MAT327 Big List

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September 4, 2019

This is a large, constantly growing list of problems in basic point set topology. This list will include many of the exercises given in the lecture notes. These problems are drawn from or inspired by many sources, including but not limited to:

- *Topology, Second Edition*, by James Munkres.
- *Counterexamples in Topology*, by Steen and Seebach.
- *Foundations of Topology*, by C. Wayne Patty.
- *Problems and Theorems in Classical Set Theory*, by Komjath and Totik
- Course notes and assignments by Micheal Pawliuk.
- Course notes and assignments by Peter Crooks.

They are divided into sections by topic, and rated with my opinion of their difficulty, from one to three stars, or with a † for the especially challenging ones.

Contents

1 Topologies

*1. Fix $a < b \in \mathbb{R}$. Show explicitly that the interval (a, b) is open in $\mathbb{R}_{\text{usual}}$. Show explicitly that the interval $[a, b)$ is not open in $\mathbb{R}_{\text{usual}}$.

*2. Let X be a set and let $\mathcal{B} = \{\, \{x\} : x \in X \,\}$. Show that the only topology on X that contains $\mathcal{B}$ as a subset is the discrete topology.

*3. Fix a set X, and let $\mathcal{T}_{\text{co-finite}}$ and $\mathcal{T}_{\text{co-countable}}$ be the co-finite and co-countable topologies on X, respectively.

(a) Show explicitly that $\mathcal{T}_{\text{co-finite}}$ and $\mathcal{T}_{\text{co-countable}}$ are both topologies on X.

(b) Show that $\mathcal{T}_{\text{co-finite}} \subseteq \mathcal{T}_{\text{co-countable}}$.

(c) Under what circumstances does $\mathcal{T}_{\text{co-finite}} = \mathcal{T}_{\text{co-countable}}$?

(d) Under what circumstances does $\mathcal{T}_{\text{co-countable}} = \mathcal{T}_{\text{discrete}}$?

*4. Let $(X, \mathcal{T}_{\text{co-countable}})$ be an infinite set with the co-countable topology. Show that $\mathcal{T}_{\text{co-countable}}$ is closed under *countable* intersections. Give an example to show that it need not be closed under *arbitrary* intersections.

*5. Let X be a nonempty set, and fix an element $p \in X$. Recall that

$$\mathcal{T}_p := \{\, U \subseteq X \,:\, p \in U \,\} \cup \{\emptyset\}.$$

is called the particular point topology at p on X. Show explicitly that $\mathcal{T}_p$ is a topology on X.

*6. Recall that the ray topology on $\mathbb{R}$ is:

$$\mathcal{T}_{\text{ray}} := \{\, (a, \infty) \,:\, a \in \mathbb{R} \,\} \cup \{\emptyset, \mathbb{R}\}.$$

Show explicitly that $\mathcal{T}_{\text{ray}}$ is a topology on $\mathbb{R}$. Be sure to think carefully about unions.

*7. Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be a set with the property that for every $x \in A$, there is an open set $U_x \in \mathcal{T}$ such that $x \in U_x \subseteq A$. Show that A is open.

*8. Let $(X, \mathcal{T})$ be a topological space, and let $f : X \to Y$ be an injective (but not necessarily surjective) function. Is $\mathcal{T}_f := \{\, f(U) : U \in \mathcal{T} \,\}$ necessarily a topology on Y? Is it necessarily a topology on the range of f?

*9. Let X be a set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Is $T_1 \cup \mathcal{T}_2$ a topology on X? What about $\mathcal{T}_1 \cap \mathcal{T}_2$? If yes, prove it. If not, give a counterexample.

*10. Let X be an infinite set. Show that there are infinitely many distinct topologies on X.

**11. Fix a set X, and let ϕ be a property that subsets A of X can have. For example, ϕ could be "A is countable", or "A is finite". ϕ could be "A contains p" or "A doesn't contain p" for a fixed point $p \in X$. If $X = \mathbb{R}$, ϕ could be "A is an interval" or "A contains uncountably many irrational numbers less than π". Define

$$\mathcal{T}_{\text{co-}\phi} = \{\, U \subseteq X \,:\, U = \emptyset, \text{ or } X \setminus U \text{ has } \phi \,\}.$$

Under what assumptions on ϕ is $\mathcal{T}_{\text{co-}\phi}$ a topology on X? Which topologies we have seen so far can be described in this way, using which ϕ's?

**12. Let $\{\, \mathcal{T}_\alpha \,:\, \alpha \in I \,\}$ be a collection of topologies on a set X, where I is some indexing set. Prove that there is a unique finest topology that is refined by all the $\mathcal{T}_\alpha$'s.

That is, prove that there is a topology $\mathcal{T}$ on X such that

(a) $\mathcal{T}_\alpha$ refines $\mathcal{T}$ for every $\alpha \in I$.

(b) If $\mathcal{T}'$ is another topology that is refined by $\mathcal{T}_\alpha$ for every $\alpha \in I$, then $\mathcal{T}$ is finer than $\mathcal{T}'$.

**13. This extends Exercise 8. Show with examples that the assumption that f is injective is necessary. That is, give an example of a topological space $(X, \mathcal{T})$ and a non-injective function $f : X \to Y$ such that $\mathcal{T}_f$ is a topology, and also give an example where $\mathcal{T}_f$ is not a topology.

(Hint: You can do both in $\mathbb{R}_{\text{usual}}$.)

**14. Working in $\mathbb{R}_{\text{usual}}$:

(a) Show that every nonempty open set contains a rational number.

(b) Show that there is no uncountable collection of pairwise disjoint open subsets of $\mathbb{R}$.

2 Bases for topologies

*1. Show explicitly that the collection $\mathcal{B} = \{ (a,b) \subseteq \mathbb{R} : a < b \}$ is basis, and that it generates the usual topology on $\mathbb{R}$.

*2. Show that the collection $\mathcal{B}_{\mathbb{Q}} := \{ (a,b) \subseteq \mathbb{R} : a, b \in \mathbb{Q}, a < b \}$ is a basis for the usual topology on $\mathbb{R}$.

*3. Some exercises about the Sorgenfrey line.

Recall that the collection $\mathcal{B} = \{ [a,b) \subseteq \mathbb{R} : a < b \}$ is a basis which generates $\mathcal{S}$, the Lower Limit Topology. The space $(\mathbb{R}, \mathcal{S})$ is called the Sorgenfrey line. We will reference several of these exercises later when we explore properties of this space.

(a) Show that every nonempty open set in $\mathcal{S}$ contains a rational number.

(b) Show that the interval $(0,1)$ is open in the Sorgenfrey line.

(c) More generally, show that for any $a < b \in \mathbb{R}$, (a,b) is open in the Sorgenfrey line.

(d) Is the interval $(0,1]$ open $\mathcal{S}$?

(e) Show that the $\mathcal{S}$ strictly refines the usual topology on $\mathbb{R}$.

(f) Show that the real numbers can be written as the union of two disjoint, nonempty open sets in $\mathcal{S}$.

(g) Let $\mathcal{B}_{\mathbb{Q}} := \{ [a,b) \subseteq \mathbb{R} : a, b \in \mathbb{Q}, a < b \}$. Show that $\mathcal{B}_{\mathbb{Q}}$ is *not* a basis for the Lower Limit Topology.

*4. Recall that the collection $\mathcal{B} = \{ \{x\} : x \in X \}$ is a basis for the discrete topology on a set X. If X is a finite set with n elements, then clearly $\mathcal{B}$ also has n elements. Is there a basis with *fewer* than n elements that generates the discrete topology on X?

*5. Let $X = [0,1]^{[0,1]}$, the set of all functions $f : [0,1] \to [0,1]$. Given a subset $A \subseteq [0,1]$, let

$$U_A = \{ f \in X : f(x) = 0 \text{ for all } x \in A \}.$$

Show that $\mathcal{B} := \{ U_A : A \subseteq [0,1] \}$ is a basis for a topology on X.

**6. Let $\mathcal{B}$ be a basis on a set X, and let $\mathcal{T}_{\mathcal{B}}$ be the topology it generates. Show that

$$\mathcal{T}_{\mathcal{B}} = \bigcap \{ \mathcal{T} \subseteq \mathcal{P}(X) : \mathcal{T} \text{ is a topology on } X \text{ and } \mathcal{B} \subseteq \mathcal{T} \}.$$

That is, show that $\mathcal{T}_{\mathcal{B}}$ is the intersection of all topologies that contain $\mathcal{B}$.

**7. Let $\{ \mathcal{T}_\alpha : \alpha \in I \}$ be a collection of topologies on a set X, where I is some indexing set. Prove that there is a unique coarsest topology that refines all the $\mathcal{T}_\alpha$'s.

That is, prove that there is a topology $\mathcal{T}$ on X such that

(a) $\mathcal{T}$ refines $\mathcal{T}_\alpha$ for every $\alpha \in I$.

(b) If $\mathcal{T}'$ is another topology that refines $\mathcal{T}_\alpha$ for every $\alpha \in I$, then $\mathcal{T}$ is coarser than $\mathcal{T}'$.

**8. Let $m, b \in \mathbb{Z}$ with $m \neq 0$. Recall that a set of the form $Z(m,b) = \{\, mx + b \,:\, x \in \mathbb{Z} \,\}$ is called an arithmetic progression. Show that the collection $\mathcal{B}$ of all arithmetic progressions is a basis on $\mathbb{Z}$.

The topology $\mathcal{T}_{\text{Furst}}$ that $\mathcal{B}$ generates is called the Furstenberg topology. This is an interesting topology we will use later to give a very novel proof of the infinitude of the primes.

(a) Describe the open sets of this topology (qualitatively).

(b) Show that every nonempty open set in $\mathcal{T}_{\text{Furst}}$ is infinite.

(c) Let $U \in \mathcal{B}$ be a basic open set. Show that $\mathbb{Z} \setminus U$ is open.

(d) Show that for any pair of distinct integers m and n, there are disjoint open sets U and V such that $m \in U$ and $n \in V$.

For the next three problems, we're going to define a new idea. In lectures we said that a basis can be a convenient way of specifying a topology so we don't have to list out all the open sets. Now we'll extend this idea by defining a convenient way of specifying a basis. This isn't always more convenient than just listing the elements of the basis, but often is.

Definition 2.1. *Let X be a set. A collection $\mathcal{S} \subseteq \mathcal{P}(X)$ is called a subbasis on X if the collection of all finite intersections of elements of S is a basis on X. That is, if*

$$\mathcal{B} := \{\, S_1 \cap \cdots \cap S_n \,:\, n \in \mathbb{N},\, S_1, \ldots, S_n \in \mathcal{S} \,\}$$

is a basis on X. This basis is called the basis generated by $\mathcal{S}$.

The topology generated by the basis generated by $\mathcal{S}$ is called the topology generated by $\mathcal{S}$, as you might expect.

**9. Show that the collection

$$\mathcal{S} := \{\, (-\infty, b) \,:\, b \in \mathbb{R} \,\} \cup \{\, (a, \infty) \,:\, a \in \mathbb{R} \,\}$$

is a subbasis that generates the usual topology on $\mathbb{R}$.

**10. Let $\mathcal{S}$ be a collection of subsets of a set X that covers X. That is, $X = \bigcup \mathcal{S}$. Show that $\mathcal{S}$ is a subbasis on X. Give an example of a subbasis on $\mathbb{R}$ that *does not* generate the usual topology on $\mathbb{R}$.

**11. For a prime number p, let $S_p = \{\, n \in \mathbb{N} \,:\, n \text{ is a multiple of } p \,\}$.

(a) Show that $\mathcal{S} := \{\, S_p \,:\, p \text{ is prime} \,\} \cup \{\{1\}\}$ is a subbasis on $\mathbb{N}$.

(b) Describe the open sets in the topology generated by $\mathcal{S}$ (qualitatively).

***12. Fix an infinite subset A of $\mathbb{Z}$ whose complement $\mathbb{Z} \setminus A$ is also infinite. Construct a topology on $\mathbb{Z}$ in which:

(a) A is open.

(b) Singletons are never open (ie. for all $n \in \mathbb{Z}$, $\{n\}$ is not open).

(c) For any pair of distinct integers m and n, there are disjoint open sets U and V such that $m \in U$ and $n \in V$.

3 Closed sets, closures, and dense sets

*1. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ a basis for $\mathcal{T}$. Let $A \subseteq X$. Show that $x \in \overline{A}$ if for every *basic* open set U containing x, $U \cap A \neq \emptyset$.

*2. Let $(X, \mathcal{T})$ be a topological space, and $A, B \subseteq X$. Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Is it true that $\overline{A \cap B} = \overline{A} \cap \overline{B}$? Prove it or give a counterexample.

*3. Let $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. In $\mathbb{R}_{\text{usual}}$, prove that $\overline{A} = A \cup \{0\}$.

*4. Prove all the claims that are left unproved in Example 2.3 of the lecture notes on Closures.

*5. Prove Proposition 3.4 from the lecture notes on Closures. That is, if $(X, \mathcal{T})$ is a topological space at $A \subseteq X$, show that

$$\overline{A} = \bigcap \{ C \subseteq X : C \text{ is closed, and } A \subseteq C \} .$$

*6. Prove Proposition 4.2 from the lecture notes on Closures. That is, if $(X, \mathcal{T})$ be a topological space and $D \subseteq X$, show that D is dense if and only if for every nonempty open set $U \subseteq X$, $D \cap U \neq \emptyset$.

*7. In an arbitrary topological space, is the union of two dense sets necessarily dense? What about the intersection of two dense sets? For both questions, prove it or give a counterexample.

For some of the exercises in this section, we are going to need a new definition. This definition is dual to the definition of closure in some sense, though it will not be quite as useful for us.

Definition 3.1. *Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. We define the interior of A in $(X, \mathcal{T})$, denoted* $\text{int}(A)$*, by:*

$$x \in \text{int}(A) \textit{ if and only if there is an open set } U \textit{ containing } x \textit{ such that } U \subseteq A.$$

*8. Show for any topological space $(X, \mathcal{T})$ and any $A \subseteq X$, that $\text{int}(A)$ is open.

*9. Going further than the previous exercise, show that

$$\text{int}(A) = \bigcup \{ U \subseteq X : U \text{ is open, and } U \subseteq A \} .$$

Many texts will take the equation above as the definition of the interior of a set.

*10. Show that a subset A of a topological space X is open if and only if $A = \text{int}(A)$.

*11. Find the interiors and closures of the following sets in the given spaces:

(a) $(0, 1]$ in $\mathbb{R}_{\text{usual}}$.

(b) $(0,1]$ in the Sorgenfrey Line.

(c) $(0,1]$ in $(\mathbb{R}, \mathcal{T}_{\text{indiscrete}})$.

(d) $(0,1]$ in $(\mathbb{R}, \mathcal{T}_{\text{discrete}})$.

(e) $(0,1]$ in $(\mathbb{R}, \mathcal{T}_{\text{ray}})$.

(f) $(0,1]$ in $(\mathbb{R}, \mathcal{T}_{\text{co-finite}})$.

(g) The set E of even numbers in $(\mathbb{Z}, \mathcal{T}_{\text{co-finite}})$.

(h) $\mathbb{Q}$ in $\mathbb{R}_{\text{usual}}$.

(i) $\mathbb{Q}$ in the Sorgenfrey Line.

(j) $\mathbb{Q} \times \mathbb{Q}$ in $\mathbb{R}^2_{\text{usual}}$.

(k) $\{ (x,y,z) \in \mathbb{R}^3 \, : \, x = 0 \}$ in $\mathbb{R}^3_{\text{usual}}$.

Another new definition, intimately connected to interiors and closures.

Definition 3.2. *Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. We define the boundary of A in $(X, \mathcal{T})$, denoted $\partial(A)$, by*

$x \in \partial(A)$ *if and only if for every open set U containing x, $U \cap A \neq \emptyset$ and $U \cap (X \setminus A) \neq \emptyset$.*

Intuitively, these are the points that are close to both A and its complement.

*12. Let A be a subset of a topological space X. Show that $\partial(A) = \overline{A} \cap \overline{X \setminus A} = \overline{A} \setminus \text{int}(A)$.

*13. Let A be a subset of a topological space X. Show that $\overline{A} = A \cup \partial(A)$ and $\text{int}(A) = A \setminus \partial(A)$.

**14. Let A be a subset of a topological space X. Show that $X = \text{int}(A) \sqcup \partial(A) \sqcup \text{int}(X \setminus A)$.

(The $\sqcup$ is the "disjoint union" symbol. It denotes the same operation as the usual $\cup$, but specifies that the sets being unioned are disjoint. So to give a full solution, you must show that the three sets above are disjoint, and that their union is X.)

**15. A subset A of a topological space X is called regular open if $\text{int}(\overline{A}) = A$. Regular open sets play an important role in set theoretic topology.

(a) Show that in $\mathbb{R}_{\text{usual}}$, any open interval (a,b) is regular open.

(b) Let A be a subset of a topological space X. Is it true that $\text{int}(\overline{A}) = \overline{\text{int}(A)}$? If not, is there containment one way or the other?

(c) Show that the intersection of two regular open sets is again regular open (in any topological space).

(d) Is the union of two regular open sets again regular open? Prove it or give a counterexample.

(e) Given a subset A of a topological space X, let $A^\perp = X \setminus \overline{A}$. Show that a set A is regular open if and only if $(A^\perp)^\perp = A$.

**16. Let A be a subset of $\mathbb{R}^n$ with its usual topology. Show that $x \in \overline{A}$ if and only if there exists a sequence of elements of A that converges to x.

**17. We have already learned that $\mathbb{Q}$ is dense in $\mathbb{R}$ with its usual topology. Is $\mathbb{Q} \setminus \{0\}$ dense? How about if you remove finitely many points from $\mathbb{Q}$? Is there an infinite set of points you can remove from $\mathbb{Q}$ that leaves the resulting set dense?

**18. Let $(X, \mathcal{T})$ be a topological space, and let D_1 and D_2 be dense open subsets of X. Prove that $D_1 \cap D_2$ is dense and open. Give an example in $\mathbb{R}_{\text{usual}}$ to show that this does not extend even to countably infinite intersections. That is, give an example of a collection $\{ D_n : n \in \mathbb{N} \}$ of dense open subsets of $\mathbb{R}_{\text{usual}}$ such that $\bigcap_{n=1}^{\infty} D_n$ is not open (as you will soon see, such an intersection must be dense).

**19. Recall the Furstenberg topology $\mathcal{T}_{\text{Furst}}$ on $\mathbb{Z}$, introduced in the exercises from the previous section. To remind you, this is the topology on $\mathbb{Z}$ generated by the basis consisting of all infinite arithmetic progressions in $\mathbb{Z}$. Earlier, you proved that every nonempty open subset in $\mathcal{T}_{\text{Furst}}$ is infinite. You also proved that for every basic open set U in $\mathcal{T}_{\text{Furst}}$, $\mathbb{Z} \setminus U$ is open. We now know this is the same as saying every basic open subset is closed.

You are going to use this topology to give a slick, elegant proof that there are infinitely many prime numbers.

(a) Using the notation from when this topology was first introduced, show that

$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \text{ is prime}} Z(p, 0).$$

(b) Assume for the sake of contradiction that there are only finitely many primes. Deduce from this assumption that $\mathbb{Z} \setminus \{-1, 1\}$ is closed.

(c) Find a contradiction resulting from the previous part, and conclude that there must be infinitely many primes.

***20. Prove that if $\{ D_n : n \in \mathbb{N} \}$ is a collection of dense open subsets of $\mathbb{R}_{\text{usual}}$, then $\bigcap_{n=1}^{\infty} D_n$ is dense.

