MAT137 - Term 2, Week 4

- Reminder: Your seventh problem set is due **next Thursday, February 1st**, by 11:59pm.

- This lecture will assume you have watched all the videos on integration techniques up to and including video 9.14.

- Today we're talking about:
  - Substitution
  - Integration by parts
  - Integrating products of trig functions
  - Trigonometric substitutions

- Before next week's lecture, please watch all of the remaining videos from playlist 9.

- I've added some comments after the lecture, with red text.

Substitution

Recall from the videos that the technique of substitution is derived from integrating the chain rule.

Here's the chain rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x).$$

Integrating this yields:

$$f(g(x)) + C = \int f'(g(x))\, g'(x)\, dx.$$

Substitution

$$f(g(x)) + C = \int f'(g(x))\, g'(x)\, dx.$$

To use this formula to compute the antiderivative $\int h(x)\, dx$, you must find two functions f' and g such that

$$h(x) = f'(g(x))\, g'(x).$$

Once you have found these functions, all you need to figure out is f (which is an antiderivative of f').

Sometimes this is easy, like in the case of

$$\int 2x(x^2 + 1)^7\, dx.$$

In this case $g(x)$ should be $x^2 + 1$ and $f'(x)$ should be x^7, so $f(x) = \dfrac{x^8}{8}$.

Substitution

Sometimes (most times, sadly) it isn't quite so easy, and we have to adjust some things.

Exercise: Compute $\int \cos^2(7x)\sin(7x)\,dx$.

Substitution notation

When using the substitution rule, we will usually use the notation

$$u = g(x),$$

and

$$du = g'(x)\,dx.$$

With this notation, the substitution rule says:

$$f(u) + C = \int f'(u)\,du,$$

which is something we already know from the FTC.

This process amounts to changing the variable from x to something that's more convenient for us to integrate with.

Substitution

Some strategy for using substitution.

1. Look at your integrand, and try to find an occurence of a function $g(x)$ such that its derivative $g'(x)$ appears as a multiplicative factor. You may need to try several things before one works. Remember that if you're just missing a multiplicative constant, you can easily adjust for it manually.
2. Let $u = g(x)$ be your new variable, and then compute $du = g'(x)\,dx$.
3. Express the whole integrand in terms of u and du.
4. Compute the antiderivative (which will be doable, if you chose u well).
5. Put everything back in terms of x at the end.

Substitution exercises.

Exercise: Consider the integral $\int e^{7x+5}\,dx$. What should our u be?

Exercise: Consider the integral $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$. What should our u be?

Exercise: Consider the integral $\int \frac{x}{1+x^4}\,dx$. What should our u be? (We have seen this one before.)

Exercise: Compute the integral $\int \cot(x)\log(\sin(x))\,dx$ using substitution.

Trickier exercise: Compute $\int x\sqrt[3]{x+3}\,dx$.

Definite integrals and substitution

We must be a bit careful when using substitution to compute definite integrals.

Question: Consider the definite integral:

$$\int_0^{\frac{\pi}{2}} \sin^2(x)\,\cos(x)\,dx.$$

Which of the following does it equal?

1. $\frac{1}{3}\left[\sin^3(x)\right]_0^{\frac{\pi}{2}}$
2. $\frac{1}{3}\left[u^3\right]_0^{\frac{\pi}{2}}$
3. $\frac{1}{3}\left[\sin^3(x)\right]_0^{1}$
4. $\frac{1}{3}\left[u^3\right]_0^{1}$

A useful theorem about odd functions

Note that we didn't write a proof for this in class, though I did start you off on the proof.

Theorem

Let f be a continuous function defined on all of $\mathbb{R}$. If f is odd, then for any positive real number a,

$$\int_{-a}^{a} f(x)\,dx = 0.$$

1. Convince yourself that this theorem is true by drawing a picture.
2. Make sure you have a definition of "odd function" to work with.
3. Express the definite integral as the sum of two definite integrals in a natural way, then use substitution to prove the theorem.

Integration by parts

Here's the integration by parts formula, as derived in one of the videos:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Usually we use notation similar to what we used with the substitution rule. We let $u = f(x)$ and $v = g(x)$.

Then accordingly we write $du = f'(x)\,dx$ and $dv = g'(x)\,dx$.

With this notation, the formula looks like this:

$$\int u\,dv = uv - \int v\,du.$$

Integration by parts

$$\int u\,dv = uv - \int v\,du.$$

Note that while the substitution rule actually computed antiderivatives for us, this rule does not.

It simply turns our antiderivative into

$\langle$*something*$\rangle$ *minus* $\langle$*another antiderivative*$\rangle$.

The "art" of using this formula is choosing u and v in such a way that the new antiderivative on the right side is easier to compute.

Integration by parts examples

Note that the first exercise on this page is slightly harder than I had originally intended. The choice of parts for integration by parts is very natural, but the integral you're left with requires a bit of cleverness. I left you with a hint towards that integral during lecture.

Integration by parts will help you compute the following antiderivatives (though it won't necessarily be the only thing you need to do):

Exercise: Compute $\int x^2 \arcsin(x)\,dx$.

Exercise: Compute $\int \log(x)\,dx$.

Exercise: Compute $\int \sin(\log(x))\,dx$.

A reduction formula

You're going to prove a result that will allow you to compute the antiderivative of any positive integer power of $\log(x)$, by proving something called a *reduction formula*.

A reduction formula is a formula that expresses one integral in terms of a strictly simpler integral of the same sort.

Exercise: Let $n > 2$ be an integer. Use integration by parts to come up with a formula of this form:

$$\int \big(\log(x)\big)^n \, dx = \big[\text{SOMETHING}\big] + \big[\text{CONSTANT}\big] \int \big(\log(x)\big)^{n-1} \, dx.$$

Exercise: Use the formula you derived to compute the following antiderivative:

$$\int \big(\log(x)\big)^{1000} \, dx.$$

Integrals of certain combinations of trig functions

In this section we are going to talk about some general methods for dealing with certain combinations of trig functions.

There are no concepts to learn here. We will be using substitution and integration by parts, along with some cleverness with trig identities.

- The Pythagorean identities:
  - $\sin^2(x) + \cos^2(x) = 1$.
  - $\tan^2(x) + 1 = \sec^2(x)$.
- The angle addition identities:
  - $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.
  - $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$.
- The double angle formulas (which are easy consequences of the previous two):
  - $\sin(2x) = 2\sin(x)\cos(x)$.
  - $\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$.

Products of trig functions

Exercise: Compute $\int \cos^7(x)\,dx$.

Exercise: Let k be a positive integer. Write the following antiderivative as another antiderivative that you know how to compute:

$$\int \sin^{2k+1}(x)\,dx$$

Slightly more complicated

What if there are sines and cosines!?

Exercise: Compute $\int \sin^5(x)\cos^{17}(x)\,dx$. (Or at least write it in terms of something you know how to do.)

Hint: The idea is the same as before. Use the Pythagorean identity to express the integral as

$$\int \Big(\text{stuff in terms of } \sin(x)\Big)\cos(x)\,dx$$

or

$$\int \Big(\text{stuff in terms of } \cos(x)\Big)\sin(x)\,dx.$$

Tangents and secants

Exercise: Compute $\int \sec^{12}(x)\,dx$.

Exercise: Compute $\int \tan^{7}(x)\sec^{7}(x)\,dx$.

Can you generalize these results?

Similar methods allow us to integrate

- Any even power of $\sec(x)$,
- Any product of the form $\tan^{m}(x)\sec^{n}(x)$ in which...
  - ...at least one of m or n is odd, or
  - ...n is even.