0} \frac{f(x)}{g(x)}$ doesn't exist, and doesn't equal $\pm\infty$.

Indeterminate forms

For this reason, if

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x),$$

we say that $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ is <u>indeterminate of type $\frac{0}{0}$</u>.

The same is true if

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty.$$

In this case, we say that $\lim_{x \to 0} \frac{f(x)}{g(x)}$ is <u>indeterminate of type $\frac{\infty}{\infty}$</u>.

L'Hôpital's rule is a tool for dealing with limits of these two types.

Some intuition

Here's some intuition for the statement of L'Hôpital's rule.

Suppose L_1 and L_2 are lines with slopes m_1 and m_2, respectively. Also suppose they both have zeros at $x = 7$.

Then $\displaystyle\lim_{x \to 7} \frac{L_1(x)}{L_2(x)}$ is indeterminate of type $\frac{0}{0}$.

Of course, we can just write down their equations easily:

$$L_1(x) = m_1(x-7) \quad \text{and} \quad L_2(x) = m_2(x-7).$$

and we can evaluate the limit easily:

$$\lim_{x \to 7} \frac{L_1(x)}{L_2(x)} = \lim_{x \to 7} \frac{m_1}{m_2} = \frac{m_1}{m_2} = \lim_{x \to 7} \frac{L_1'(x)}{L_2'(x)}$$

Some intuition

For more general functions (not all functions though), if

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x),$$

and in addition f and g are differentiable near (and at) a, and $g'(a) \neq 0$, then f and g are closely-approximated by their tangent lines at a:

$$L_1(x) = f'(a)(x-a) \quad \text{and} \quad L_2(x) = g'(a)(x-a).$$

So we *might* expect to get:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}.$$

THIS IS NOT A PROOF!!! ...I secretly assumed many things.

L'Hôpital's rule

This theorem is tricky to state, because there are many cases.

Theorem

Let $a \in \mathbb{R}$, and let f and g be functions defined at and near a.

Suppose that

- $\lim_{x \to a} \frac{f(x)}{g(x)}$ *is indeterminate of type* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$.
- f *and* g *are differentiable near* a *(except possibly at* a*).*
- g *is never 0 near* a *(except possibly at* a*).*
- g' *is never 0 near* a *(except possibly at* a*).*
- $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ *exists, or is* $\pm\infty$.

Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

L'Hôpital's rule

In the theorem we just stated, "near a" means "on an open interval containing a".

The theorem also holds for limits as $x \to \infty$ or $x \to -\infty$, in which case "near a" is replaced with "on an interval of the form (p, ∞) or $(-\infty, p)$ for some $p \in \mathbb{R}$", respectively.

Examples

1. Use L'Hôpital's rule to compute $\lim_{x \to 0} \frac{x^2 - 7x}{e^x - 1}$.

2. Compute $\lim_{x \to \infty} \frac{x^2}{e^x}$.

3. Compute $\lim_{x \to 0} \frac{2x - \sin(2x)}{x \sin(x)}$. (We didn't see this one in lecture.)

Homework: Show that for any natural number N, $\lim_{x \to \infty} \frac{x^N}{e^x} = 0$.

Warnings

L'Hôpital's rule is very powerful, but with great power comes great responsibility.

Warning 1: The hypotheses are all important.

Example: Evaluate $\lim_{x\to\infty} \frac{x+\sin(x)}{x}$.

INCORRECT PROOF.

The top and bottom both $\to \infty$, so this is indeterminate of type $\frac{\infty}{\infty}$. So:

$$\lim_{x\to\infty} \frac{x+\sin(x)}{x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1+\cos(x)}{1} = \lim_{x\to\infty} \left[1+\cos(x)\right].$$

The last limit doesn't exist, so $\lim_{x\to\infty} \frac{x+\sin(x)}{x}$ doesn't exist. □

(It's easy to check that the original limit does exist and equal 1.)

Warnings

Warning 2: L'Hôpital's rule doesn't always help.

Example: Evaluate $\lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

The top and bottom both $\to \infty$, so this is indeterminate of type $\frac{\infty}{\infty}$. So:

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \overset{\text{L'H}}{=} \cdots\cdots$$

These equalities are all true, they just don't go anywhere.

L'Hôpital's rule is just another tool you can use; it doesn't magically solve all problems.

(**Exercise:** Compute this limit.)

Warnings

Warning 3: Don't blindly apply it without simplifying things if you can.

[Contrived] Example: Compute $\lim_{x \to 0^+} \frac{\log(x)}{\left(\frac{1}{x}\right)}$.

$$\lim_{x \to 0^+} \frac{\log(x)}{\left(\frac{1}{x}\right)} \stackrel{\text{L'H}}{=} \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} \stackrel{\text{L'H}}{=} \lim_{x \to 0^+} \frac{\left(-\frac{1}{x^2}\right)}{\left(\frac{2}{x^3}\right)} \stackrel{\text{L'H}}{=} \lim_{x \to 0^+} \frac{\left(\frac{2}{x^3}\right)}{\left(\frac{-6}{x^4}\right)} \stackrel{\text{L'H}}{=} \cdots\cdots$$