This is a special case of a famous result called the Baire Category Theorem. You will need the result of Exercise 16 above and you might need to recall the definition of a Cauchy sequence. This is hard, but probably not as hard as you might expect for a "famous" theorem. Try to extend the argument you used to prove that the intersection of two (and therefore finitely many) dense open sets is dense open in a clever way.

Give an example in $\mathbb{R}_{\text{usual}}$ to show that this result does not extend to arbitrary (ie. uncountable) intersections.

†21. This is a famous problem called the Kuratowski 14-set problem.

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Prove that at most 14 distinct sets can be obtained by successively applying the operations of closures and complements to A any (finite) number of times. Give an example of a subset $A \subseteq \mathbb{R}$ with its usual topology such that 14 different sets can be obtained in this way.

Feel free to do some research on this problem online. Many detailed write-ups are available on the subject. It is really more of an algebraic problem then a topological one, but I include it because the result is so striking.

4 Countability

This section contains several problems that while very interesting, are not directly pertinent to this course. You are welcome to just attempt the one- and two-star problems if you are not interested in furthering your knowledge of countability beyond what is necessary to succeed in MAT327. That said, this is a favourite subject of mine, so I have included some of my favourite ideas in this section, mostly in the three-star problems.

*1. Construct an explicit bijection $f : \mathbb{N} \to \mathbb{Z}$ (ie. give a formula for such a bijection).

*2. Prove Proposition 4.11 from the lecture notes on Countability.

While doing this, try to isolate which of the implications involve some amount of the Axiom of Choice. If you're not familiar with the Axiom of Choice at the moment, wait until after we've discussed it in lecture before thinking about this.

*3. Let A be a countable set. Prove that $\text{Fin}(A) := \{ X \subseteq A : X \text{ is finite} \}$ is countable.

*4. Let A and B be countable sets. Prove that $A \times B$ is countable. Then show that the Cartesian product of finitely many countable sets is countable.

*5. Complete the proof of Corollary 5.5 in the lecture notes on Countability. That is, if A_n, $n \in \mathbb{N}$ are all countably infinite, mutually disjoint sets and $f_n : \mathbb{N} \to A_n$ witnesses that A_n is countable, prove that $g : \mathbb{N} \times \mathbb{N} \to \bigcup_{n \in N} A_n$ defined by $g(n, i) = f_n(i)$ is a bijection. Conclude from this that a countable union of countable sets is countable.

*6. Show that $\mathcal{B}_\mathbb{Q} = \{ (a, b) \subseteq \mathbb{R} : a, b \in \mathbb{Q} \}$ is countable, and conclude that $\mathbb{R}_{\text{usual}}$ is second countable.

*7. A topological space $(X, \mathcal{T})$ is said to have <u>the countable chain condition</u> (usually we just say "$(X, \mathcal{T})$ is ccc") if there are no uncountable collections of mutually disjoint open subsets of X. In an exercise from section 1, you proved that $\mathbb{R}_{\text{usual}}$ is ccc. Prove that any separable space is ccc.

*8. Prove the uncountable pigeonhole principle. That is, if A is an uncountable set and A_n, $n \in \mathbb{N}$ are mutually disjoint subsets of A such that $A = \cup_{n \in \mathbb{N}} A_n$, show that at least one of the A_n must be uncountable.

**9. Show that $\mathcal{B} = \{ B_\epsilon(x) \subseteq \mathbb{R}^n : x \in \mathbb{Q}^n \text{ and } 0 < \epsilon \in \mathbb{Q} \}$ is countable, and conclude that $\mathbb{R}^n_{\text{usual}}$ is second countable.

**10. In a previous exercise, you showed that the collection $\mathcal{B}_\mathbb{Q}$ of intervals $[a, b)$ with rational endpoints is not a basis for the Sorgenfrey line. Take this a step further by proving that there is *no* countable basis for the Sorgenfrey line. That is, prove that the Sorgenfrey line is not second countable.

**11. Suppose $A \subseteq \mathbb{R}$ is countable. Show that there exists a real number x such that $A \cap (x + A) = \emptyset$. (Here, $x + A = \{ x + a : a \in A \}$.)

**12. Show that the set $2^{\mathbb{N}}$ of all functions $f : \mathbb{N} \to \{0, 1\}$ is uncountable.

(Hint: Use the fact that $\mathcal{P}(\mathbb{N})$ is uncountable.)

**13. Let $f_n : \mathbb{N} \to \mathbb{N}$ ($n \in \mathbb{N}$) be a fixed collection of functions. Construct a function $g : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$\lim_{k \to \infty} \frac{g(k)}{f_n(k)} = \infty$$

(In words, construct a function g that increases faster than *all* of the f_n's.)

***14. Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions. Define the set $B(f, g) := \{ k \in \mathbb{N} : f(k) = g(k) \}$. That is, $B(f, g)$ is the set of numbers on which f and g agree.

Your task: Construct a family $\{f_n\}_{n \in \mathbb{N}}$ of functions $\mathbb{N} \to \mathbb{N}$ with the property that for any $g : \mathbb{N} \to \mathbb{N}$ and any $N \in \mathbb{N}$, there is an f_n in your family such that $|B(f_n, g)| > N$.

***15. We have already proved that $(0, 1)$ is uncountable and that $\mathcal{P}(\mathbb{N})$ is uncountable. Prove that these two sets are of the same cardinality. That is, construct a bijection $f : \mathcal{P}(\mathbb{N}) \to (0, 1)$ as explicitly as possible.

***16. The following is a beautifully slick proof that the real numbers are uncountable.

We define a game for two players. Ahead of time, fix a subset $A \subseteq [0, 1]$.

Player I starts by choosing a number $a_1 \in (0, 1)$. Player II responds by choosing a number $b_1 \in (a_1, 1)$. The process then repeats inside the interval $[a_1, b_1]$: Player I chooses a number $a_2 \in (a_1, b_1)$, and Player II chooses a number $b_2 \in (a_2, b_1)$. The game continues in this way for all n: at stage $n + 1$, Player 1 chooses a number $a_{n+1} \in (a_n, b_n)$, and Player II responds by choosing a number $b_{n+1} \in (a_{n+1}, b_n)$.

At the end of the game, the two players have created a sequence of nested, closed intervals $C_n = [a_n, b_n]$. The sequence $\{a_n\}$ is increasing by construction and bounded above (by 1, for example), and so it converges by the Monotone Sequence Theorem. Let a be its limit. We say that Player I wins the game if $a \in A$, and Player II wins the game if $a \notin A$.

Prove that if A is countable, then Player II has a winning strategy. Then prove as an immediate corollary that $[0, 1]$ is uncountable.

***17. In this exercise, you will prove a very useful fact from set theory called the Δ-System Lemma.

Definition 4.1. *Let A be a set. A collection $\mathcal{A}$ of finite subsets of A is called a* <u>*Δ-system*</u> *if there is a finite set $r \subseteq A$ such that $a \cap b = r$ for any distinct $a, b \in \mathcal{A}$. The set r, which may be empty, is called the* <u>root</u> *of the Δ-system.*

Prove the following:

Theorem 4.2 (Δ-System Lemma)**.** *Let A be an uncountable set, and let $\mathcal{A}$ be an uncountable collection of finite subsets of A. Prove that there is an uncountable $\mathcal{B} \subseteq \mathcal{A}$ that is a Δ-system.*

Hint: First use the uncountable pigeonhole principle to reduce to the situation in which every element of $\mathcal{A}$ has size n for some fixed $n \in \mathbb{N}$. Then proceed by induction on n. At some point, you will need to use transfinite induction, or something like it. If you are not comfortable with that, try to give an intuitive justification of what you have to do, and discuss it with me afterwards.

***18. Call a subset $X \subseteq \mathbb{R}^2$ a "Y-set" if it is the union of three straight line segments that share a common endpoint. (That is, a set that looks like a capital Y, except the three lines can meet at any angles and can each be of any finite length.)

Prove that any collection of mutually disjoint Y-sets is countable.

More generally, what other capital letters of the English alphabet have this property? For example, is there an uncountable collection of mutually disjoint circles (ie. O-sets) in $\mathbb{R}^2$? What about B-sets? H-sets?

†19. (Please note, this problem is very hard, not to mention outside the scope of this course. The correct solution—at least the one I know—requires a great deal of ingenuity, as well as some knowledge I do not expect any students in this class to have. If you want to work on this problem, talk to me and I will be very happy to give you some topics to read about. If you don not want to get into material outside the scope of this course—and that's totally fine—but still want to get something useful out of this problem, you should try to come up with some reasonable guesses for strategies, and prove why none of them work.)

We define another game, played on $\mathbb{R}$. This one is simpler to define than the previous one. Player I starts by picking an uncountable set $X_1 \subseteq \mathbb{R}$. Player II responds by picking an uncountable set $X_2 \subseteq X_1$. The process then repeats, forming a chain of sets $\mathbb{R} \supseteq X_1 \supseteq X_2 \supseteq X_3 \cdots$.

Player I is declared the winner if $X := \bigcap_{n \in \mathbb{N}} X_n$ is nonempty, and Player II is declared the winner if $X = \emptyset$.

Prove that Player II always has a winning strategy.

5 Sequence convergence and first countability

⋆1. Prove that every Hausdorff space is T_1, and that every T_1 space is T_0.

⋆2. Let $(X, \mathcal{T})$ be a T_1 topological space, and let $x \in X$. Show the constant sequence $x, x, x, x, \ldots$ converges to x and to no other point.

⋆3. Let $(X, \mathcal{T})$ be a topological space. Prove that the following are equivalent.

(a) $(X, \mathcal{T})$ is T_1.

(b) For every $x \in X$, $\{x\}$ is closed.

(c) Every constant sequence in X converges only to its constant value.

(d) Every finite subset of X is closed.

(e) For every subset $A \subseteq X$, $A = \bigcap \{ U \subseteq X : U \text{ is open and } A \subseteq U \}$.

⋆4. Let X be uncountable. Show that $(X, \mathcal{T}_{\text{co-countable}})$ is not first countable.

⋆5. Let X be a finite set. Show that the only T_1 topology on X is the discrete topology.

⋆6. Let X be a set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two distinct topologies on X such that $\mathcal{T}_1 \subseteq \mathcal{T}_2$. If $(X, \mathcal{T}_1)$ is Hausdorff, does that imply that $(X, \mathcal{T}_2)$ is Hausdorff? What about the other direction? In both cases, prove it or give a counterexample.

⋆7. Show that every second countable topological space is both separable and first countable.

⋆⋆8. Which of the spaces have seen so far in the course are first countable? Which are Hausdorff? This is an exercise you should make a point of doing whenever we define new properties of topological spaces. Essentially, you should be mentally constructing a table like the following one, and filling in "yes" or "no" whenever we learn something new. Most of these have been resolved by examples in the notes or earlier problems, so there isn't much to think about in most cases.

	T_0	T_1	Hausdorff	separable	1st countable	2nd countable	ccc
$\mathbb{R}_{\text{usual}}$							
$(\mathbb{R}, \mathcal{T}_{\text{co-finite}})$							
$(\mathbb{R}, \mathcal{T}_{\text{co-countable}})$							
$(\mathbb{R}, \mathcal{T}_{\text{discrete}})$							
$(\mathbb{R}, \mathcal{T}_{\text{ray}})$							
$(\mathbb{R}, \mathcal{T}_7)$							
Sorgenfrey Line							
$(\mathbb{N}, \mathcal{T}_{\text{co-finite}})$							
$(\mathbb{N}, \mathcal{T}_{\text{co-countable}})$							
$(\mathbb{N}, \mathcal{T}_{\text{discrete}})$							

**9. A subset A of a topological space $(X, \mathcal{T})$ is called a <u>G_δ set</u> if it equals a countable intersection of open subsets of X. In other words, $A \subseteq X$ is a G_δ set if there exist open sets U_n, $n \in \mathbb{N}$, such that $A = \bigcap_{n \in \mathbb{N}} U_n$. Show that in a first countable T_1 space, $\{x\}$ is a G_δ set for every $x \in X$.

The property that every singleton is a G_δ set is occasionally of specific interest in set theoretic topology. We often talk about spaces that have "points G_δ". We will not consider this property much in our course though.

**10. A topological space is called a <u>G_δ space</u> if every closed set is G_δ. Give an example of a space that is not a G_δ space.

**11. Let $(X, \mathcal{T})$ be a first countable topological space, and let $A \subseteq X$. Prove that $x \in \overline{A}$ if and only if there is a sequence of elements of A converging to x.

**12. Let $(X, \mathcal{T})$ be a first countable topological space. Show that every $x \in X$ has a countable *nested* local basis. That is, for every $x \in X$, show that there is a local basis $\mathcal{B}_x = \{ B_n : n \in \mathbb{N} \}$ such that $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$.

**13. Is every countable topological space (ie. one in which the underlying set is countable) separable? What about ccc? Can a countable space be first countable but not second countable?

***14. Construct a topological space $(X, \mathcal{T})$ which is countable (ie. X is countable) but not first countable.

(Note, no topological space we have seen so far does this. You really have to construct a new space from scratch. Feel free to do some research on the subject, but not before you spend some time thinking about it.)

6 Continuous functions and homeomorphisms

*1. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be continuous in the first-/second-year calculus sense. That is, for every $a \in \mathbb{R}^n$ and for every $\epsilon > 0$ there is a $\delta > 0$ such that $x \in B_\delta(a) \subseteq \mathbb{R}^n$ implies $f(x) \in B_\epsilon(f(a)) \subseteq \mathbb{R}^k$. Show that f is continuous in the topological sense we have defined.

(This proof is in the notes, more or less, but go through it yourself to make sure you understand.)

*2. Let X be a set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Show that the identity function $\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ given by $\mathrm{id}(x) = x$ is continuous if and only if $\mathcal{T}_1$ refines $\mathcal{T}_2$.

*3. Show that addition and multiplication, thought of as functions from $\mathbb{R}^2$ to $\mathbb{R}$ with their usual topologies, are continuous functions. (That is, show that the map $+ : \mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x + y$ is continuous). For this problem it will be useful to use one of the equivalent definitions of continuity given in the lecture notes.

*4. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous when the domain and codomain both have the usual topology, but not continuous when they both have the ray topology or the Sorgenfrey/lower limit topology.

*5. Prove that all the properties in Proposition 6.2 in the lecture notes on continuous functions are indeed topological invariants. (All of these proofs should be *very* straightforward from the definitions, so if you find yourself having to get creative at all you are likely overthinking things.)

*6. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, let $D \subseteq X$ be dense in X, and suppose $f : X \to Y$ is continuous and surjective. Show that $f(D) = \{ f(d) \in Y : d \in D \}$ is dense in Y. Conclude that a surjective, continuous image of a separable space is separable.

(This proof is essentially in the lecture notes.)

*7. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and suppose that $f : X \to Y$ is a bijection. Show that the following are equivalent:

(a) f^{-1} is continuous.
(b) f is open.
(c) f is closed.

(The equivalence between (b) and (c) is Proposition 4.3 in the lecture notes on continuous functions. Again, these proofs should be immediate from the definitions and some elementary properties of functions and sets. If you find yourself having to get creative, you are overthinking.)

**8. Characterize the continuous functions from $\mathbb{R}_{\text{co-countable}}$ to $\mathbb{R}_{\text{usual}}$, and from $(\mathbb{R}, \mathcal{T}_7)$ to $\mathbb{R}_{\text{usual}}$ (recall that $\mathcal{T}_7$ is the particular point topology at 7).

**9. This is a result I mentioned several times while talking about sequences (and in the notes on nets). With our experience with first countable spaces now, it should be relatively straighforward.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and let $f : X \to Y$.

(a) Suppose f is continuous, and $\{x_n\}_{n=1}^{\infty}$ is a sequence in X converging to a point x. Show that $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x)$.

(b) Suppose $(X, \mathcal{T})$ is first countable, and suppose for each $x \in X$ and each sequence $\{x_n\}_{n=1}^{\infty}$ that converges to x, $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x)$. Show that f is continuous.

**10. Let $f : \mathbb{R}_{\text{usual}} \to \mathbb{R}_{\text{usual}}$ be continuous. Let $g : \mathbb{R}_{\text{usual}} \to \mathbb{R}^2_{\text{usual}}$ be defined by $g(x) = (x, f(x))$. Show that g is continuous.

(Hint: Use the previous exercise. You don't have to, but it might be easier.)

**11. Let $(X, \mathcal{T})$ be a topological space and let $f, g : X \to \mathbb{R}_{\text{usual}}$ be continuous functions. Prove that $f + g$ (ie. $(f + g)(x) = f(x) + g(x)$) is continuous. Do the same for $f(x)g(x)$.

(Hint: I would suggest not doing this directly from the definition of continuity. Also, the function $h : X \to \mathbb{R}^2$ defined by $h(x) = (f(x), g(x))$ might be useful to consider.)

**12. Let $(X, \mathcal{T})$ be a topological space, and let $(Y, \mathcal{U})$ be a Hausdorff topological space. Let $A \subseteq X$ be a nonempty subset. Suppose $f, g : X \to Y$ are continuous functions that agree on all points of A (ie. $f(a) = g(a)$ for all $a \in A$). Show that they must agree on all points of $\overline{A}$.

Conclude that $A_{f,g} := \{\, x \in X \;:\; f(x) = g(x) \,\}$ is closed.

(Note that neither of the spaces need be first countable, so sequences should not be involved here.)

Also conclude that if $f, g : X \to Y$ are continuous functions, $D \subseteq X$ is dense, Y is Hausdorff, and $f(x) = g(x)$ for all $x \in D$, then $f = g$. One consequence of this fact is that in order to define a continuous function $f : \mathbb{R}_{\text{usual}} \to \mathbb{R}_{\text{usual}}$, it suffices to define it on the rationals.

***13. Show that $\mathbb{R}_{\text{usual}}$ and $\mathbb{R}^2_{\text{usual}}$ are not homeomorphic.

(This problem will be very easy later in the course, but I want you to think about it now. Note that no topological invariant we have defined thus far will distinguish them.)

†14. Show that $\mathbb{R}^n_{\text{usual}}$ is not homeomorphic to $\mathbb{R}^m_{\text{usual}}$ for any $n \neq m$ where $n, m > 1$.

At this point in the course I do not expect anyone in the course to be able to solve this problem without *substantial* creativity. As with some previous problems of this difficulty, I would like you to spend some time think about it. In particular, if you asked someone who cared about vector space structure (we do not care about that in this course) this question, they would say "the two spaces have different dimensions, so they cannot be isomorphic as vector spaces". This is because "having dimension n" is an invariant property of vector spaces.

The natural question is then whether there is a topological notion of dimension. This is an interesting question, because topological spaces are much more interesting than vector spaces (particularly than finite-dimensional vector spaces). There are a few such notions, but defining them is quite tricky. It seems obvious that $\mathbb{R}^n_{\text{usual}}$ should have dimension n, but what about other topologies on the reals? Do $\mathbb{R}_{\text{usual}}$ and the Sorgenfrey Line feel they should have the same dimension? What about $\mathbb{R}_{\text{usual}}$ and $\mathbb{Q}$? These are hard questions, and should give you some feeling for how much work it takes to define a topological notion of dimension.

7 Subspaces

*1. Let $(X, \mathcal{T})$ be a topological space and let Y be a subspace of X. Show that if U is an open subset of Y and Y is an open subset of X, then U is an open subset of X.

