MAT327 - Lecture 1

Wednesday, May 8th, 2019

Office Hours: Wednesday at 4-5PM and Thursday at 1-3PM in HU1018 (This is on the tenth floor, not the first floor.)

In prior discussions, you will have seen what are called the **open intervals** in $\mathbb{R}$, which look like (a, b). You will also have seen closed intervals that look like $[a, b]$.

Recall also that a function $f : \mathbb{R}^n \to \mathbb{R}^k$ is said to be **continuous** at a point $a \in \mathbb{R}^n$ if:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}^n, \|x - a\| < \delta \implies \|f(x) - f(a)\| < \epsilon$$

In the one-dimensional case where we're talking about a function $f : \mathbb{R} \to \mathbb{R}$, this becomes:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

That is, if x is close to a, then $f(x)$ is close to $f(a)$. Alternatively, if $V \subseteq \mathbb{R}$ is an open interval, then $f^{-1}(V)$ is an open interval.

We want to be able to generalize this idea to spaces that don't have a nice notion of distance.

Another question we want to answer in this course, *what does it mean for a subset of $\mathbb{R}^n$ to be "nice"?* This is where we get the notion of **closedness** and **boundedness** in $\mathbb{R}^n$. We will later generalize this idea to what's called **compactness**.

Definition : Topology

Given a set X, a **topology** on X is a specified collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$, called a collection of **open sets**, with the following properties:

1. $\emptyset$ and X are both in $\mathcal{T}$.
2. If $U_1, U_2, \ldots U_n$ are in $\mathcal{T}$, so is their intersection $\bigcap_{i=1}^{n} U_i$. [a]
3. Given some arbitrary subset $\mathcal{T}' \subseteq \mathcal{T}$, the union $\bigcup \mathcal{T}' = \bigcup_{U \in \mathcal{T}} U$ is also in $\mathcal{T}$.

[a] In a lot of the following proofs, we'll just take two elements of $\mathcal{T}$ and show that their intersection is also in $\mathcal{T}$. Since we are always working with a finite number of open sets, we can always apply an easy induction argument to extend this.

A set X together with a topology $\mathcal{T}_X$ on X is called a **topological space,** denoted $(X, \mathcal{T}_X)$. This is similar to how we would think of a vector space as a set V, together with a field F, with two binary operations $+$ and $\cdot$ that satisfy some eight properties we learned in kindergarten. Formally, we would write this vector space as $(V, F, +, \cdot)$.

When talking about topologies on $\mathbb{R}^n$, we might simply write $\mathbb{R}^n_{\text{some topology}}$.

We will often talk about different topologies on the same set X. If we say a set $U \subseteq X$ is open, we should specify which topology on X we're talking about.

Now let's define some topologies. The first one is one we've worked with before, even if we might not have known its name.

Definition : Euclidean Metric on $\mathbb{R}^n$

We define the **Euclidean Metric** on $\mathbb{R}^n$, $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, as:

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

This is the usual notion of "distance" we are used to on $\mathbb{R}^n$. Generalizing the idea of open intervals, we can define what are called **open balls** in $\mathbb{R}^n$.

Definition : Open Balls

Given a point $x \in \mathbb{R}^n$, we define the **open ball of radius** ϵ centered at x as:

$$B_\epsilon(x) = \{y \in \mathbb{R}^n : d(x, y) < \epsilon\}$$

Note that open balls in $\mathbb{R}$ are just open intervals of the form $(x - \epsilon, x + \epsilon)$. This gives us what we need to define the usual topology.

Definition : Usual Topology on $\mathbb{R}^n$

We now define:

$$\mathcal{T}_{\text{usual}} = \{U \subseteq \mathbb{R}^n : \forall x \in U, \exists \epsilon > 0 \text{ such that } B_\epsilon(x) \subseteq U\}$$

That this defines a topology is easy to show. The finite intersections of these sets are again open, and so are their arbitrary unions. However, their infinite intersections are not necessarily open. Consider for example, the following subsets of $\mathbb{R}$:

$$\left\{ \left(-\frac{1}{n}, \frac{1}{n} \right) : n \in \mathbb{N} \right\}$$

The infinite intersection of all of these sets is just $\{0\}$, which is not open. Indeed, there does not exist any $\epsilon > 0$ such that $(0 - \epsilon, 0 + \epsilon) \subseteq \{0\}$.

Example : Open Sets in $\mathbb{R}$

The set
$$(1,3) \cup (5,7) \cup (10,\infty)$$
is open in $\mathbb{R}$. We will later see that every open set in $\mathbb{R}^n$ is a union of open balls.

We will see later that the open balls we defined earlier (and the open intervals in $\mathbb{R}$) form what's called a **basis** for the usual topology. In the same sense that a vector space might have a basis, open balls "generate" the usual topology. We will explore this more next time.