*2. Let A be a subspace of X. For any $B \subseteq A$, show that $\mathrm{cl}_A(B) = A \cap \mathrm{cl}_X(B)$, where $\mathrm{cl}_X(B)$ denotes the closure of B computed in X, and similarly $\mathrm{cl}_A(B)$ denotes the closure of B computed in the subspace topology on A.

*3. Let $f : X \to Y$ be a continuous function, and let A be a subspace of X. Show that the restriction of f to A, $f \restriction A : A \to Y$, is continuous.

*4. Let $f : X \to Y$ be a homeomorphism, and let A be a subspace of X. Show that $B := f(A)$ is a subspace of Y, and $f \restriction A : A \to B$ is a homeomorphism.

*5. Let B be a subspace of Y, and let $f : X \to B$ be a continuous function. Show that $f : X \to Y$ is continuous (we are not altering the function here at all, just expanding the space we think of as its codomain).

*6. Prove the Pasting Lemma. That is, let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $A, B \subseteq X$ be both closed (or both open) subsets of X such that $X = A \cup B$, thought of as subspaces. Suppose $f : A \to Y$ and $g : B \to Y$ are continuous functions that agree on $A \cap B$ (ie. $f(x) = g(x)$ for all $x \in A \cap B$). Define $h : X \to Y$ by

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

Show that h is continuous.

*7. Prove that the properties of being countable, first countable, second countable, T_0, and T_1 are all hereditary.

*8. We saw in the lecture notes that separability is not hereditary. Show that an *open* subspace of a separable space is separable. That is, show that if $(X, \mathcal{T})$ is a separable space and $U \subseteq X$ is open, then U with its subspace topology is separable.

(A property that is inherited by every *open* subspace is sometimes called weakly heriditary.)

*9. Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Let $i : A \to X$ be the inclusion function, defined by $i(x) = x$. Show that the subspace topology A inherits from X is the coarsest topology on A such that i is continuous.

*10. Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. A is called sequentially closed if the limit point of every convergent sequence $\{x_n\} \subseteq A$ is in A. We already know that every closed set is sequentially closed. $(X, \mathcal{T})$ is called sequential if every sequentially closed subset is closed.

Show that the property of being sequential is a topological invariant.

**11. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $A \subseteq Y$ be a subspace (with its subspace topology inherited from Y). If there is a homeomorphism $f : X \to A$, we say that X is embedded in Y, and that f is an embedding.

Give examples of topological spaces X and Y such that X is embedded in Y and Y is embedded in X, but which are not homeomorphic.

**12. This exercise is a prelude to a subject we will talk about at the end of the course. For the moment, you can use it to solve one of the harder problems from earlier in the notes.

(a) In a topological space $(X, \mathcal{T})$, the sets X and $\emptyset$ are always both clopen (that is, both closed and open). These two are called trivial clopen subsets of X. Show that the property of having a nontrivial clopen subset is a topological invariant.

(b) In a topological space $(X, \mathcal{T})$ with no nontrivial clopen subsets, a point $p \in X$ is called a cut point if $X \setminus \{p\}$ (with its subspace topology) has a nontrivial clopen subset. Show that for any $n \in \mathbb{N}$, the property of having n cut points is a topological invariant.

(c) You may assume that $\mathbb{R}_{\text{usual}}$ has no nontrivial clopen subsets. Prove that no two of $(0, 1)$, $[0, 1)$, and $[0, 1]$ with their subspace topologies inherited from $\mathbb{R}_{\text{usual}}$ are homeomorphic.

(d) Show that $\mathbb{R}_{\text{usual}}$ is not homeomorphic to $\mathbb{R}^n_{\text{usual}}$ for all $n > 1$.

**13. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $f : X \to Y$ be a function. f is called a local homeomorphism if for every $x \in X$ there is an open set $U \subseteq X$ containing x and an open set $V \subseteq Y$ such that $f \restriction U : U \to V$ is a homeomorphism (where U and V have their subspace topologies).

(a) Show that every homeomorphism is a local homeomorphism.

(b) Show that every local homeomorphism is continuous and open, and conclude that a bijective local homeomorphism is a homeomorphism.

(c) Give an example of a map $f : X \to Y$ between topological spaces that is a local homeomorphism but not a homeomorphism.

***14. In the last problem of the previous section we saw that it might be useful to define a "dimension" for topological spaces. There are at least three reasonable definitions of dimensions that I know. Here we will define one called the inductive dimension (or sometimes the small inductive dimension).

Let $(X, \mathcal{T})$ be a topological space. We define the inductive dimension $\text{ind}(X)$ recursively as follows.

(a) $\text{ind}(\emptyset)$ is defined to equal -1.
(b) $\text{ind}(X) \leq n$ if for every $x \in X$ and every open set U containing x, there exists an open set V such that $x \in \overline{V} \subseteq U$ and $\text{ind}(\partial V) \leq n-1$. (Recall that ∂V is the boundary of V, here considered as a subspace of X.)
(c) $\text{ind}(X) := n$ if $\text{ind}(X) \leq n$ and $\text{ind}(X) \not\leq n-1$.
(d) $\text{ind}(X) := \infty$ if $\text{ind}(X) \not\leq n$ for all $n \in \mathbb{N}$.

This definition looks extremely complicated, but once you work with it a bit you will begin to like it. Here are some exercises to start you off.

(a) Prove that the inductive dimension of a topological space is a topological invariant.
(b) Prove that if $(X, \mathcal{T})$ has a basis consisting of clopen sets, then $\text{ind}(X) = 0$. This result allows you to conclude that some spaces we have seen are are 0-dimensional in this sense. Which ones?
(c) Prove that $\text{ind}(\mathbb{R}_{\text{usual}}) = 1$, $\text{ind}(\mathbb{R}^2_{\text{usual}}) = 2$, and $\text{ind}(\mathbb{R}^3_{\text{usual}}) = 3$.

***15. This is for the budding category theorists out there. It is not actually that hard, just a bit of a specialized problem. What we are going to do here is give an alternative characterization of the subspace topology in terms of what is called a <u>universal property</u>.

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Let $i : A \to X$ be the inclusion function, defined by $i(x) = x$. Suppose $\mathcal{T}_A$ is a topology on A. Show that $\mathcal{T}_A$ is the subspace topology inherited from X if and only if it has the following property:

For any topological space $(Y, \mathcal{U})$ and any function $f : (Y, \mathcal{U}) \to (A, \mathcal{T}_A)$, f is continuous if and only if $i \circ f : Y \to X$ is continuous.

†16. Construct an example to demonstrate that the property of being sequential is not hereditary.

If you are interested in doing this problem, you will likely need some extra guidance. Please feel free to talk to me about it and I will give you some things to read about.

8 Finite products

There isn't really too much interesting to ask about finite products at this stage. We will *more than* make up for this when we start talking about infinite products, which are very interesting.

*1. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. Show that $X \times Y \simeq Y \times X$.

*2. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let A and B be subsets of X and Y, respectively. Show that $\overline{A \times B} = \overline{A} \times \overline{B}$ and $\operatorname{int}(A \times B) = \operatorname{int}(A) \times \operatorname{int}(B)$.

*3. Prove that the following topological properties are all finitely productive.

   (a) T_0 and T_1.
   (b) Separable.
   (c) First countable.
   (d) Second countable.
   (e) Finite (ie. the underlying set of the space is finite).
   (f) Countable.
   (g) The property of being a discrete space.
   (h) The property of being an indiscrete space.

*4. Prove explicitly that $\mathbb{R}^2_{\text{usual}} = (\mathbb{R}_{\text{usual}})^2$. (Equal, not just homeomorphic.)

*5. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let A and B be subspaces of X and Y respectively. Show that the product topology on $A \times B$ defined from the subspace topologies on A and B is equal to the subspace topology $A \times B$ inherits from the product topology on $X \times Y$.

*6. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let π_1 and π_2 be the usual projections. Show that they are both open functions.

*7. Let $(X, \mathcal{T})$ be a topological space. Define a subset of $X^2 = X \times X$ called the diagonal by

$$\Delta := \left\{ (x, x) \in X^2 \,:\, x \in X \right\}.$$

Show that $(X, \mathcal{T})$ is homeomorphic to Δ with its subspace topology inherited from the product topology on X^2.

*8. This is one of the all-time classic topology problems. Let $(X, \mathcal{T})$ be a topological space. Prove that $(X, \mathcal{T})$ is Hausdorff if and only if Δ (as defined in the previous problem) is closed in the product topology.

**9. Show that $(\mathbb{R}_{\text{Sorgenfrey}})^2$ is separable but not hereditarily separable.

9 Stronger separation axioms

*1. Show that a topological space $(X, \mathcal{T})$ is regular if and only if for every point $x \in X$ and every open set U containing x, there is an open set V such that $x \in V \subseteq \overline{V} \subseteq U$. (Be sure to draw a picture.)

*2. Show that a topological space $(X, \mathcal{T})$ is normal if and only if for every open set U and every closed set $C \subseteq U$, there is an open set V such that $C \subseteq V \subseteq \overline{V} \subseteq U$. (Again, be sure to draw a picture.)

*3. Prove that regularity is a topological invariant.

*4. Prove that regularity is finitely productive. (Hint: This is easiest to do using the alternative characterization of regularity given in the first problem.)

*5. Prove that normality is a topological invariant.

*6. Prove that every *closed* subspace of a normal topological space is normal.

**7. Show that $\mathbb{R}_{\text{Sorgenfrey}}$ is normal.

At this point, we will give some new definitions of finer properties, and ask some questions about them afterwards.

Definition 9.1. *A topological space $(X, \mathcal{T})$ is said to be $\underline{T_{2.5}}$ (often also called a <u>Urysohn space</u>) if for any two distinct points $x, y \in X$, there are open sets U and V containing x and y respectively such that $\overline{U} \cap \overline{V} = \emptyset$.*

Definition 9.2. *Let $(X, \mathcal{T})$ be a topological space. We say two points $x, y \in X$ can be <u>separated by a continuous function</u> if there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ (here, $[0, 1]$ is understood to have its subspace topology induced by $\mathbb{R}_{usual}$).*

Similarly, if $A \subseteq X$ and $x \notin A$, we say x and A can be separated by continuous function if there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(a) = 1$ for all $a \in A$.

Finally, we say that two disjoint subsets A and B of X can be separated by continuous functions if there exists a continuous $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(y) = 1$ for all $y \in B$.

Definition 9.3. *Let $(X, \mathcal{T})$ be a topological space.*

- *$(X, \mathcal{T})$ is said to be <u>completely Hausdorff</u> if every pair distinct points can be separated by a continuous function.*

- *$(X,\mathcal{T})$ is said to be completely regular if every $x \in X$ and every closed set $C \subseteq X$ not containing x can be separated by a continuous function. $(X,\mathcal{T})$ is said to be $T_{3.5}$ (or sometimes a Tychonoff space) if it is completely regular and T_1.*

Here it seems like we might define completely normal to be a space in which any two closed sets can be separated by a continuous function. However, that property turns out to be equivalent to normality. This result is called Urysohn's Lemma, and the proof is one of the most interesting and most nontrivial proofs we will discuss in this class. For now, you may use this result without proof, which we state here for clarity:

Theorem 9.4 (Urysohn's Lemma)**.** *A topological space is normal if and only if every pair of disjoint, nonempty closed sets can be separated by a continuous function.*

To make up for this unusual feature of the hierarchy we are constructing, we have more definitions of stronger properties than normality:

Definition 9.5. *Let $(X,\mathcal{T})$ be a topological space.*

- *Two subsets A and B of X are called separated if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.*
- *$(X,\mathcal{T})$ is called completely normal if whenever A and B are separated subsets of X, then there are disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.*

  A space that is completely normal and T_1 is sometimes called T_5.
- *$(X,\mathcal{T})$ is called perfectly normal if whenever C and D are disjoint, nonempty, closed subsets of X, there exists a continuous function $f : X \to [0,1]$ such that $C = f^{-1}(0)$ and $D = f^{-1}(1)$.*

  A space that is perfectly normal and T_1 is sometimes called T_6.

*8. Show that every completely regular space is regular.

*9. Prove that every T_3 space is $T_{2.5}$, and that every $T_{2.5}$ space is T_2.

*10. Prove that every perfectly normal space is normal, and that every completely normal space is normal.

*11. Show that all of the properties we just defined are topological invariants.

**12. Show that every completely Hausdorff space is $T_{2.5}$.

**13. Show that complete regularity is hereditary, and finitely productive.

**14. Show that a topological space is completely normal if and only if it is hereditarily normal.

(In particular, note that since we have already mentioned that normality is not hereditary, this result shows that complete normality is strictly stronger than normality.)

**15. Two very useful ways of characterizing perfect normality are given by the following theorem. For now, you may assume this theorem without proof.

Theorem 9.6. *Let $(X, \mathcal{T})$ be a topological space. Then the following are equivalent.*

(a) X is perfectly normal.

(b) For every closed $C \subseteq X$, there is a continuous function $f : X \to [0, 1]$ such that $C = f^{-1}(0)$.

(We would refer to this property by saying "every closed set is a zero set".)

(c) X is normal, and every closed subset of X is G_δ.

(Recall that a subset of a topological space is called $\underline{G_\delta}$ if it is equal to a countable intersection of open sets.)

Using this theorem, show that the property of being perfectly normal is hereditary, and conclude from the previous exercise that every perfectly normal space is completely normal.

*16. From the previous exercises in this and an earlier section, conclude that

$$T_6 \Rightarrow T_5 \Rightarrow T_4 \Rightarrow T_{3.5} \Rightarrow T_3 \Rightarrow T_{2.5} \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0.$$

(This is as full as this hierarchy ever gets.)

***17. In this exercise you're going to prove the theorem just above.

First, prove the equivalence between (a) and (b). This should be very simple.

Next, recall from an earlier exercise that a subset of a topological space is called $\underline{G_\delta}$ if it is equal to a countable intersection of open sets. Prove that a topological space $(X, \mathcal{T})$ is perfectly normal (in the sense of Definition 9.5) if and only if it is normal and every closed subset of X is G_δ.

(You will need to use Urysohn's Lemma for at least one direction.)

***18. Construct a topological space to show that $T_2 \not\Rightarrow T_{2.5}$. (Hint: Try some ideas similar to the Furstenburg topology.)

It is also true that $T_{2.5} \not\Rightarrow T_3$, though the example is not something anyone could reasonably come up with. Take a look at Example 78 in *Counterexamples in Topology* for more information.

10 Orders and ω_1

*1. Consider the following two ways of defining a topology on on any subset of $\mathbb{R}$. If $X \subseteq \mathbb{R}$, we can give X its subspace topology inherited from $\mathbb{R}_{\text{usual}}$, which we will call $\mathcal{T}_X^{\mathbb{R}}$, or we can give it the order topology (since $(X, \leq)$ is a linear order), which we will call $\mathcal{T}_X^{O}$.

Give an example of a set $X \subseteq \mathbb{R}$ such that $(X, \mathcal{T}_X^{\mathbb{R}}) \simeq (X, \mathcal{T}_X^{O})$, and another example such that $(X, \mathcal{T}_X^{\mathbb{R}}) \not\simeq (X, \mathcal{T}_X^{O})$. Be sure to prove your claims as explicitly as possible.

*2. Let $(L, \leq)$ be a linear order. A subset $X \subseteq L$ is called convex if for every $x < y \in X$, we have $(x, y) \subseteq X$ (where this interval is always defined in L). Colloquially, X is convex if it has no gaps as an order under $\leq$.

(a) Give an example of a convex and a non-convex subset of $\mathbb{R}$.

(b) Give an example of a convex and a non-convex subset of $\mathbb{Q}$.

(c) Show that the intersection of two convex sets is convex.

(d) Is the union of two convex sets convex?

(e) Show that if $X \subseteq L$ is convex, then its subspace topology inherited from the order topology on L is equal (not just homeomorphic, but equal) to the order topology on X when seen as a linear order on its own.

*3. Let $(W, \leq)$ be a well-order, and let $S \subseteq W$ be non-empty. Show that S has a *unique* least element.

*4. Let $(W, \leq)$ be an infinite well-order. Show that there is a "copy" of the well-order $\mathbb{N}$ at the "bottom" of W.

(Hint: Show that there is a least element of W, then a second-least element of W, etc.)

*5. Let $(W, \leq)$ be a linear order. Show that it is a well-order if and only if it contains no infinite, decreasing chains.

*6. Show that a subset of ω_1 is countable if and only if it is bounded.

*7. Let $m = \min(\omega_1)$. Show that $\{m\}$ is clopen in ω_1.

*8. Show that ω_1 is not discrete.

(Hint: Find a "copy" of $\omega + 1$ in ω_1.)

*9. **Key fact.** Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of elements of ω_1. Show that there is a $\alpha \in \omega_1$ such that

$$\bigcup_{n\in\mathbb{N}} \mathrm{pred}(\alpha_n) = \mathrm{pred}(\alpha).$$

(This is a property of ω_1 as a well-order, not as a topological space.) This property is a stronger way of saying that every countable subset of ω_1 is bounded above. It plays a key role in many proofs about ω_1.

*10. Using the same notation as the previous problem, show that any increasing sequence $\{\alpha_n\}$ converges to α.

This connects the key fact from the previous exercise to the topology on ω_1. Henceforth, given a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of elements of ω_1, we will refer to the α guaranteed by the key fact as $\lim \alpha_n$.

**11. Let $X = [0,1] \times [0,1]$ be the unit square, and let $\preceq$ be the lexicographical order on X induced from the usual linear order on both copies of $[0,1]$. That is, for (x_1, y_1) and (x_2, y_2) in X, define

$$(x_1, y_1) \preceq (x_2, y_2) \text{ if and only if } x_1 < x_2 \text{ or both } x_1 = x_2 \text{ and } y_1 \leq y_2.$$

Before proceeding any further, be sure to draw a picture of this space and think about what the ordering looks like.

Let $\mathcal{T}$ be the order topology defined by $\preceq$. Consider the following questions about the $(X, \mathcal{T})$.

(a) For any $a \in [0,1]$, show that any "vertical slice" of the form $V_a = \{ (a, y) \in X \; : \; y \in [0,1] \}$ is not open.

(b) Is this space second countable? First countable?

(c) Is this space separable or ccc?

**12. In this problem we will explore convexity a little more. Recall the property we defined in Problem 10.2 just above.

(a) Let $(W, \leq)$ be a linear order, and let $f : [0,1] \to W$ be a continuous function (where $[0,1]$ has its usual topology and W has its order topology). A function of this form essentially traces out a path in W. Show that the range of f is convex.

(Hint: Recall the fact that $[0,1]$ has no nontrivial clopen subsets.)

(b) Let $(X, \mathcal{T})$ be the space from the previous exercise (the lexicographical order on the unit square). Suppose that $f : [0,1] \to X$ is continuous, $f(0) = (0,0)$, and $f(1) = (1,1)$.

Show that f is surjective.

(c) Show that there are no functions with the properties described in part (b).

**13. The lecture notes mentioned that well-orders are exactly what you need to do induction. You will show that here.

Let $(W, \leq)$ be a well-order. Suppose $S \subseteq W$ has the following two properties:

(a) $\min(W) \in S$.

(b) If $y \in W$ is such that $x \in S$ for all $x < y$, then $y \in S$.

Show that $S = W$.

**14. Show that no subspace of $\mathbb{R}_{\text{usual}}$ is homeomorphic to ω_1.

**15. Show that no subspace of ω_1 is homeomorphic to $\mathbb{Q}$ (with its subspace topology from $\mathbb{R}_{\text{usual}}$).

(Hint: You will have to make up your own topological invariant, but you can do it with some quite simple ones.)

**16. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in ω_1 with the property that for each n, $\alpha_n \leq \beta_n \leq \alpha_{n+1}$. Show that $\lim \alpha_n = \lim \beta_n$.

**17. Let $A, B \subseteq \omega_1$ be disjoint, closed sets. Show that one of A or B must be countable.

(Hint: Use the result of the previous exercise.)

**18. An element $\alpha \in \omega_1$ is called a successor if $\text{pred}(\alpha)$ has a maximal element (maximal in the sense of the order on ω_1). If α is not a successor and $\text{pred}(\alpha) \neq \emptyset$, α is called a limit. (These names come from the set theoretic definition of ω_1.)

Show that α is a successor if and only if $\{\alpha\}$ is open.

**19. Show that in the space $\omega_1 + 1$, the set $\{\Omega\}$ is closed but not G_δ. (Recall that a subset of a topological space is called a G_δ set if it is equal to a countable intersection of open sets.) Conclude, using the result of an exercise from the previous section, that $\omega_1 + 1$ is not perfectly normal.

***20. Show that every continuous function $f : \omega_1 \to \mathbb{R}$ is eventually constant. That is, there is an $\alpha \in \omega_1$ and an $x \in \mathbb{R}$ such that $f(\beta) = x$ for all $\alpha < \beta \in \omega_1$.

This is a lovely fact. There are two ways of approaching this that I know. The "cleaner" proof goes through a very important set theoretical fact called *The Pressing Down Lemma*, which you are encouraged to look up. Feel free to talk to me about it.

The other approach is more hands-on, and therefore a little trickier, but does not involve any concepts you do not know yet. Here is a nudge in the right direction:

First, note that it suffices to prove that any continuous function $f : \omega_1 \to (0, 1)$ is eventually constant. So suppose f is a continuous function of this sort, fix $n \in \mathbb{N}$, and divide $[0, 1]$ into closed intervals of length $\frac{1}{2^n}$. Take preimages of these intervals, and apply the fact about disjoint closed subsets of ω_1.

11 The Axiom of Choice and Zorn's Lemma

As mentioned in class and in the lecture notes, you will never be tested *specifically* on the Axiom of Choice or Zorn's Lemma, though you will be expected to do simple proofs using Zorn's Lemma. This section is mostly designed for those students interested in learning a little more about the Axiom of Choice and its equivalents. You are very welcome to come to me for clarification or guidance with the harder questions in this section.

I recommend that every student **do at least the first problem** in this section. The concepts introduced there will come up again in an important proof later in the course, at which time you will be expected to be familiar with them.

**1. For this question we will recall some definitions first presented in the supplementary notes about nets and filters.

Definition 11.1. *Let X be a set. A nonempty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is called a* <u>*filter on X*</u> *if the following three properties are satisfied:*

(a) $\emptyset \notin \mathcal{F}$.

(b) $\mathcal{F}$ is closed upwards: if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

(c) $\mathcal{F}$ is closed under finite intersections: if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

A filter $\mathcal{F}$ on a set X is called an <u>*ultrafilter*</u> *if it is not properly contained in any other filter on X.*

Your task: Use Zorn's Lemma to prove that every filter on a set X is contained in (or "can be extended to", as is often said) an ultrafilter.

Hint: A filter $\mathcal{F}$ on X is a set of subsets of X. In other words, $\mathcal{F} \in \mathcal{P}(\mathcal{P}(X))$. Let $\mathbb{F} \subseteq \mathcal{P}(\mathcal{P}(X))$ be the collection of all filters on X. Then $(\mathbb{F}, \subseteq)$ is a partial order, to which you can apply Zorn's Lemma—after you prove it satisfies the hypotheses of Zorn's Lemma, of course. This **is not** the partial order that will solve this problem, but it is a good starting point. As a warm up exercise, first show that a filter $\mathcal{F} \in \mathbb{F}$ is an ultrafilter if and only if it is a maximal element in this partial order.

**2. Here is another simple use of Zorn's Lemma. First, a definition.

Definition 11.2. *A subset $A \subseteq \mathbb{R}$ is called* <u>*distance special*</u> *if the usual distance function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $d(x, y) = |x - y|$ is injective when restricted to the set $\{ (a, b) \in A \times A : a < b \}$. That is, $d(a, b) = d(c, d) > 0$ implies that $a = c$ and $b = d$, for all $a, b, c, d \in A$.*

For example, the set $\{ 2^n : n \in \mathbb{N} \} \subseteq \mathbb{R}$ is distance special.

Your task: Use Zorn's Lemma to show that every uncountable set $B \subseteq \mathbb{R}$ contains an uncountable distance special subset.

Hint: At one point in the proof, you may find the result of problem 4.11 useful.

**3. Show that Zorn's Lemma implies the Axiom of Choice.

Hint: Let $\mathcal{A} = \{ A_\alpha : \alpha \in I \}$ be a nonempty collection of nonempty sets. Think of a choice function on $\mathcal{A}$ as a function $f : I \to \bigcup \mathcal{A}$ such that $f(\alpha) \in A_\alpha$ for all $\alpha \in I$. Let $\mathbb{P}$ be the collection of all functions f such that $\mathrm{dom}(f) \subseteq I$ and $f(\alpha) \in A_\alpha$ for all $\alpha \in \mathrm{dom}(f)$. Define an ordering $\leq$ on $\mathbb{P}$ by saying $f \leq g$ if and only if g extends f as a function (ie. g has a larger domain, and agrees with f on all points in the domain of f).

Then carefully apply Zorn's lemma to the partial order $(\mathbb{P}, \leq)$.

**4. Show that Zorn's Lemma implies the Well-Ordering Principle.

Hint: The basics of this proof are outlined in the lecture notes on the Axiom of Choice.

***5. Prove that the Well-Ordering Principle implies Zorn's Lemma.

This proof is not actually hard, it just requires transfinite induction. Here is a hint on how to get started. Let $(\mathbb{P}, \leq)$ be a partial order satisfying the hypotheses of Zorn's Lemma. Well-order $\mathbb{P}$, then inductively build a chain whose upper bound must be a maximal element of the partial order.

***6. This question will guide you through a proof that the Axiom of Choice implies Zorn's Lemma. As with the previous problem this proof is not a very difficult one, other than the fact that it involves transfinite induction and ordinals. I am going to do most of the work for you. Your job is essentially to convince yourself that these things make sense. You do need to have some experience with ordinals though.

This is a proof by contradiction. So suppose $(\mathbb{P}, \leq)$ is a counterexample to Zorn's Lemma. That is, a partial order in which every chain has an upper bound, but which has no maximal elements.

(a) Show explicitly from the Axiom of Choice that there exist functions f and g such that for every chain $\mathcal{C} \subseteq \mathbb{P}$, $f(\mathcal{C})$ is an upper bound for $\mathcal{C}$, and such that for every $p \in \mathbb{P}$, $g(p) > p$.

(b) Using the functions from (a) and transfinite induction, define a "sequence" $\{p_\alpha\}_{\alpha \in \mathbf{Ord}}$ (**Ord** is the class of ordinals) in $\mathbb{P}$. Let p_0 be arbitrary. For successor ordinals $\alpha + 1$, let $p_{\alpha+1} = g(p_\alpha)$. For limit ordinals α, let $p_\alpha = f(\{ p_\beta : \beta < \alpha \})$.

Convince yourself that this all makes sense.

(c) Show that this "sequence" is increasing. That is, $\alpha < \beta$ implies $p_\alpha < p_\beta$. Conclude that the map $a \mapsto p_\alpha$ is injective.

(d) We have now constructed an injection **Ord** $\to \mathbb{P}$. This is a contradiction, because **Ord** is not a set.

12 Metric spaces and metrizability

*1. Show that the functions defined in each part of Example 2.2 in the lecture notes on Metric Spaces and Metrizability are actually metrics.

Hint: One part of this is actually kind of tricky: showing that $d_2(f,g) = \sqrt{\int_0^1 |f(x)-g(x)|^2\,dx}$ satisfies the triangle inequality. For that you will need the following version of Hólder's Inequality:

$$\int_0^1 |f_1(x)f_2(x)|\,dx \le \left(\int_0^1 |f_1(x)|^2\,dx\right)^{\frac{1}{2}} \left(\int_0^1 |f_2(x)|^2\,dx\right)^{\frac{1}{2}}$$

*2. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be metrizable spaces. Prove that $X_1 \times X_2$ with its product topology is metrizable. (Hint: Complete the proof of Proposition 4.3 in the lecture notes.)

*3. Prove that every metrizable space is Hausdorff.

*4. Let $X = C[0,1]$ be the set of all continuous functions $f : [0,1] \to \mathbb{R}$ (where the domain and codomain have their usual topologies). In question 12.1 above, you showed that the function $d_1 : X \times X \to \mathbb{R}$ given by

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\,dx$$

is a metric, so we can consider the metric topology on X. Show that the function $F : X \to \mathbb{R}$ given by

$$F(f) = \int_0^1 f(x)\,dx$$

is continuous (where X has the metric topology generated by d_1, and $\mathbb{R}$ has its usual topology).

*5. This exercise completes the proof of Proposition 7.2 from the lecture notes.

Let (X,d) be a metric space. Define $\overline{d} : X \times X \to \mathbb{R}$ by $\overline{d}(x,y) = \min\{1, d(x,y)\}$. Also define $d_0 : X \times X \to \mathbb{R}$ by

$$d_0(x,y) = \frac{d(x,y)}{1 + d(x,y)}.$$

(a) Prove that $\overline{d}$ and d_0 are metrics.

(b) Prove that $(X, \overline{d})$ and (X, d_0) are bounded metric spaces. (This is as easy as it looks.)

(c) Prove that $\overline{d}$ and d_0 both generate $\mathcal{T}$.

*6. Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $f : X_1 \to X_2$ be a surjective isometry (which, to remind you, is a function that preserves distances). Prove that f is a homeomorphism of topological spaces (where X_1 and X_2 have their metric topologies generated by d_1 and d_2 respectively).

*7. Let (X, d) be a metric space, and let $A, B \subseteq X$ be nonempty subsets, and let $x \in X$. Define

$$d(x, A) = \inf \{ d(x, a) \, : \, a \in A \} ,$$

$$d(A, B) = \inf \{ d(a, b) \, : \, a \in A, \, b \in B \} .$$

(a) Show that both of these notions are well-defined.

(b) Show that if $A \cap B \neq \emptyset$, then $d(A, B) = 0$.

(c) Give an example showing that the implication in the previous part does not reverse. That is, give an example of a metric space (X, d) and two *disjoint* subsets $A, B \subseteq X$ such that $d(A, B) = 0$.

(d) Show that these definitions are connected in the following ways:

- $d(x, A) = d(\{x\}, A)$.
- $d(A, B) = \inf \{ d(a, B) \, : \, a \in A \} = \inf \{ d(A, b) \, : \, b \in B \}$.

(e) Show that $A \subseteq X$ is closed (in the topology generated by d) if and only if $d(x, A) \neq 0$ for all $x \in X \setminus A$.

**8. Prove that every metrizable space is regular.

(Hint: This exercise and the next one are two-star problems if you're seeing the idea involved for the first time. For us though, you can mirror the proofs that $\mathbb{R}_{\text{usual}}$ is regular and normal.)

**9. Prove that every metrizable space is normal.

**10. This is a multi-part, guided exercise in which you are going to prove that every metrizable ccc space is second countable, completing the proof of Proposition 5.3 from the lecture notes. You will actually prove that a metrizable ccc space is separable, and then that a metrizable separable space is second countable.

Let $(X, \mathcal{T})$ be a metrizable, topological space with the countable chain condition. Let d be a metric on X that generates the topology $\mathcal{T}$. All ϵ-balls we refer to below will be with respect to this metric.

(a) Fix $n \in \mathbb{N}$. Show that there exists a *maximal* collection of mutually disjoint $\frac{1}{n}$-balls in X. (Maximal with respect to inclusion. That is, show that there is a collection $\mathcal{B}_n$ of mutually disjoint $\frac{1}{n}$-balls in X that is not properly contained in any larger such collection.)

(Hint: Use Zorn's Lemma.)

(b) For each $n \in \mathbb{N}$, suppose $\mathcal{B}_n$ is a collection as in the previous part. Let $C_n \subseteq X$ be the collection of centres of the balls in $\mathcal{B}_n$, and finally let $C = \bigcup_{n \in \mathbb{N}} C_n$. Show that C is a countable, dense subset of X. Conclude that $(X, \mathcal{T})$ is separable.

(Hint: First, show that C is countable. Next, to show it's dense, do a proof by contradiction. Suppose C is not dense, and fix an open set U such that $C \cap U = \emptyset$. After a little bit of work, you can contradict the maximality of one of the $\mathcal{B}_n$'s.)

Conclude from this and the previous part that any metrizable ccc space is separable.

(c) Now that we know that $(X, \mathcal{T})$ is separable. Prove that every separable metrizable space is second countable. (Hint: Use the fact that every point in a metric space has a particularly nice countable local basis.)

**11. In this exercise you are going to learn about two types of convergence of sequences of functions that are very important to analysis, and prove an important fact about one of them.

Definition 12.1. *Let X be a set and let $(Y, \mathcal{U})$ be a topological space. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions $f_n : X \to Y$. We say that the sequence converges pointwise to a function $f : X \to Y$ if the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ converges to $f(x)$ for all $x \in X$.*

Definition 12.2. *Let X be a set and let (Y, d) be a metric space. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions $f_n : X \to Y$. We say that the sequence converges uniformly to a function $f : X \to Y$ if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \epsilon$ for all $n > N$ and for all $x \in X$.*

(a) Let X be a set and (Y, d) a metric space, also thought of as a topological space with the topology generated by d. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of functions $f_n : X \to Y$ that converges uniformly to a function f. Show that $\{f_n\}$ also converges pointwise to f.

(b) Find a sequence of *continuous* functions $f_n : [0, 1] \to [0, 1]$ that converges pointwise to the zero function, but does not converge uniformly to anything. (Here, $[0, 1]$ has its usual topology generated by its usual metric.)

(The reason this exercise is interesting is that if $\{f_n(x)\}$ converges to some point p_x for all x, it is tempting to define a function f by $f(x) = p_x$ and conclude that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f. The example you come up with shows that this need not be the case.)

(c) This result is the big payoff of defining uniform convergence.

Let $(X, \mathcal{T})$ be a topological space and (Y, d) a metric space (also thought of as a topological space with the topology generated by d). Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of continuous functions $f_n : X \to Y$ that converges *uniformly* to a function f. Show that f must also be continuous.

(Hint: This is an "$\epsilon/3$ proof".)

**12. This proof is quite easy, but it requires two new definitions. Neither of them are tricky.

Definition 12.3. *Let X be a set and $f : X \to X$ a function. A point $x \in X$ is called a fixed point of f if $f(x) = x$.*

Definition 12.4. *Let (X, d) be a metric space. A function $f : X \to X$ is called a contraction if there is an $\alpha \in (0, 1)$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$.*

So, a contraction is just a map that decreases the distance between all pairs of points, by at least a fixed amount.

(a) Give an example of a metric space (X, d) and a map $f : X \to X$ such that $d(f(x), f(y)) < d(x, y)$ for every distinct $x, y \in X$, but such that f is not a contraction.

(b) Let (X, d) be a metric space. Prove that every contraction $f : X \to X$ has at most one fixed point.

**13. Let F be a fixed, finite set of points in $\mathbb{R}^2$. Prove that there is a *unique* closed ball of minimal radius containing all the points in F.

This result is true in $\mathbb{R}^n$ for all n. You may find it interesting to think about how you would have to modify your proof in higher dimensions (if at all).

If you're a computer scientist, you should try to design an efficient algorithm that finds this ball. Try to do better than $O(n^2)$, where n is the number of points in F.

***14. Prove the following lovely theorem. It is not actually that difficult, but it requires a somewhat novel idea.

Theorem 12.5 (Contraction Mapping Theorem)**.** *Let (X, d) be a complete metric space, also thought of as a topological space with the topology generated by d. Let $f : X \to X$ be a contraction. Then f has* exactly *one fixed point.*

Hint: The core idea of the proof is illustrated by this image.

***15. Prove the Baire Category theorem, now that we have the terminology to state it in full generality.

That is, prove that if (X, d) is a complete metric space thought of as a topological space with the topology generated by d, then the intersection of countably many dense, open subsets of X is dense.

13 Urysohn's Lemma

This section is mostly a placeholder to keep the Big List sections aligned with the Lecture Notes sections.

*1. Show that the Tietze Extension Theorem implies Urysohn's Lemma.

**2. Let (X, d) be a metric space, thought of as a topological space with the topology generated by d. Then we know that X is metrizable and therefore normal. Prove Urysohn's Lemma for this space directly, using an associated bounded metric $\overline{d}$.

14 Arbitrary Products

14.1 Problems from the Lecture Notes

This section simply collects together all the exercises mentioned in section 14 of the lecture notes.

*1. Let I be a nonempty indexing set, and let X be a nonempty set. Let $\mathcal{X} = \{ X_\alpha : \alpha \in I \}$ be the indexed family of sets such that $X_\alpha = X$ for all $\alpha \in I$. Show that $\prod_{\alpha \in I} X_\alpha = X^I$.

*2. Fully convince yourself that the definition of an arbitrary Cartesian product extends the usual definition of "ordered n-tuples" for finite products.

*3. Let I be a nonempty indexing set, and let $\mathcal{X} = \{ X_\alpha : \alpha \in I \}$ be a collection of sets such that $X_\alpha = \emptyset$ for at least one $\alpha \in I$. Show that $\prod_{\alpha \in I} X_\alpha = \emptyset$.

*4. Using the identification $n = \{0, 1, \ldots, n-1\}$ we mentioned above, show that $n^k = |n^k|$. What we mean here is that on the left side is usual exponentiation of natural numbers, while on the right side we have the size of the set of all functions $f : k \to n$.

   This is the most general way to define exponentiation of natural numbers, in the sense that it generalizes to infinite sets and cardinal numbers.

*5. Fully convince yourself that the definition of projection functions given in Definition 2.8 of the lecture notes on Arbitrary Products extends the definition of projection functions we already had for finite products.

*6. Justify the use of the word "equivalently" in Definition 4.1, which defines the product topology on $\mathbb{R}^{\mathbb{N}}$.

*7. Show that the functions d_1 and d_2 defined in Proposition 4.8 are in fact metrics on $\mathbb{R}^{\mathbb{N}}$.

*8. Show explicitly that $\mathcal{T}_{\text{prod}} \subseteq \mathcal{T}_{\text{unif}} \subseteq \mathcal{T}_{\text{box}}$ (as defined on $\mathbb{R}^{\mathbb{N}}$).

*9. Show explicitly that both of the $\subseteq$'s in the previous exercise are strict (by exhibiting open sets).

*10. Show that for an arbitrary product of topological spaces, $\mathcal{T}_{\text{prod}} \subseteq \mathcal{T}_{\text{box}}$, and that this containment is strict for infinite products.

*11. Show that the property of being Hausdorff is productive.

   Hint: Essentially the same proof we did for $\mathbb{R}^{\mathbb{N}}$ works in general here.

*12. Show that the property of being discrete is finitely productive but not countably productive. In particular, you can show that any finite product of discrete spaces is discrete, but that any countable product of nonempty discrete spaces with more than one point is not discrete.

**13. Directly show that $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ is regular.

Hint: Use the alternative definition of regularity (the one involving something like $x \in V \subseteq \overline{V} \subseteq U$). First show it for $x = (0,0,0,\dots)$ and $U = (-1,1) \times (-1,1) \times \cdots$, then convince yourself that your proof for this case easily generalizes to all other cases.