Now we'll explore two extreme examples. Unlike the above topology, we can define the following two topologies on any set X, for reasons that will soon be obvious.

Definition : Discrete Topology

Given a set X, we define the **discrete topology** on X as $\mathcal{T}_{\text{discrete}} = \mathcal{P}(X)$.

That is, every subset of X is open.

That both $\emptyset$ and X are in $\mathcal{T}_{\text{discrete}}$ is immediate. It's also not too much of a stretch that the finite intersections of subsets of X form a subset of X, and the same goes for arbitrary unions. In fact, even the arbitrary intersections of open sets is open in the discrete topology. This isn't always true.

Definition : Indiscrete Topology

We define the **indiscrete topology** on a set X, sometimes called the **trivial topology**, as $\mathcal{T}_{\text{indiscrete}} = \{\emptyset, X\}$.

The discrete topology turns out to be pretty useful for a variety of things, but the indiscrete topology is almost completely useless. The same way 2 ruins the conjecture that all prime numbers are odd, the indiscrete topology ruins a lot of nice theorems that would otherwise be true.

Now we'll define another topology on $\mathbb{R}$. This one's both easy to define and easy to work with, usually we'll only have one or the other.

Definition : Ray Topology on $\mathbb{R}$

We define $\mathcal{T}_{\text{ray}} = \{(a,\infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.

Note that we need to manually add in $\emptyset$ and $\mathbb{R}$ as neither of these can be written as an interval (a,∞).

Property (1) of being a topology is immediate. Property (2) is also quite simple;

given two non-trivial open sets (a,∞) and (b,∞), their intersection is (a,∞) if $a<b$ and (b,∞) if $b<a$. If $a=b$ there's nothing to do.

For arbitrary unions, let $\{U_i : i \in I\}$ be an arbitrary collection of open sets. It shouldn't be too hard to convince yourself that this collection is of the form $\{(i,\infty) : i \in I\}$ where $I \subseteq \mathbb{R}$. (Where we're assuming for simplicity that neither $\mathbb{R}$ nor $\emptyset$ appear in this collection.) This is because we have at most one unique open set for every real number.

Let $U = \bigcup_{i\in I}(i,\infty)$. If I is unbounded below we simply have that $U = \mathbb{R}$. Otherwise, let $j = \inf\{i \in I\}$ for which we claim that $U = (j,\infty)$.

($\subseteq$) Let $x \in U$. Then $x \in U_i = (i,\infty)$ for some $i \in I$. Since $j \leq i$ for all $i \in I$, we have that $x \in (j,\infty)$.

($\supseteq$) Now let $x \in (j,\infty)$. Let $\epsilon = x - j$. By definition of the infimum, there exists some $i \in I$ such that:

$$j \leq i < j + \epsilon = j + (x-j) = x$$

implying that $j \leq i < x$, so $x \in (i,\infty) = U_i$.

We conclude that $\mathcal{T}_{\text{ray}}$ is closed under arbitrary unions, and hence a topology on $\mathbb{R}$. ■

Now we'll define two more topologies.

Definition : Co-finite Topology

Given a set X, we define the **co-finite topology on** X as:

$$\{U \subseteq X : X \setminus U \text{ is finite}\} \cup \{\emptyset\}$$

Again, we needed to add in $\emptyset$ manually, as this set is not in the set by default. Particularly, if X is infinite. Picturing the open sets in this topology isn't as easy as the previous ones, but the way I've found works best is to think of it as "all but finitely many points of X". This isn't too bad to picture in $\mathbb{R}$, for example.

To prove this is a topology, property (1) is again immediate. For finite intersection, suppose that U and V are such that both $X \setminus U$ and $X \setminus V$ are finite. We need to show that $U \cap V$ is open. If this intersection is empty we are done, otherwise we need to show that $X \setminus (U \cap V)$ is also finite. Note that:

$$X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$$

Which is just a union of two finite sets, hence finite.

For arbitrary unions. Suppose $\{U_i : i \in I\}$ is a collection of open sets with indexing set I. Then:

$$X \setminus \left(\bigcup_{i\in I} U_i \right) = \bigcap_{i \in I} X \setminus U_i$$

This intersection is a subset of $X \setminus U_i$ for each $i \in I$. But each $X \setminus U_i$ is finite, so we are done. A subset of a finite set is definitely finite. ■

Along with this last one, we define the **co-countable topology**.

Definition : Co-countable Topology

We define the **co-countable topology** on a set X as:

$$\{U \subseteq X : X \setminus U \text{ is countable}\} \cup \{\emptyset\}$$

Showing that this is a topology is essentially the same proof we just did.