**14. Directly show that $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ and $\mathbb{R}^{\mathbb{N}}_{\text{unif}}$ are not ccc (and therefore not separable or second countable).

Hint: There are several ways to do this, but here's my favourite. Let $x \in \mathbb{Z}^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$ be a sequence with all integer values. Find an open set around x that contains no other sequences with all integer values. Recall (or prove) that $\mathbb{Z}^{\mathbb{N}}$ is uncountable to finish the proof.

**15. Directly show that $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$ is first countable (we know it is since it's metrizable, but here you're going to show it directly).

Hint: First recall from BL 4.4 that the set $\text{Fin}(\mathbb{N}) := \{\, A \subseteq \mathbb{N} : A \text{ is finite} \,\}$ is countable. Using the fact that $\mathbb{R}_{\text{usual}}$ is first countable, for each $a \in \mathbb{R}$ fix a countable local basis $\mathcal{B}_a$ at a (in $\mathbb{R}_{\text{usual}}$). Now fix $x = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}}$. For $A \in \text{Fin}(\mathbb{N})$, let

$$\mathcal{B}_{x,A} = \left\{ \prod_{n \in \mathbb{N}} U_n \,:\, U_n \in \mathcal{B}_{x_n} \text{ for all } n \in A, \text{ and } U_n = \mathbb{R} \text{ for all } n \in \mathbb{N} \setminus A \right\}.$$

Show that $\mathcal{B}_{x,A}$ is countable for all $A \in \text{Fin}(\mathbb{N})$, and then that $\mathcal{B}_x := \bigcup_{A \in \text{Fin}(\mathbb{N})} \mathcal{B}_{x,A}$ is a countable local basis at x.

**16. Repeat the proof that $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$ is metrizable, but using the metric d_2.

Warning: This proof is more tedious than the proof using d_1. I present both metrics because I believe students will be more comfortable with a metric defined like d_2, but d_1 is definitely the easier one to use.

**17. Let $\{y_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$ Show that the sequence converges to a point $x = (x_1, x_2, x_3, \dots)$ if and only if the "coordinate sequence" $\{\pi_k(y_n)\}_{n \in \mathbb{N}}$ in $\mathbb{R}$ converges to x_k for all $k \in \mathbb{N}$.

**18. Let $(X, \mathcal{T})$ be a topological space, and let $f : X \to \mathbb{R}^{\mathbb{N}}$ be a function, where $\mathbb{R}^{\mathbb{N}}$ has the product topology. Show that f is continuous if and only if $\pi_k \circ f : X \to \mathbb{R}$ is continuous for all $k \in \mathbb{N}$.

Note, these two proofs are actually quite easy. I list them here as a two-star problems because of the notation, which may be tricky for those seeing it for the first time.

**19. Repeat the two proofs above, but in the context of arbitrary products. In other words, prove Propositions 7.13 and 7.14 from the lecture notes.

Hint: The proofs should be almost identical.

**20. Fill in the details of the proofs for Proposition 7.10. In particular, show that separability and metrizability are countably productive.

14.2 Other problems on products

*1. The purpose of this exercise is to work on your intuition about $\mathbb{R}^{\mathbb{N}}_{\text{unif}}$. Recall that the uniform metric d_u on $\mathbb{R}^{\mathbb{N}}$ is defined by

$$d_u(x, y) = \sup\left\{ \overline{d}(x_n, y_n) \,:\, n \in \mathbb{N} \right\}.$$

In the notes we said that an ϵ-ball around x in this metric is sort of like a "tube" around x, but this is not *quite* true. Fix $x \in \mathbb{R}^{\mathbb{N}}$ and $\epsilon > 0$, and define a subset:

$$\begin{aligned} U(x, \epsilon) &= \prod_{n \in \mathbb{N}} (x_n - \epsilon, x_n + \epsilon) \\ &= (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times (x_3 - \epsilon, x_3 + \epsilon) \times \cdots \end{aligned}$$

This subset obviously contains x, and should be what you think of when we say "a tube around x". However:

(a) Show that $U(x, \epsilon) \neq B_\epsilon(x)$ (where $B_\epsilon(x)$ is the ϵ-ball around x according to d_u).

(b) Show that $U(x, \epsilon)$ is not open in $\mathbb{R}^{\mathbb{N}}_{\text{unif}}$.

(c) All is not lost, however. Show that for $\epsilon \leq 1$,

$$B_\epsilon(x) = \bigcup_{0<\delta<\epsilon} U(x, \delta).$$

*2. For each of the following sequences in $\mathbb{R}^{\mathbb{N}}$, determine whether they converge in $\mathbb{R}^{\mathbb{N}}_{\text{box}}$, $\mathbb{R}^{\mathbb{N}}_{\text{unif}}$, and $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$.

$$\begin{aligned} a_1 &= (1, 0, 0, 0, 0, \dots) & b_1 &= (1, 1, 1, 1, 1, \dots) \\ a_2 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, \dots) & b_2 &= (0, 2, 2, 2, 2, \dots) \\ a_3 &= (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, 0, \dots) & b_3 &= (0, 0, 3, 3, 3, \dots) \\ a_4 &= (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, 0, \dots) & b_4 &= (0, 0, 0, 4, 4, \dots) \\ &\vdots & &\vdots \end{aligned}$$

$$\begin{aligned} c_1 &= (1, 1, 1, 1, 1, \dots) & d_1 &= (1, 1, 1, 1, 1, \dots) \\ c_2 &= (0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \dots) & d_2 &= (0, 1, 1, 1, 1, \dots) \\ c_3 &= (0, 0, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \dots) & d_3 &= (0, 0, 1, 1, 1, \dots) \\ c_4 &= (0, 0, 0, \tfrac{1}{4}, \tfrac{1}{4}, \dots) & d_4 &= (0, 0, 0, 1, 1, \dots) \\ &\vdots & &\vdots \end{aligned}$$

$$
\begin{aligned}
e_1 &= (1,0,0,0,0,\dots) & f_1 &= (1,0,0,0,0,\dots)\\
e_2 &= (0,1,0,0,0,\dots) & f_2 &= (0,\tfrac{1}{2},0,0,0,\dots)\\
e_3 &= (0,0,1,0,0,\dots) & f_3 &= (0,0,\tfrac{1}{3},0,0,\dots)\\
e_4 &= (0,0,0,1,0,\dots) & f_4 &= (0,0,0,\tfrac{1}{4},0,\dots)\\
&\vdots & &\vdots
\end{aligned}
$$

*3. For each of the following functions $f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$, determine whether they are continuous or open when $\mathbb{R}^{\mathbb{N}}$ has the box, uniform, and product topologies. (One of these may be a bit trickier than the usual one-star problem.)

$$
\begin{aligned}
f_1(t) &= (t,t,t,t,t,\dots)\\
f_2(t) &= (t,2t,3t,4t,5t,\dots)\\
f_3(t) &= (t,\tfrac{1}{2}t,\tfrac{1}{3}t,\tfrac{1}{4}t,\tfrac{1}{5}t,\dots)\\
f_4(t) &= (t,t^2,t^3,t^4,t^5,\dots)\\
f_5(t) &= (t,\sqrt{t},\sqrt[3]{t},\sqrt[4]{t},\sqrt[5]{t},\dots)
\end{aligned}
$$

(Assume f_5 is defined on $[0,\infty)$.)

*4. Recall that we say two sets A and B are of the same <u>cardinality</u> if there exists a bijection $f : A \to B$. In this case, we write $|A| = |B|$.

Let $(X, \mathcal{T})$ be a topological space. Show that if $|I| = |J|$, then X^I is homeomorphic to X^J (where both sets have the product topology). Also show with an example that the converse is not true.

**5. Let $\mathbb{R}_0 \subseteq \mathbb{R}^{\mathbb{N}}$ be the collection of all sequences that are eventually constantly zero. (To be formal, we mean $x = (x_1, x_2, x_3, \dots) \in \mathbb{R}_0$ if and only if $\exists N \in \mathbb{N}$ such that $x_n = 0$ for all $n > N$.)

Compute the closure of $\mathbb{R}_0$ in the box, uniform, and product topologies on $\mathbb{R}^{\mathbb{N}}$.

**6. A sequence $x = (x_1, x_2, x_3, \dots) \in \mathbb{R}^{\mathbb{N}}$ is called <u>square summable</u> if $\sum_{n=1}^{\infty} x_n^2$ converges (in the first year calculus sense). The collection of all square summable sequences in $\mathbb{R}^{\mathbb{N}}$ is called ℓ_2 (pronounced "little ell two", to differentiate it from $L_2 \subseteq \mathbb{R}^{\mathbb{R}}$—the space of all square integrable functions on $\mathbb{R}$—which is another matter). The function

$$d(x,y) = \sum_{n=1}^{\infty} (x_n - y_n)^2$$

defines a metric on ℓ_2 (you do not have to prove this). This metric defines a topology on ℓ_2 in the usual way which we will denote by $\mathcal{T}_{\ell_2}$. In this problem, we will denote by $\mathcal{T}_{\text{box}}$, $\mathcal{T}_{\text{unif}}$ and $\mathcal{T}_{\text{prod}}$ the subspace topologies ℓ_2 inherits from the box, uniform, and product topologies on $\mathbb{R}^{\mathbb{N}}$, respectively.

(a) Show that $\mathcal{T}_{\text{unif}} \subseteq \mathcal{T}_{\ell_2} \subseteq \mathcal{T}_{\text{box}}$.

(b) Show that $\mathbb{R}_0$ (defined in the previous problem) is a subset of ℓ_2. Show that the four topologies $\mathbb{R}_0$ inherits from the four topologies on ℓ_2 we are considering are all distinct.

(Note that by previous exercises, you already know that $\mathcal{T}_{\text{prod}} \subseteq \mathcal{T}_{\text{unif}} \subseteq \mathcal{T}_{\ell_2} \subseteq \mathcal{T}_{\text{box}}$.)

(c) Let $H \subseteq \mathbb{R}^{\mathbb{N}}$ be defined by

$$H = \prod_{n\in\mathbb{N}} [0, \tfrac{1}{n}].$$

H is called the <u>Hilbert cube</u>. Show that $H \subseteq \ell_2$. Having shown this, there are again four different topologies H can inherit from ℓ_2. Determine which of them, if any, coincide.

**7. In the lecture notes we saw that if I is an uncountable indexing set, then $\mathbb{R}^I$ with the product topology is not first countable. We proved this by defining a subset $A \subseteq \mathbb{R}^I$ and a point $x \in \overline{A}$ such that no sequence from A converges to x.

In this exercise, give a direct proof that $\mathbb{R}^I$ is not first countable by fixing an element $f \in \mathbb{R}^I$ and a countable collection $\{ U_n : n \in \mathbb{N} \}$ of open subsets of the product topology on $\mathbb{R}^I$, each containing f, and finding an open set U containing f such that $U_n \not\subseteq U$ for all $n \in \mathbb{N}$.

**8. We have already seen that the Hausdorff property is arbitrarily productive. In this exercise, show that the converse is true. That is, let I be a nonempty indexing set and let $\mathcal{X} = \{ (X_\alpha, \mathcal{T}_\alpha) : \alpha \in I \}$ be a collection of nonempty topological spaces. Suppose their Cartesian product $X := \prod_{\alpha\in I} X_\alpha$ (with the product topology) is Hausdorff. Show that $(X_\alpha, \mathcal{T}_\alpha)$ is Hausdorff for each $\alpha \in I$.

Hint: First, show that each X_α is homeomorphic to a subspace of X in a natural way.

**9. In this and another exercise, we will explore how productive separability is. In the notes we mentioned that separability somehow cares about the exact size of the product in question. We have shown already that $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$ is separable, and more generally that separability is countably productive.

In this exercise, your task is to show that $\mathbb{R}^{\mathbb{R}}_{\text{prod}}$ is separable.

Recall that $\mathbb{R}^{\mathbb{R}}$ is the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Using the definition of the product topology, a basic open set in this topology is defined by picking a finite number of points $x_1, x_2, \ldots, x_n \in \mathbb{R}$, and an equal finite number of open sets $U_1, U_2, \ldots, U_n$ in $\mathbb{R}_{\text{usual}}$, and considering the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x_k) \in U_k$ for all $k = 1, \ldots, n$.

Hint: Consider the collection of all step functions with finitely many steps, rational step heights, and whose steps are all on rational intervals.

**10. The previous exercise along with the second last one in this section say that uncountable products of separable spaces can be separable. This may strike you as odd, and that's fine; we are getting cloe to Continuum Hypothesis territory here. What might make you feel a little better is the result of this exercise.

Let I be an uncountable indexing set, and let $\mathcal{X} = \{ (X_\alpha, \mathcal{T}_\alpha) : \alpha \in I \}$ be a collection of topological spaces, each of which is T_1 and contains two or more points. Let $X = \prod_{\alpha \in I} X_\alpha$ be their product (with its product topology). You will show that X has a non-separable subspace. The proof is very similar to the proof that $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ is not first countable.

Fix a point $f \in X$ for the duration of this problem. Define a subset $A \subseteq X$ by:

$$A := \{ g \in X : g(\alpha) = f(\alpha) \text{ for all but countably many } \alpha \in I \}.$$

You will show that A is not separable with its subspace topology. So fix any countable set $D = \{g_1, g_2, g_3, \dots\} \subseteq A$.

(a) Show that there exists an $\alpha \in I$ such that $g_n(\alpha) = f(\alpha)$ for all $n \in \mathbb{N}$.

(b) Find an open subset $U \subseteq X$ such that $g_n \notin U \cap A$ for all $n \in \mathbb{N}$.

(c) Conclude that D is not dense, and the conclude that A is not separable.

**11. This is a very important exercise, in which we are going to define a topology on the power set of an arbitrary set X. The last two parts, which are probably the most interesting parts, are going to be somewhat mind-bending. They are not difficult proofs, just tricky ones to get your head around.

So, let X be an infinite set. Recall that there is a natural bijection between $\mathcal{P}(X)$ and $\{0,1\}^X$, via indicator functions. That is, for a subset $A \subseteq X$, define its <u>indicator function</u> $i_A : X \to \{0,1\}$ by $i_A(x) = 1$ if and only if $x \in A$.

(a) Show that the mapping $\phi : \mathcal{P}(X) \to \{0,1\}^X$ defined by $A \mapsto i_A$ is a bijection.

(b) This bijection allows us to define a topology on $\mathcal{P}(X)$ in the natural way: Say that $\mathcal{A} \subseteq \mathcal{P}(X)$ is open if and only if $\phi(\mathcal{A}) \subseteq \{0,1\}^X$ is open (where $\{0,1\}^X$ has the product topology induced by the discrete topology on all of the factors). Call this topology $\mathcal{T}$.

Write down a natural-feeling basis for $\mathcal{T}$.

(c) Describe what it means for a sequence $\{A_n\}_{n \in \mathbb{N}}$ of subsets of $\mathbb{N}$ to converge to a subset $A \subseteq \mathbb{N}$ in this topology.

(d) Recall from the section on the Axiom of Choice the definition of a filter and an ultrafilter. As a warm-up, consider the following fact (which you do not have to prove, just think about it until you believe it).

Proposition 14.1. *Let $\mathcal{U} \subseteq \mathcal{P}(X)$ be a filter. Then $\mathcal{U}$ is an ultrafilter if and only if for every $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.*

Now, there are two sorts of ultrafilters.

Definition 14.2. *An ultrafilter $\mathcal{U}$ on X is called principal, if there is an element $a \in X$ such that $\mathcal{U}$ is of the form $\mathcal{U} = \{ A \subseteq X : a \in A \}$. Equivalently, an ultrafilter is principal if and only if it contains a finite set. An ultrafilter that is not a principal ultrafilter is called non-principal.*

(e) Let $\mathcal{U}$ be a principal ultrafilter on $\mathbb{N}$. Determine whether $\mathcal{U}$, as a subset of $\mathcal{P}(\mathbb{N})$ with the topology $\mathcal{T}$ we defined above, is open, closed or neither.

(f) Do the same but for a non-principal ultrafilter on $\mathbb{N}$.

**12. The earlier exercise about determining whether or not certain sequences converge in the box topology probably led you to believe that it's very hard for a sequence to converge in the box topology. Obviously constant sequences converge, and sequences that basically "live in" $\mathbb{R}^n$ for some n converge, such as the following sequence:

$$\begin{aligned} y_1 &= (1, 1, 0, 0, 0, \dots) \\ y_2 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, \dots) \\ y_3 &= (\tfrac{1}{3}, \tfrac{1}{3}, 0, 0, 0, \dots) \\ y_4 &= (\tfrac{1}{4}, \tfrac{1}{4}, 0, 0, 0, \dots) \\ &\vdots \end{aligned}$$

This exercise should confirm this intuition, by showing you that almost any interesting sequence fails to converge in the box topology.

(a) Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}^{\mathbb{N}}$ with the property that every x_n (thought of as a sequence in $\mathbb{R}$) has infinitely many nonzero terms. Show that $\{x_n\}$ does not converge to $\overline{0}$.

Hint: Diagonalize!

(b) Looking back again at the sequences in the earlier exercise, we see that the condition in the previous part can be improved upon, since the sequence $e_1, e_2, e_3, \dots$ from that exercise does not converge in the box topology but every element of the sequence has only one nonzero term.

Try to find a stronger condition than the one in part (a) that guarantees a sequence in $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ does not converge to $\overline{0}$. Make sure your condition applies to $e_1, e_2, e_3, \dots$.

***13. This and the next exercise are difficult because they are notationally intensive, but they are not very difficult proofs. I will guide you through most of the work.

In this first exercise, you are going to generalize the fact that $\mathbb{R}^{\mathbb{R}}$ is separable. Instead of showing that a product of copies of $\mathbb{R}$ indexed by $\mathbb{R}$ is separable, you will show that any product of separable spaces indexed by $\mathbb{R}$ is separable.

Let $\mathcal{X} = \{ (X_\alpha, \mathcal{T}_\alpha) : \alpha \in \mathbb{R} \}$ be a collection of separable topological spaces. Your goal is to show that $X := \prod_{\alpha\in\mathbb{R}} X_\alpha$ is separable with its product topology. You will exploit the fact that $\mathbb{R}$ is second countable, and use that fact to define a dense set in a similar way to how we showed separability is countably productive earlier in the notes.

For each $\alpha \in \mathbb{R}$, fix a countable dense subset $D_\alpha \subseteq X_\alpha$, and fix an enumeration of it:

$$D_\alpha = \{x_1^\alpha, x_2^\alpha, x_3^\alpha, \dots\}$$

Also let $\mathcal{B} = \{ (a,b) \subseteq \mathbb{R} : a, b \in \mathbb{Q} \}$ be the usual countable basis for the usual topology on $\mathbb{R}$.

Let $F \subseteq \mathcal{B}$ be a finite set of mutually disjoint intervals with rational endpoints, and let f be any function $f : F \to \mathbb{N}$. (If you prefer, instead of such an f you can just think of a finite list of natural numbers of the same length as the size of F.)

(a) Show that the set E of all such pairs (F, f) as we have defined them is countable.

(b) For each pair (F, f), define an element $g \in X$ by:

$$g_{F,f}(\alpha) = \begin{cases} x^\alpha_{f(B)} & \alpha \in B \text{ for some } B \in F \\ x_1^\alpha & \text{otherwise} \end{cases}$$

Convince yourself that this is a well-defined element of X.

(c) Show that the set $D := \{ g_{F,f} : (F, f) \in E \}$ is a countable dense subset of X.

***14. In this exercise you will show that the previous result is almost "strict", in the sense that almost any product of separable spaces indexed by sets *larger* than $\mathbb{R}$ is not separable.

Let $\mathcal{X} = \{ (X_\alpha, \mathcal{T}_\alpha) : \alpha \in I \}$ be a collection of separable Hausdorff topological spaces, each with two or more points. You will show that if $X := \prod_{\alpha\in I} X_\alpha$ is separable, then $|I| \le |\mathbb{R}|$, which you will recall means that I has cardinality smaller than or equal to the cardinality of $\mathbb{R}$, or in other words that there is an injection $I \to \mathbb{R}$.

(The contrapositive of this result is that if your indexing set has larger cardinality than $\mathbb{R}$, then your product is not separable.)

For each $\alpha \in I$, let U_0^α and U_1^α be disjoint, nonempty open subsets of X_α (which we can find since X_α is Hausdorff and has more than one point). Let $D \subseteq X$ be a countable dense set. For each $\alpha \in I$, let:

$$D_\alpha := \pi_\alpha^{-1}(U_0^\alpha) \cap D = \{ f \in X : f(\alpha) \in U_0^\alpha \} \cap D.$$

(a) Show that $D_\alpha \neq \emptyset$ for each $\alpha \in I$.

(b) Show that if $\alpha \neq \beta$, then $D_\alpha \neq D_\beta$. (Hint: This is where U_1^α comes into play.)

(c) By the previous two parts, the map $I \to \mathcal{P}(D)$ defined by $\alpha \mapsto D_\alpha$ is an injection. Recalling the fact that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$ finishes the proof. Convince yourself of this.

15 Urysohn's metrization theorem

*1. Recall that a topological space $(X, \mathcal{T})$ is called completely regular if every $x \in X$ and every closed set $C \subseteq X$ not containing x can be separated by a continuous function. This definition first appeared in section 9 of the Big List. $(X, \mathcal{T})$ is said to be $T_{3.5}$ if it is completely regular and T_1.

Examine the proof of Urysohn's Lemma (*Lemma*, not metrization theorem) and determine why we can't do a similar proof to show that every regular space is completely regular.

**2. Carefully examine the first proof of Urysohn's metrization theorem from the notes, and convince yourself that we actually proved this slightly more general fact:

Let X be a T_1 topological space. Let I be some indexing set, and suppose there is a family $\{ f_\alpha : X \to [0,1] \, : \, \alpha \in I \}$ of continuous functions such that for each $a \in X$ and each open set U containing a, there is an $\alpha \in I$ such that $f_\alpha(a) > 0$ and $f_\alpha(x) = 0$ for all $x \in X \setminus U$. Then the function $F : X \to [0,1]^I$ defined by

$$F(x) = g_x : I \to [0,1], \text{ where } g_x(\alpha) = f_\alpha(x),$$

is an embedding of X into $[0,1]^I$.

**3. Here is the big payoff. This is a powerful embedding theorem (actually a full *characterization*) for completely regular spaces, and one of the reasons completely regular spaces are so beloved by topologists.

Theorem 15.1. *A topological space $(X, \mathcal{T})$ is $T_{3.5}$ if and only if it is homeomorphic to a subspace of $[0,1]^I$ for some indexing set I.*

Prove this theorem.

16 Compactness

16.1 Problems from the lecture notes

*1. Show that every compact space is Lindelöf, and find some examples of Lindelöf spaces that are not compact.

*2. Show that a discrete topological space is compact if and only if it is finite, and Lindelöf if and only if it is countable.

*3. Show that ω_1 is not Lindelöf.

*4. Show that every compact Hausdorff space is normal.

Hint: Use the fact that every compact Hausdorff space is regular to speed up your proof.

*5. Show that if $(X, \mathcal{T})$ is Hausdorff and $K \subseteq X$ is compact, then K is closed.

Hint: This proof is almost the same as the proof that every compact Hausdorff space is regular.

*6. Let $(X, \mathcal{T})$ be a compact topological space and let $f : X \to \mathbb{R}_{\text{usual}}$ be a continuous function. Show that $f(X)$ is bounded.

*7. Prove the Extreme Value Theorem from first year calculus. That is, show that every continuous function $f : [a, b] \to \mathbb{R}$ achieves a minimum and maximum.

**8. Show that $\omega_1 + 1$ is compact.

**9. Finish the proof of the general form of the Heine-Borel theorem. That is, prove Theorem 4.7 from the lecture notes.

16.2 Other problems on compactness

*1. Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a basis for the topology. Prove that X is compact if and only if every open cover of X consisting of *basic* open sets has a finite subcover.

(Interesting note: You can even replace "basic" with "subbasic", but this fact is highly nontrivial; it is called the Alexander Subbase Theorem.)

*2. Show that any finite union of compact subsets of a topological space is compact, and that an arbitrary intersection of compact subsets of a Hausdorff topological space is compact.

*3. Let (X, d) be a metric space, thought of as a topological space with the metric topology generated by d. Let $C, K \subseteq X$ be disjoint subsets such that C is closed and K is compact. Show that $d(C, K) > 0$, where recall:

$$d(C, K) = \inf \{ d(c, x) : c \in C, x \in K \} .$$

*4. Show that every compact metrizable space is second countable. Conclude that every compact metrizable space is hereditarily separable and hereditarily ccc.

*5. Show that every compact metrizable space is sequentially compact. That is, let $(X, \mathcal{T})$ be a compact metrizable space, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Show that it has a convergent subsequence.

Hint: First, take care of the case where $S = \{ x_n : n \in \mathbb{N} \}$ is finite. Now assume the sequence has no convergent subsequences. Show that for each $x \in \overline{S}$, there is an $\epsilon_x > 0$ such that $B_{\epsilon_x}(x) \cap \overline{S} = \{x\}$. Then show that $\mathcal{U} = \left\{ B_{\epsilon_x}(x) : x \in \overline{S} \right\} \cup \{X \setminus \overline{S}\}$ is a cover of X with no finite subcover.

**6. Determine which of the following spaces are compact. (This exercises ranges from one- to two-star difficulty.)

(a) $\mathbb{R}_{\text{co-finite}}$.

(b) $\mathbb{R}_{\text{co-countable}}$.

(c) $(\mathbb{R}, \mathcal{T}_7)$ (the particular point topology at 7).

(d) $(\mathbb{R}, \mathcal{T}_{\text{ray}})$.

(e) $(-\infty, \alpha]$, as subspace of ω_1 with its order topology, for any $\alpha \in \omega_1$.

(f) A subset $A \subseteq \omega_1$ with no largest element.

After you have done that, try to characterize the compact subsets of each space.

**7. In an earlier exercise in this section, you showed that the intersection of compact subsets of a Hausdorff space is again compact. Give an example to show that Hausdorffness is necessary in that result. That is, construct a non-Hausdorff topological space $(X, \mathcal{T})$ with two compact subsets A and B such that $A \cap B$ is not compact.

You will need to construct a space here, but you don't have to work too hard. The easiest way to do it is to start with $\mathbb{N}_{\text{discrete}}$ and add some points and new open sets containing those points.

**8. Let $(L, \leq)$ be a linear order, thought of as a topological space with its order topology. Show that L is compact if and only if every nonempty subset of L has a least upper bound and a greatest lower bound.

Hint: This is very similar to the "creeping along" proof that $[0, 1]$ is compact.

**9. Let $(X, \mathcal{T})$ be a topological space, let $(Y, \mathcal{U})$ be a compact Hausdorff space, and let $f : X \to Y$ be a function. Show that f is continuous if and only if the graph of f, which is the set

$$\Gamma_f = \{ (x, f(x)) \in X \times Y \,:\, x \in X \}$$

is closed.

Hint: The $(\Leftarrow)$ direction is the tricky one. Prove this intermediary lemma first, then the result follows:

Lemma. *Let $(X, \mathcal{T})$ be a topological space, and $(Y, \mathcal{U})$ a compact space. Then the first projection function $\pi_1 : X \times Y \to X$ is closed.*

**10. In this exercise we are going to define a new topological space called the Cantor set that is one of the most classic (and most counterintuitive) examples in point set topology. You are strongly encouraged to draw yourself a picture as we go.

Let $C_0 = [0, 1]$, with its usual topology. Let C_1 be the set obtained by deleting the open middle third of C_1, or in other words let $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Let C_2 be the set obtained by deleting the open middle thirds of both of the intervals in C_1, or in other words let

$$C_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1].$$

Continue this process inductively, with C_n being the set obtained from C_{n-1} by deleting the open middle third of each of the intervals in C_{n-1}.

Then the Cantor set $\mathcal{C}$ is defined as $\mathcal{C} := \bigcap_{n \in \mathbb{N}} C_n$.

Since $\mathcal{C}$ is a subset of $\mathbb{R}$ it is second countable and metrizable, and therefore also first countable, T_4, separable and ccc.

(a) Show that $\mathcal{C} \neq \emptyset$.

(b) Show that $\mathcal{C}$ is uncountable.

(c) Show that $\mathcal{C}$ is closed, and conclude from this that $\mathcal{C}$ is compact.

(d) A subset A of a topological space $(X, \mathcal{T})$ is called perfect if for every $x \in A$ and every open set U containing x, U contains an element of A other than x, or in other words $(U \cap A) \setminus \{x\} \neq \emptyset$. Another way of saying this is that A has no "isolated points".

Show that $\mathcal{C}$ is perfect. (Another example of a perfect set with which you are already familiar is $\mathbb{Q}$, as a subset of $\mathbb{R}_{\text{usual}}$.)

(e) Show that $\text{int}(\mathcal{C}) = \emptyset$.

(f) Show that the outer measure of $\mathcal{C}$ is zero. (If you have not heard of outer measure, feel free to skip this problem or ask me about it.)

(g) Convince yourself that the Cantor set consists precisely of those elements of $[0, 1]$ that have a ternary expansion (that is, a "decimal" expansion in base 3) containing no 1s.

(Note that a number can have more than one decimal expansion in base 3, since for example the real number $\frac{1}{3}$—which is in $\mathcal{C}$—has the two expansions 0.1 and 0.0222... in base 3. So another way to phrase this question is: Show that $[0, 1] \setminus \mathcal{C}$ is the collection of real numbers in $[0, 1]$ *all* of whose base 3 expansions contain a 1.)

After you have done these exercises, take a moment to think about these results together and convince yourself that $\mathcal{C}$ is weird. It is not unusual to satisfy any one of these properties; we know lots of uncountable subsets of the reals, lots of compact subsets of the reals, we know $\mathbb{Q}$ is perfect, and we know lots of sets with empty interiors. It is weird that $\mathcal{C}$ has all of these properties simultaneously.

We will explore the Cantor set a little more after we study connectedness.

**11. Let $X := \{0, 1\}^{\mathbb{N}}$ with the product topology induced by the discrete topology in each factor. Show that X compact directly (ie. without using Tychonoff's theorem) by showing that X is homeomorphic to the Cantor set.

Hint: By the last part of the previous exercise, every element a of $\mathcal{C}$ can be written as

$$a = \sum_{n=1}^{\infty} \frac{a_n}{3^n},$$

for some sequence $\{a_n\}_{n \in \mathbb{N}}$ of 0s and 2s.

**12. The previous result, that $\{0, 1\}^{\mathbb{N}}$ is homeomorphic to the Cantor set, is a very important one in topology. The Cantor set $\mathcal{C}$ (which from this point onwards we will think of interchangeably as both $\{0, 1\}^{\mathbb{N}}$ and as the subset of the reals we constructed earlier) turns out to contain all the information of a great number of topological spaces. In this exercise we will explore this a little bit.

(a) Show that $\mathcal{C}$ is homeomorphic to $\mathcal{C}^{\mathbb{N}}$ (with the product topology). Thinking of $\mathcal{C}$ as $\{0,1\}^{\mathbb{N}}$, this proof should be very short, given things you already know about products.

(b) Prove that $[0,1]$ (with its usual topology) is a continuous image of $\mathcal{C}$. (That is, show that there is a continuous surjection $f : \mathcal{C} \to [0,1]$.)

Hint: Think of $\mathcal{C}$ as a subset of $\mathbb{R}$ whose elements are given by sums as in the previous question, and consider the map:

$$\sum_{n=1}^{\infty} \frac{a_n}{3^n} \mapsto \sum_{n=1}^{\infty} \frac{a_n}{2^{n+1}}.$$

(c) Conclude from the previous two parts that $H := [0,1]^{\mathbb{N}}$ is a continuous image of $\mathcal{C}$.

(d) Let $(X, \mathcal{T})$ be any compact metrizable topological space. Show that X is homeomorphic to a subset of H.

Hint: By a previous exercise in this section, X is separable. Let $D = \{\, x_n : n \in \mathbb{N} \,\}$ be a countable dense subset of X. Assuming that $\mathcal{T}$ is generated by a metric d that is bounded by 1, let $F : X \to H$ be defined by:

$$F(x) = (d(x, x_1), d(x, x_2), d(x, x_3), \dots).$$

First, show that F is continuous (this is easy). By a proposition from the lecture notes, it then suffices to show that F is injective.

(e) Conclude from the previous parts that **every compact metrizable space is a continuous image of the Cantor set**. This is a very powerful result. By now I hope you are convinced that the Cantor set is very important.

**13. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $f : X \to Y$ be a closed, surjective, continuous function with the additional property that for every $y \in Y$, $f^{-1}(\{y\})$ is a compact subset of X. Such a map is called a <u>perfect</u> map. As you might expect from the name, perfect maps are pretty nice.

(a) Show that if X is Hausdorff, then Y is Hausdorff.

(b) Show that if X is second countable, then Y is second countable.

(c) Show that if Y is compact, then X is compact.

(d) Show that perfect maps are "rigid" in the sense that they cannot be extended continuously to closures: Let X be Hausdorff and $f : X \to Y$ be a perfect map. Suppose Z is a Hausdorff space that contains X as a dense proper subspace. Show that there is no continuous function $F : Z \to Y$ such that $F(x) = f(x)$ for all $x \in X$.

(Hint: Do a proof by contradiction.)

**14. Let (X, d) be a metric space which is compact when thought of as a topological space with its metric topology generated by d, and let $f : X \to \mathbb{R}_{\text{usual}}$ be a continuous function. Show that f is uniformly continuous.

**15. This exercise defines a condition on topological spaces that is of particular interest to algebraic geometers. As you will see, this property is *much* stronger than compactness.

Definition 16.1. *A topological space $(X, \mathcal{T})$ is called Noetherian if every ascending chain of open sets stabilizes. That is, if $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ is an ascending sequence of open sets, then there is an $N \in \mathbb{N}$ such that $U_n = U_m$ for all $n, m > N$.*

(a) Show that the property of being Noetherian is hereditary.

(b) Show that every Noetherian space is compact.

Hint: Let $(X, \mathcal{T})$ be Noetherian and let $\mathcal{U}$ be an open cover of X. Consider the collection $\mathcal{A}$ of all finite unions of elements of $\mathcal{U}$. Apply Zorn's Lemma to $\mathcal{A}$, and show that the maximal element is X.

(c) Conclude from the previous two exercises that every Noetherian space is hereditarily compact.

The property of being hereditarily compact is *very* strong, and can be counterintuitive. Every open subset of a hereditarily compact space is compact, for example. The following exercise should give you some impression of how strong it is.

(d) Show a topological space is Hausdorff and hereditarily compact if and only if it is finite and discrete.

(e) Show that any Hausdorff Noetherian space is finite and discrete.

***16. This is for the student interested in category theory. These exercises are not particularly difficult, but they are enough outside the scope of this course (and notationally heavy enough) that no one should feel obligated to do them. Still, they are pretty cool.

Let $\{ (X_n, \mathcal{T}_n) : n \in \mathbb{N} \}$ be a collection of topological spaces, and assume that for each $n > 1$, $f_n : X_n \to X_{n-1}$ is a continuous function. The sequence

$$X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} \cdots,$$

is called an inverse system of topological spaces, which we denote by $\langle (X_n, \mathcal{T}_n), f_n \rangle$, or simply $\langle X_n, f_n \rangle$ if the topologies on each space are clear. (In general, an inverse system is defined on a directed set rather than the naturals; we are examining a simpler special case in this problem.)

In this context, the sequence above is a generalization of a sequence $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$ of nested spaces (each space would have its subspace topology from the topology on X_1, and the maps would be inclusions).

We define the inverse limit of this sequence, denoted by $\varprojlim X_n$, as the set:

$$\varprojlim X_n := \left\{ \{x_n\}_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n \ : \ f_n(x_n) = x_{n-1} \text{ for all } n > 1 \right\}$$

(which is a topological space with its subspace topology inherited from the product topology on $\prod_{n\in\mathbb{N}} X_n$). This is a very powerful way of defining new spaces from old ones.

(a) Prove that if the $\langle X_n, f_n \rangle$ is a nested sequence, meaning $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$, and $f : X_n \to X_{n-1}$ is the natural inclusion map for every $n > 1$, then $\varprojlim X_n$ is homeomorphic to $\bigcap_{n\in\mathbb{N}} X_n$ (with its subspace topology inherited from $\mathcal{T}_0$).

(b) Let $(X, \mathcal{T})$ be a topological space, and let $\langle X_n, f_n \rangle$ be the constant inverse system, where $X_n = X$ and $f_n = \text{id}$ for all n. Show that $\varprojlim X_n \simeq X$

(c) In this part, we use the notation of the initial definitions in this section.

One of the reasons this construction is useful is that inverse limits satisfy the following universal property.

If $(Y, \mathcal{U})$ is a topological space and $g_n : Y \to X_n$ is a continuous function with the property that $g_{n-1} = f_n \circ g_n$ for all $n > 1$, then there exists a unique continuous function $u : Y \to \varprojlim X_n$ such that $g_n = \pi_n \circ u$ (where π_n is the usual projection function, restricted to $\varprojlim X_n \subseteq \prod_{n\in\mathbb{N}} X_n$).

Prove this. Then try to interpret this property in the context of specific inverse systems of topological spaces. For example, in the context of part (a), this says that $A \subseteq \bigcap_{n\in\mathbb{N}} X_n$ if and only if $A \subseteq X_n$ for all n.

This universal property actually *characterizes* the inverse limit. A category theorist would most likely take this as a definition of the inverse limit, then carry out the construction we did above as a proof that such objects exist and are unique in the category of topological spaces.

(d) Prove that if $\langle X_n, f_n \rangle$ is an inverse system of nonempty compact Hausdorff spaces, then $\varprojlim X_n$ is a nonempty compact Hausdorff space.

(By part (a), this exercise is a *significant* generalization of the fact that the intersection of a sequence of nested, closed intervals in $\mathbb{R}$ with decreasing lengths is a closed interval.)

(e) This exercise will show you that compact Hausdorff spaces are much nicer than compact non-Hausorff spaces.

For each $n \in \mathbb{N}$, we define a basis $\mathcal{B}_n$ on $\mathbb{N}$ by the following:

$$\mathcal{B}_n = \{\{1\}, \{2\}, \ldots, \{n\}, \{n+1, n+2, \ldots\}\}.$$

It is easy to see that this is a basis. Let $\mathcal{T}_n$ be the topology it generates.

i. Show that $(\mathbb{N}, \mathcal{T}_n)$ is compact but not Hausdorff for every n.

ii. Show that the identity map $\mathrm{id} : \mathbb{N} \to \mathbb{N}$ given by $\mathrm{id}(x) = x$ is continuous when viewed as a map from $(\mathbb{N}, \mathcal{T}_n)$ to $(\mathbb{N}, \mathcal{T}_{n-1})$ for all $n > 2$. (This is also very easy.)

iii. The previous exercise shows us that $\langle(\mathbb{N}, \mathcal{T}_n), \mathrm{id}\rangle$ is an inverse system. Let $(X, \mathcal{T}) = \; ds\lim_{\longleftarrow}(\mathbb{N}, \mathcal{T}_n)$ be its inverse limit. This space is a very simple space with which you are already familiar. What is it?

iv. Conclude that an inverse limit of compact non-Hausdorff spaces need not be compact.

(f) If you have two inverse systems $\langle X_n, f_n\rangle$ and $\langle Y_n, g_n\rangle$, and you have some continuous functions $\phi_n : X_n \to Y_n$ for every n, you should expect there to be a way of defining a continuous function $\varprojlim X_n \to \varprojlim Y_n$. It turns out you can do this only if the ϕ_n's play nicely with the f_n's and g_n's, in the sense that:

$$\phi_{n-1} \circ f_n = g_n \circ \phi_n \tag{1}$$

for all $n > 1$. This is confusing to look at, so the correct way to think of it is in terms of what mathematicians call a diagram, like the following:

$$\begin{array}{ccccccccc}
\cdots & \xleftarrow{f_{n-1}} & X_{n-1} & \xleftarrow{f_n} & X_n & \xleftarrow{f_{n+1}} & X_{n+1} & \xleftarrow{f_{n+2}} & \cdots \\
 & & \big\downarrow{\scriptstyle \phi_{n-1}} & & \big\downarrow{\scriptstyle \phi_n} & & \big\downarrow{\scriptstyle \phi_{n+1}} & & \\
\cdots & \xleftarrow{g_{n-1}} & Y_{n-1} & \xleftarrow{g_n} & Y_n & \xleftarrow{g_{n+1}} & Y_{n+1} & \xleftarrow{g_{n+2}} & \cdots
\end{array}$$

The statement (1) above amounts to saying that this diagram commutes, meaning that if you start at a point and go to another point along two different paths of arrows, the resulting compositions of arrows are equal.

So, we say that a function $\Phi = (\phi_n)_{n\in\mathbb{N}} : \langle X_n, f_n\rangle \to \langle Y_n, g_n\rangle$ is continuous provided the functions ϕ_n are continuous for all n, and that they obey the condition outlined above. Finally, we define an induced function $\phi : \varprojlim X_n \to \varprojlim Y_n$ by

$$\phi(\{x_n\}_{n\in\mathbb{N}}) = \{\phi_n(x_n)\}_{n\in\mathbb{N}}$$

Prove that this map ϕ is well defined, in the sense that it actually maps into $\varprojlim Y_n$.

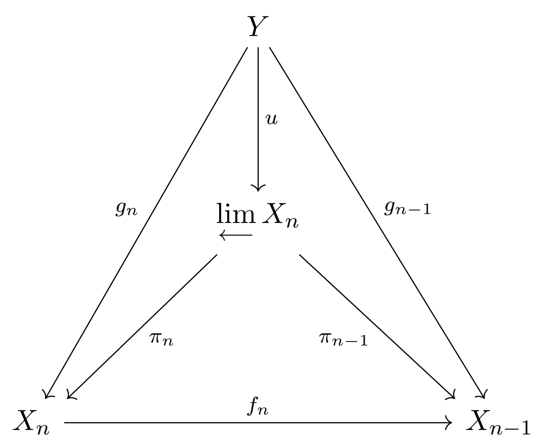
(g) Prove that if Φ is continuous, then ϕ is continuous.

(h) If you are still interested in learning more about this, talk to me. There is *a lot* of room to generalize these definitions to get very powerful tools. For example, in this discussion we only mentioned inverse systems indexed by $\mathbb{N}$, when in fact we can create a similar object indexed by any directed set (recall that definition from my supplementary notes on nets and filters). This allows us to prove some nice results, like that every topological spaces is an inverse limit of spaces whose topologies are finite.

There are also direct limits, which are like inverse limits except all the arrows go the other way. Direct limits are generalizations of unions the same way inverse limits are generalizations of intersections.

While this all looks complicated, inverse and direct limits are a very basic tool for a lot of higher mathematics. They can be defined in any *category*, and not just for topological spaces.

PS: Diagrams are a great way of looking at certain relationships. For example, the universal property of inverse limits described in part (c) is more readily illustrated by saying that following diagram commutes for all $n > 1$.



17 Tychonoff's theorem and properties related to compactness

17.1 Problems related to Tychonoff's theorem

*1. A collection $\mathcal{A}$ of subsets of a set X is said to have the countable intersection property (CIP) if for every countable subcollection $\mathcal{C} \subseteq \mathcal{A}$, $\bigcap \mathcal{C} \neq \emptyset$. Note that a collection of sets with the CIP also has the FIP. Show that a topological space $(X, \mathcal{T})$ is Lindelöf if and only if every collection of closed subsets of X with the countable intersection property has a nonempty intersection.

**2. Let X be a set (which you should imagine being very large). A transitive binary relation on X is a subset, $R \subseteq X^2 = X \times X$ such that if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$. Note that by this definition, $R \in \mathcal{P}(X^2)$. Let $\mathcal{R}$ be the collection of all transitive binary relations on X, so $\mathcal{R} \subseteq \mathcal{P}(X^2)$.

Thinking of $\mathcal{P}(X^2)$ as a topological space with the topology defined in BL 14.2.11, show that $\mathcal{R}$ is compact.

Hint: Recall that in defining the topology on $\mathcal{P}(X^2)$, we identify the set with $\{0, 1\}^{X^2}$, which is compact by Tychonoff's theorem. Clearly this space is Hausdorff (being a product of copies of the Hausdorff space $\{0, 1\}$), and therefore it suffices to show that $\mathcal{R}$ is closed. You should actually show that its complement is open.

**3. Let X be a set. A linear order $\leq$ on X is a transitive binary relation on X, and so the set $\mathcal{L} \subseteq \mathcal{P}(X^2)$ of all linear orders on X is a subset of the set $\mathcal{R}$ from the previous problem. Show that $\mathcal{L}$ is compact.

**4. We already know that the Axiom of Choice implies Tychonoff's theorem, because we used it (in the form of Zorn's Lemma) in an important way in the proof. In this exercise, you will show that Tychonoff's theorem implies AC.

Assume Tychonoff's theorem, and let $\mathcal{A} = \{ A_\alpha : \alpha \in I \}$ be a nonempty collection of nonempty sets. We want to show that there is a choice function on $\mathcal{A}$.

(a) Show that there is a choice function on $\mathcal{A}$ if and only if $\prod_{\alpha \in I} A_\alpha \neq \emptyset$. (This is an exercise you did back when we discussed AC, but remind yourself of it here.)

(b) Let $\heartsuit$ be a symbol that that is not in any of the A_α's. For each $\alpha \in I$, we are going to define a topology on $A_\alpha \cup \{\heartsuit\}$. Let $\mathcal{T}_\alpha$ be:

$$\mathcal{T}_\alpha = \{ U \in \mathcal{P}(A_\alpha \cup \{\heartsuit\}) : U = \emptyset, U = \{\heartsuit\} \text{ or } (A_\alpha \cup \{\heartsuit\}) \setminus U \text{ is finite} \}.$$

That is, $\mathcal{T}_\alpha$ is the co-finite topology on $A_\alpha \cup \{\heartsuit\}$, with $\{\heartsuit\}$ also declared to be open. Convince yourself that this is a topology on $\mathcal{A}_\alpha \cup \{\heartsuit\}$.

(c) Prove that $(A_\alpha \cup \{\heartsuit\}, \mathcal{T}_\alpha)$ is a compact topological space for each $\alpha \in I$. Then by Tychonoff's theorem, $X := \prod_{\alpha \in I}(A_\alpha \cup \{\heartsuit\})$ is compact with its product topology.

(d) Let $\mathcal{C}$ be the following collection of subsets of X:

$$\mathcal{C} = \left\{ \pi_\alpha^{-1}(A_\alpha) \,:\, \alpha \in I \right\}.$$

Verify that each set in $\mathcal{C}$ is a closed subset of X.

(e) This is the part where this business with ♡ helps us. Show that $\mathcal{C}$ has the finite intersection property.

Hint: Let $C_1, \ldots, C_n$ be elements of $\mathcal{C}$. Then for each i there is an $\alpha_i \in I$ such that $C_i = \pi_{\alpha_i}^{-1}(A_{\alpha_i})$. We know that each C_i is nonempty (since the projection maps are all surjective) and since we are only looking at finitely many nonempty sets, we can use finite choice to pick an element $b_i \in A_{\alpha_i}$ for each $i = 1, \ldots, n$. Define an element $f \in X$ by:

$$f(\alpha) = \begin{cases} b_i & \alpha = \beta_i \text{ for some } i = 1, \ldots, n \\ \heartsuit & \text{otherwise} \end{cases}$$

Show that $f \in C_1 \cap \cdots \cap C_n$.

Morally, in this proof we are trying to choose an element from each A_α. We cannot do that without AC, but we can definitely pick the element ♡ from each $A_\alpha \cup \{\heartsuit\}$. We put ♡ into each of those sets precisely so we can pick it here when necessary, leaving us with only finitely many "non-obvious" choices to make.

(f) Conclude from the previous part that $\bigcap \mathcal{C} \neq \emptyset$. Finally, conclude from this that $\prod_{\alpha \in I} A_\alpha \neq \emptyset$.

***5. In this exercise you are going to produce another proof of Tychonoff's theorem, this time using the Well-Ordering Principle rather than Zorn's Lemma (which you should recall was used in the proof given in the lecture notes to extend a filter to an ultrafilter). This exercise will involve transfinite induction. This is much closer to (but not the same as) the original proof of Tychonoff's theorem.

(a) Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces with X compact, and let $\mathcal{B}$ be the usual basis for the product topology on $X \times Y$. Let $\mathcal{A} \subseteq \mathcal{B}$ be a collection of basic open sets with the property that no finite subcollection of $\mathcal{A}$ covers $X \times Y$. Show that there is an $x \in X$ such that no finite subcollection of $\mathcal{A}$ covers $\{x\} \times Y$.

(This is very similar, and follows from, Lemma 5.2 in the notes on compactness.)

(b) Now, for remainder of the problem, let I be a nonempty indexing set and let $\mathcal{X} = \{ (X_\alpha, \mathcal{T}_\alpha) \,:\, \alpha \in I \}$ be a collection of compact topological spaces. Let $X = \prod_{\alpha \in I} X_\alpha$ be their product, which we want to show is compact with its product topology. Let $\mathcal{B}$ be the usual basis for the product topology. Also, fix a well-ordering $\leq$ of I with the property that $(I, \leq)$ has a largest element (this is no loss of generality, since you can fix any well-order, then take the least element and "move it to the top" of the order).

The proof proceeds by induction (transfinite induction, in general) on this well-order. Fix $\beta \in I$, and for each $\gamma < \beta$, fix some $p_\gamma \in X_\gamma$. For each $\alpha < \beta$, define the subset $Y_\alpha \subseteq X$ by:

$$Y_\alpha = \{ x \in X : \pi_\gamma(x) = p_\gamma \text{ for all } \gamma \leq \alpha \}.$$

Note that these sets are nested, in the sense that if $\alpha_1 < \alpha_2$, then $Y_{\alpha_1} \supseteq Y_{\alpha_2}$. Finally, let $Z_\beta = \bigcap_{\alpha<\beta} Y_\alpha$.

Show that if $\mathcal{A} \subseteq \mathcal{B}$ is a finite collection of basic open sets that covers Z_β, then $\mathcal{A}$ also covers Y_α for some $\alpha < \beta$.

Hint: Recall that in a well-order, an element β either has an immediate predecessor in the order or it does not. The first case is basically immediate. For the other case, for each $U \in \mathcal{A}$ let $I_U \subseteq I$ be the set

$$I_U = \{ \alpha < \beta : \pi_\alpha(U) \neq X_\alpha \}.$$

By definition of the product topology, $\bigcup_{U\in\mathcal{A}} I_U$ is finite. Choose $\alpha = \max \bigcup_{U\in\mathcal{A}} I_U$.

(c) Now, we prove that X is compact by contrapositive. So let $\mathcal{U} \subseteq \mathcal{B}$ be a collection of basic open sets such that no finite subcollection of $\mathcal{U}$ covers X, and we'll show that $\mathcal{U}$ is not a cover of X.

Show by transfinite induction that one can choose points $p_\alpha \in X_\alpha$ for all $\alpha \in I$ such that for each $\beta \in I$, the set Y_β (as defined earlier, with this new choice of p_α's used for all of them) cannot be covered by a finite subcollection of $\mathcal{U}$.

Hint: Base case: If $\beta = \min I$, Then we can regard X as $X_\beta \times \prod_{\beta<\alpha\in I} X_\alpha$, and find the point p_β using the result of part (a).

Suppose now that we have found the p_α's that work for all $\alpha < \beta$, and we attempt to find p_β. By the contrapositive of part (b) you can conclude that no finite subcollection of $\mathcal{U}$ covers Z_β. Then show that you can use part (a) to find p_β.

(d) Conclude from the previous part that if $\beta = \max I$ (which exists by assumption), then the element $f \in X$ defined by $f(\alpha) = p_\alpha$ is an element of X that is not covered by any finite subcollection from $\mathcal{U}$. Show that this implies $\mathcal{U}$ is not a cover.

This proof provides us with an interesting way to look at Tychonoff's theorem. We already know that AC, Tychonoff's theorem, and the Well-Ordering Principle are all equivalent, and this proof allows us to conclude that any product of compact sets *indexed by a set that can be well-ordered* is compact. This means that that in a world without AC, many large products of compact sets are still compact. For example any product of compact sets indexed by any ordinal is compact.

17.2 Problems on properties related to compactness

None of this material will be tested in this class, but I include it for some flavour. It is interesting that all of these notions collapse to the same thing in a metric space.

Recall the following properties related to compactness that we defined in the lecture notes:

1. Compactness;
2. Sequential compactness;
3. Countable compactness;
4. Limit point compactness.

We also proved (or at least stated) the following relationships between them:

- Compact $\Rightarrow$ countably compact.
- Countably compact $\Leftrightarrow$ every infinite subset has an ω-accumulation point.
- Sequentially compact $\Rightarrow$ countably compact.
- First countable + countably compact $\Rightarrow$ sequentially compact.
- Compact $\not\Rightarrow$ sequentially compact (the example I mentioned, without a complete proof, was $[0,1]^{[0,1]}$).
- Sequentially compact $\not\Rightarrow$ compact (the example I mentioned was ω_1, which is even first countable).
- Second countable $\Rightarrow$ (compact $\Leftrightarrow$ countably compact).
- First countable + T_1 + limit point compact $\Rightarrow$ sequentially compact.

The main result of this section is to prove that in a metric space, all four properties listed above, along with the property of being complete and totally bounded (which we call property 5), are equivalent. This is a two-star series of exercises.

So let (X, d) be a metric space, thought of as a topological space with its metric topology generated by d. (We have to fix a particular metric here, since property 5 depends on the exact metric we use). Recall that a metrizable space is Hausdorff (and therefore T_1), and first countable. Note also that the property that every infinite set has an ω-accumulation point is much stronger than the property of being limit point compact.

Our proof will take the following form:

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$$

and then we will separately prove that (5) is equivalent to these.

Note that we already know (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) from our list of results above.

1. Prove that (1) $\Rightarrow$ (2). This is almost entirely done by the results above.

2. This is an intermediate fact. Suppose X is limit point compact. Let $\mathcal{U} = \{ U_\alpha : \alpha \in I \}$ be a given open cover of X. Show that there is an $\epsilon > 0$ such that for all $x \in X$, $B_\epsilon(x) \subseteq U_\alpha$ for some $\alpha \in I$.

   Hint: Suppose there is no such $\epsilon > 0$. Then for each $n \in \mathbb{N}$ we can find a point $x_n \in X$ such that $B_{\frac{1}{n}}(x_n)$ is not a subset of any element of $\mathcal{U}$.

   (a) Show that $A := \{ x_n : n \in \mathbb{N} \}$ is infinite.

   (b) Since X is limit point compact, let a be a limit point of A. Then $a \in U$ for some $U \in \mathcal{U}$, and therefore there is an $\epsilon > 0$ such that $B_\epsilon(a) \subseteq U$. Derive a contradiction from here.

3. Prove that (4) $\Rightarrow$ (1).

   Hint: Use the previous part.

4. We now prove that (5) $\Leftrightarrow$ (2).

   ($\Leftarrow$). Suppose (X, d) is sequentially compact (and therefore compact).

   (a) Prove that X is totally bounded.

   (b) To show that the space is complete, note that every Cauchy sequence in X has a convergent subsequence, since X is sequentially compact. Use this to show that the whole sequence converges.

   ($\Rightarrow$). Suppose (X, d) is complete and totally bounded, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. We want to show that it has a convergent subsequence, and by completeness it suffices to show it has a Cauchy subsequence. This is a sort of diagonalization argument.

   (a) Suppose $\mathcal{U}_1$ is a cover of X by finitely many metric balls of radius 1 (which we can find since X is totally bounded). Show that there is a ball $B_1 \in \mathcal{U}_1$ that contains infinitely many members of the sequence. Call the corresponding set of indices A_1. (ie. $A_1 = \{ n \in \mathbb{N} : x_n \in B_1 \}$).

   (b) Suppose $\mathcal{U}_2$ is a cover of X by finitely many metric balls of radius $\frac{1}{2}$. Show that there is a ball $B_2 \in \mathcal{U}_2$ and an infinite subset $A_2 \subseteq A_1$ such that $x_n \in B_2$ for all $n \in A_2$.

   (c) Proceeding inductively, we construct a collection of open balls B_n of radius $\frac{1}{n}$ for each n, and a sequence of infinite subsets of $\mathbb{N}$ $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ such that $A_k = \{ k \in \mathbb{N} : k \in B_k \}$. Convince yourself we can do this.

   (d) Show that we can inductively choose numbers $n_k \in A_k$ such that

   $$n_1 < n_2 < n_3 < \cdots .$$

   (e) Show that the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of our original sequence is Cauchy.

18 Connectedness

*1. Prove Proposition 2.2 from the lecture notes (which provides a number of equivalent definitions of disconnectedness).

*2. Let X be a set, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X such that $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Does the connectedness of either one of them imply the connectedness of the other?

*3. Let $D \subseteq \mathbb{R}^2$ be countable. Show that $\mathbb{R}^2 \backslash D$ is connected (with its usual subspace topology).

Hint: Show it is path-connected.

*4. Let $(W, \leq)$ be a well-order, thought of as a topological space with its order topology. Provided that W has more than one point, show that it is disconnected.

*5. Let $(X, \mathcal{T})$ be a topological space, and let $D \subseteq X$ be a connected subset. Show that $\overline{D}$ is connected.

In particular, this implies that a topological space with a dense connected subspace is connected.

*6. Let $(X, \mathcal{T})$ be a topological space, and let $\{ A_n : n \in \mathbb{N} \}$ be connected subsets of X with the property that $A_n \cap A_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$. Show that $\bigcup_{n \in \mathbb{N}} A_n$ is connected.

*7. Show that $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ is disconnected. What about $\mathbb{R}^{\mathbb{N}}_{\text{unif}}$?

Hint: For the first question, consider the set of sequences of real numbers that converge to zero in the usual topology, as a subset of $\mathbb{R}^{\mathbb{N}}$.

*8. Show that path-connectedness is (arbitrarily) productive.

*9. Fill in the details of the proof that the Topologist's Sine Curve is connected but not path-connected.

*10. This is a definition we will return to a few times throughout these exercises:

Definition 18.1. *A topological space $(X, \mathcal{T})$ is called totally disconnected if its only connected subsets are singletons.*

(a) Show that total disconnectedness is a topological invariant.

(b) Show that a topological space $(X, \mathcal{T})$ is totally disconnected if for every pair of distinct points $a, b \in X$ there are disjoint open sets U, V such that $a \in U$, $b \in V$ and $X = U \sqcup V$.

(c) Show that total disconnectedness is hereditary.

(d) Show that total disconnectedness is (arbitrarily) productive.

(e) Intuitively, total disconnectedness feels a lot like discreteness. Show that any discrete space is totally disconnected, and come up with an example of a non-discrete space that is totally disconnected. (For the latter, you know a few spaces already that satisfy this.)

(f) Show that any zero-dimensional T_0 space is totally disconnected. (Recall that a topological space is called zero-dimensional if it admits a basis consisting of clopen sets.)

*11. Here is another useful definition.

Definition 18.2. *A topological space $(X, \mathcal{T})$ is called locally connected if for every $x \in X$ and every open set U containing x, there is an open, connected set V such that $x \in V \subseteq U$. Equivalently, $(X, \mathcal{T})$ is locally connected if it admits a basis of connected sets.*

(a) Show that there is no implication relationship between connectedness and local connectedness. In fact, find four topological spaces which are respectively:

i. Not connected or locally connected.

ii. Connected but not locally connected. (This is the tricky one, but you have already seen the example in the lecture notes.)

iii. Locally connected but not connected.

iv. Connected and locally connected.

(b) Show that every discrete space is locally connected.

(c) Show (with as trivial an example as possible) that a continuous image of a locally connected space need not be locally connected.

(d) Show that local connectedness is a topological invariant.

Can you relax this at all? That is, you just showed that if X is locally connected and $f : X \to Y$ is a homeomorphism, then Y is locally connected. Examine your proof, and see if a weaker function f will also work. You know from the previous part that a continuous surjection is not good enough.

(e) Show with an example that local connectedness is not hereditary (one of your examples from the first part should work).

(f) Give an example of a space that is locally connected but not totally disconnected (this is easy), and a space that is totally disconnected but not locally connected (this is slightly less easy).

(g) Show that a totally disconnected, locally connected topological space must be discrete.

*12. Recall from some time ago that if $(X, \mathcal{T})$ is a connected topological space, a point $p \in X$ is called a cut point if $X \setminus \{p\}$ is disconnected.

(a) Prove that homeomorphisms send cut points to cut points.

(b) For $n \in \mathbb{N}$, prove that the property of having n cut points is a topological invariant.

(c) Does $\mathbb{R}_{\text{co-finite}}$ have any cut points? What about $\mathbb{R}_{\text{co-countable}}$? What about the Topologist's Sine Curve?

(d) Consider the English capital letters X, P, Y, B, and O, and imagine them as subspaces of $\mathbb{R}^2_{\text{usual}}$. Are any of them homeomorphic to one another?

**13. Prove the direction of Theorem 2.6 that we left out. That is, prove that a connected linear order $(L, \leq)$, thought of as a topological space with its order topology, is Dedekind complete and has no gaps.

Hint: Prove the contrapositive (which will involve two cases).

**14. Prove that connectedness is productive.

Hint: A similar idea to the proof that connectedness is finitely productive will work, but there's much more bookkeeping to do:

Let $X = \prod_{\alpha \in I} X_\alpha$ be a product of connected spaces. Fix an element $f \in X$, and let

$$D_f = \{\, g \in X \;:\; g(\alpha) = f(\alpha) \text{ for all but finitely many } \alpha \in I \,\}.$$

Show that D_f is dense in X and connected, and then use an earlier exercise to conclude that X is connected.

**15. Show that the Cantor set is totally disconnected. Note that by the result of an exercise in section 16, this implies that a continuous image of a totally disconnected space need not be totally disconnected, since any compact metrizable space is a continuous image of the Cantor set.

**16. In this exercise, you will show that any compact, Hausdorff, totally disconnected space is zero-dimensional. (This may not seem so exciting, but it will lead somewhere exciting.)

Let $(X, \mathcal{T})$ be a compact, Hausdorff, totally disconnected topological space. The proof proceeds by constructing a local basis of clopen sets around each point in the space (the union of all of these is then a basis of clopen sets for the topology). Before we begin, recall that any compact Hausdorff space is normal.

Fix a point $x \in X$. Let $\mathcal{C} = \{\, A \subseteq X \;:\; x \in A, \text{ and } A \text{ is clopen} \,\}$. Then $X \in \mathcal{C}$, so $\mathcal{C} \neq \emptyset$.

(a) Show that $\mathcal{C}$ is closed under finite intersections, and that $P := \bigcap \mathcal{C}$ is a closed set that contains x.

(b) Show that for every closed set F that is disjoint from P, there is a $C \in \mathcal{C}$ such that $F \cap C = \emptyset$.

Hint: Suppose not, fixing a closed set F which is disjoint from P, and which intersects every member of $\mathcal{C}$. Show that $\mathcal{A} = \{ C \cap F : C \in \mathcal{C} \}$ is a collection of closed sets with the finite intersection property, and use the FIP definition of compactness to derive a contradiction.

(c) Show that $P = \{x\}$.

Hint: Suppose not. Then P has more than one point, and is therefore disconnected (as a subspace) since X is totally disconnected. Fix A and B, nonempty open subsets of the space P, such that $P = A \sqcup B$, and such that $x \in A$. Then A and B are closed in X (since P is a closed subspace), and so since X is normal we can find disjoint open sets U and V containing A and B respectively. Then $F := X \setminus (U \cup V)$ is a closed set that does not intersect P. Apply the previous part, and derive a contradiction.

(d) Conclude that $\mathcal{C}$ is a local basis at x.

**17. In this exercise, you are going to prove a powerful characterization theorem about compact, Hausdorff, totally disconnected spaces.

Suppose $(X, \mathcal{T})$ is a compact, Hausdorff, totally disconnected space. By the previous exercise, the space admits a basis $\mathcal{B} = \{ B_\alpha : \alpha \in I \}$ of clopen sets. You are going to show that X is homeomorphic to a subspace of $\{0,1\}^I$.

Recall that for a subset $A \subseteq X$, the corresponding <u>indicator function</u> $\chi_A : X \to \{0,1\}$ is the function

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Define a map $\phi : X \to \{0,1\}^I$ by $\phi(x)(\alpha) = \chi_{B_\alpha}(x)$. Understanding this map will likely take you a minute, so stare at it for a bit. Remember that elements of $\{0,1\}^I$ are functions $I \to \{0,1\}$, so $\phi(x)$ is defined by how it acts on each $\alpha \in I$. The way ϕ acts is by sending x to what is essentially a checklist of what elements of $\mathcal{B}$ contain x.

(a) Show that ϕ is injective. This is just unravelling of definitions, though it will be important to remember that in a T_1 space like X, a point x is equal to the intersection of all basic open sets that contain it.

(b) Prove that for each $B \in \mathcal{B}$, χ_B is continuous.

(c) Prove that ϕ is continuous. I suggest doing this by showing that the preimages of *subbasic* open sets in $\{0,1\}^I$ are open, using the previous part.

(d) Conclude that ϕ is a homeomorphism onto its range, which establishes the result.

I hope you are now getting a good feeling for the usefulness of embedding spaces into large products.

**18. Let $(X, \mathcal{T})$ be a topological space, and let Y be any set. A function $f : X \to Y$ is called locally constant if for every $x \in X$, there is an open set $U_x \subseteq X$ containing x such that f is constant on U_x. Prove that a locally constant function with a connected domain is constant.

***19. This is a problem about vector bundles. This problem is not very difficult at all, but it is (a) notationally intensive, and (b) somewhat outside the scope of this course. I put it here with three stars for interested students, just so no one feels they have to do this. The biggest challenge with this problem will likely be wrapping your mind around the definition of a vector bundle.

Let $(X, \mathcal{T})$ be a topological space. Let $(E, \mathcal{U}, \pi)$ be a triple such that:

(i) $(E, \mathcal{U})$ is a topological space;

(ii) $\pi : E \to X$ is a continuous surjection; and

(iii) for each $x \in X$, the preimage $\pi^{-1}(x)$ has the structure of a finitel-dimensional real vector space.

The preimages $\pi^{-1}(x)$ are called fibres. In the context of the rest of this problem, you should think of E as a "bundle" of these fibres. (A vector bundle is a particular case of a more general object called a fibre bundle.)

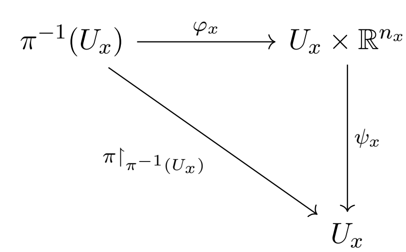
Such a triple is called a vector bundle on $(X, \mathcal{T})$ if:

For every $x \in X$ there is an open set $U_x \subseteq X$ containing x, a non-negative integer n_x, and a homeomorphism

$$\varphi_x : \pi^{-1}(U_x) \to U_x \times \mathbb{R}^{n_x}$$

(where $\mathbb{R}^{n_x}$ has its usual topology) such that the following two properties are satisfied:

(a) The following diagram commutes:



where $\psi_x : U_x \times \mathbb{R}^{n_x} \to U_x$ is the first projection function: $\psi_x(y, v) = y$. (Usually I would call this function π_1, but π is always used for the other map in a vector bundle.) To remind you, the diagram above is said to commute if

$$\psi_x \circ \varphi_x = \pi \restriction_{\pi^{-1}(U_x)} .$$

(b) For each $y \in U_x$, the map:

$$\varphi_x \restriction_{\pi^{-1}(y)} : \pi^{-1}(y) \to \psi_x^{-1}(y)$$

is an isomorphism of vector spaces.

To make sense of this, note that by definition of ψ_x, $\psi_x^{-1}(y) = \{y\} \times \mathbb{R}^{n_x}$, which we regard as a vector space by simply ignoring the constant first coordinate. Also by hypothesis we know that $\pi^{-1}(y)$ has the structure of a finite-dimensional real vector space.

Some intuition for these definitions. First of all, these definitions are usually written without the subscript x on everything. I wanted to include them to stress that all of these things depend on the particular x.

Next, observe that given a fixed integer (we'll use 7), the set $E = X \times \mathbb{R}^7$, with its product topology and the map $\pi : E \to X$ given by $\pi(x, y) = x$, is trivially a vector bundle because you can choose $U_x = x$, $n_x = 7$ and $\varphi_x = \mathrm{id}$ for all $x \in X$. This is called the trivial vector bundle of rank 7 over $(X, \mathcal{T})$.

In general, the map φ_x is often called a "local trivialization". So the idea here is that E is a big bundle of vector spaces, one sitting "over" every point in X. The existence of U_x and the map φ_x shows that *near* x, the map π looks like a simple projection $U_x \times \mathbb{R}^n \to U_x$ for some n, and in turn that *near* x, this collection of vector spaces over points looks like a trivial vector bundle of some fixed rank n_x.

One example of a vector bundle you may have seen is the tangent bundle of a smooth manifold. In this case, the topological space $(X, \mathcal{T})$ is a smooth manifold M, the fiber over a point $x \in M$ is its tangent space, and E is the disjoint union of these tangent spaces with the disjoint union topology on it. A vector field on M is an assignment to each $x \in M$ of a vector in its tangent space (in general this sort of assignment is called a section of the bundle), and so on.

The special case in which every fibre has dimension 1 is called a line bundle. Consider S^1 with its usual topology. Then there are two line bundles over S_1: the trivial bundle $S^1 \times \mathbb{R}$, which you can easily visualize as a cylinder, and one non-trivial one. This non-trivial vector bundle is called the M obius bundle, and is the most natural mathematical description of a M obius strip. It's probably the simplest concrete example of a non-trivial vector bundle.

Okay, so what's the actual problem? Let $(X, \mathcal{T})$ be a connected topological space, and let $(E, \mathcal{U}, \pi)$ be a vector bundle on $(X, \mathcal{T})$. Show that there is a single non-negative integer n such that

$$\dim\left(\pi^{-1}(x)\right) = n$$

for all $x \in X$ (where we mean vector space dimension there).

19 Compactifications

*1. Prove that local compactness is a topological invariant.

*2. Prove that a locally compact Hausdorff space is regular.

*3. Show that any open or closed subspace of a locally compact Hausdorff space is locally compact.

Hint: For the open part, use the previous exercise.

*4. Show that local compactness is finitely productive.

*5. Generalizing the previous exercise, show that if $\mathcal{X} = \{ (X_\alpha, \mathcal{T}_\alpha) : \alpha \in I \}$ is a collection of topological spaces, then $X = \prod_{\alpha \in I} X_\alpha$ is locally compact if and only if each $(X_\alpha, \mathcal{T}_\alpha)$ is locally compact and all but finitely many of them are compact.

*6. Describe (geometrically, if possible) the one point compactifications of the following topological spaces.

(a) $\mathbb{R}^2 \setminus \{\text{one point}\}$, as a subspace of $\mathbb{R}^2_{\text{usual}}$.

(b) $\mathbb{R}^2 \setminus \{\text{finitely many points}\}$, as a subspace of $\mathbb{R}^2_{\text{usual}}$.

(c) Countably many mutually disjoint open balls in $\mathbb{R}^2_{\text{usual}}$. (Far apart, like balls of radius $\frac{1}{2}$ centred at integer points on the x-axis.)

(d) The set $\mathbb{R}^2$, with its order topology induced by the lexicographical order.

(e) $\mathbb{R}_{\text{discrete}}$.

(f) The space $\mathbb{L}$ given in Example 5.3.4 of the lecture notes on Connectedness.

**7. (These are mostly one-star questions.) Determine whether the following topological spaces are locally compact.

(a) $\mathbb{R}^{\mathbb{N}}_{\text{box}}$

(b) $[0,1]^{\mathbb{N}}$ with its box topology.

(c) $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$

(d) $[0,1]^{\mathbb{N}}$ with its product topology.

(e) $(\mathbb{R}, \mathcal{T}_7)$.

(f) $\mathbb{R}_{\text{ray}}$.

(g) The set of irrational numbers, as a subspace of $\mathbb{R}_{\text{usual}}$.

(h) $\{ (x,y) \in \mathbb{R}^2 : y > 0 \} \cup \{(0,0)\}$, as a subspace of $\mathbb{R}^2_{\text{usual}}$.

(i) $\mathbb{R}_{\text{co-countable}}$

**8. Prove that $\mathbb{R}_{\text{Sorgenfrey}}$ is not locally compact.

Hint: First show that every compact subset of $\mathbb{R}_{\text{Sorgenfrey}}$ is countable, and then the result about local compactness follows easily.

**9. In this exercise you are going to prove Proposition 6.1 from the lecture notes on compactifications, about the Stone-Čech compactification of $(\mathbb{N}, \mathcal{T}_{\text{discrete}})$. Recall that we define $\beta\mathbb{N}$ to be the set of all ultrafilters on $\mathbb{N}$, and give it the topology generated by the basis $\mathcal{B} = \{ \mathcal{B}_A : A \subseteq \mathbb{N} \}$, where

$$\mathcal{B}_A := \{ \mathcal{U} \in \beta\mathbb{N} : A \in U \} .$$

Define $i : \mathbb{N} \to \beta\mathbb{N}$ by $i(n) = \mathcal{U}_n$, where $\mathcal{U}_n = \{ A \subseteq \mathbb{N} : n \in A \}$ is the principal ultrafilter at n. You will show that $\beta\mathbb{N}$ along with the map i forms the Stone-Čech compactification of the naturals.

(a) First, prove the following elementary facts about the basic open sets in $\mathcal{B}$. All of these proofs should be at most one or two lines, and should follow easily from the properties of ultrafilters you already know.

i. $B_\emptyset = \emptyset$.
ii. If $A_1 \subseteq A_2$, then $\mathcal{B}_{A_1} \subseteq \mathcal{B}_{A_2}$.
iii. For all $A_1, A_2 \subseteq \mathbb{N}$, we have $\mathcal{B}_{A_1} \cup \mathcal{B}_{A_2} = \mathcal{B}_{A_1 \cup A_2}$ and $\mathcal{B}_{A_1} \cap \mathcal{B}_{A_2} = \mathcal{B}_{A_1 \cap A_2}$.
iv. For all $A \subseteq \mathbb{N}$, we have $\mathcal{B}_{\mathbb{N} \setminus A} = \beta\mathbb{N} \setminus \mathcal{B}_A$. (In particular, every basic open set is closed.)
v. If $A_1 \neq A_2$, then $\mathcal{B}_{A_1} \neq \mathcal{B}_{A_2}$.

(b) Prove that $\mathcal{B}$ is actually a basis on $\beta\mathbb{N}$.

(c) Having now described the topology on $\beta\mathbb{N}$, prove that this space is actually the Stone-Čech compactification of $\mathbb{N}$. To do this you must show the following:

i. $i : \mathbb{N} \to \beta\mathbb{N}$ is an embedding.
ii. $i(\mathbb{N})$ is a dense subset of $\beta\mathbb{N}$.
iii. $\beta\mathbb{N}$ is a compact Hausdorff space.
iv. $\beta\mathbb{N}$ has the universal property that characterizes the Stone-Čech compactification.

Here are some hints for each part.

i. First show that i is injective. Then show that $i(\mathbb{N})$ is a discrete subspace of $\beta\mathbb{N}$. It follows almost immediately that i is continuous and open.
ii. Show that every nonempty basic open set in $\mathcal{B}$ contains a principal ultrafilter. This is pretty straightforward.

iii. For Hausdorffness: Suppose $\mathcal{U} \neq \mathcal{V} \in \beta\mathbb{N}$. Then there must be some $A \subseteq \mathbb{N}$ such that $A \in \mathcal{U}$ but $A \notin \mathcal{V}$. But then $\mathbb{N} \setminus A \in \mathcal{V}$ since $\mathcal{V}$ is an ultrafilter. Finish from here.

   For compactness, use the characterization that a space is compact if every family of closed sets with the finite intersection property has a nonempty intersection. Note that it suffices to use "basic closed sets", which are complements of basic open sets. By (iv) in part (a), these basic closed sets are themselves basic open sets. So let $\mathcal{C} = \{ \mathcal{B}_{A_\alpha} : \alpha \in I \}$ be some collection of basic clopen sets with the FIP. Show that this implies that $\{ A_\alpha : \alpha \in I \} \subseteq \mathcal{P}(\mathbb{N})$ has the FIP. Use this collection to generate a filter on $\mathbb{N}$, and finish the proof from there.

iv. Fix an embedding $f : \mathbb{N} \to Z$, where Z is a compact Hausdorff space. We want to show that there is a unique continuous function $\beta f : \beta\mathbb{N} \to Z$ such that $f = \beta f \circ i$.

   For $\mathcal{U} \in \beta\mathbb{N}$, recall from the lecture notes on the proof of Tychonoff's theorem that the collection

$$f_*(\mathcal{U}) := \left\{ B \subseteq Z : f^{-1}(B) \in \mathcal{U} \right\}$$

   is an ultrafilter on Z. Since Z is compact and Hausdorff, $f_*(\mathcal{U})$ converges to a unique point, which we will call $z_{\mathcal{U}}$. Define $\beta f : \beta\mathbb{N} \to Z$ by $\beta f(\mathcal{U}) = z_{\mathcal{U}}$. Convince yourself that this map is well-defined, and satisfies the relationship $f = \beta f \circ i$.

   Then prove that βf is continuous. This is a bit tedious, but straightforward. You will need to note at some point that Z is regular, and you will need the definition of filter convergence.