MAT327 - Lecture 4

Friday, May 17th, 2019

Recall the following definition..

Definition : Closures

Let $(X, \mathcal{T}_X)$ be a topological space, and let $A \subseteq X$. The closure of A in $(X, \mathcal{T}_X)$ is the set:

$$\overline{A} = \{x \in X : \forall \text{ open sets } U \text{ containing } x, A \cap U \neq \emptyset\}$$

Intuitively, $\overline{A}$ is the set of all points that are "close" to A (with respect to the "structure" of $\mathcal{T}_X$).

Recall also that we discussed that a set $C \subseteq X$ is closed if and only if $\overline{C} = C$ if and only if $X \setminus C$ is open.

Finally, recall the definition of density we gave.

Definition :

A set $D \subseteq X$ is said to **dense** if $\overline{D} = X$

That is, a dense set is close to the whole set. We can prove the following equivalent definition, which is a lot nicer to do proofs with:

Theorem :

Let $(X, \mathcal{T}_X)$ be a topological space, a subset $D \subseteq X$ is dense if and only if for all $U \in \mathcal{T}_X \setminus \{\emptyset\}$, $D \cap U \neq \emptyset$.

Proof. Suppose that $D \subseteq X$ is dense. Then for all $x \in X$ and open U containing x, we have that $U \cap D \neq \emptyset$.

Given some non-empty set U, this set contains a point $x \in X$, and is therefore an open neighbourhood[1] of that point, so $U \cap D \neq \emptyset$.

Conversely, suppose that for every non-empty open set U, that $U \cap D \neq \emptyset$. Pick some $x \in X$, we want to show that $x \in \overline{D}$.

If U is an open set containing x, then U is non-empty, so $U \cap D \neq \emptyset$. ■

This proof just feels wrong at every level, but it's one of those ones you have to be careful not to overthink.

[1] Don't know if I've used this word before in these notes, but an **open neighbourhood** of a point x is just an open set containing x.

Example : Dense Sets

Let $(X, \mathcal{T}_X)$ be a topological space.

1. X is dense.
2. If D is dense and $A \supseteq D$, then A is dense.
3. in $\mathbb{R}_{\text{usual}}$, $\mathbb{Q}$ is dense, and $\mathbb{R} \setminus \mathbb{Q}$ is dense. You will have seen this fact in a course like MAT157, yet I still find it very surprising. It feels like a dense set is almost as big as $\mathbb{R}$, and it feels like a countable set is as small as $\mathbb{N}$.
4. in $\mathbb{R}_{\text{Sorgenfrey}}$, $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are also dense. (This is true since every basic open set is of the form $[a, b)$, therefore containing the open interval (a, b), and we know the open interval contains both a rational and irrational number.)
5. In $\mathbb{R}_{\text{ray}}$, $\mathbb{N}$ is dense. Every open set (a, ∞) contains a rational number; infinitely many, infact. More generally, any set that is unbounded above is dense.
6. In X_{discrete}, the only dense set is X.
7. In $X_{\text{indiscrete}}$, every nonempty set is dense.
8. In X_p, $\{p\}$ is dense, but no other singleton is.
9. in $X_{\text{co-finite}}$, any infinite set is dense. Similarly, in $X_{\text{co-countable}}$, any uncountable set is dense.

Countability

The question we want to answer now is, *how do we enumerate a set?* The natural thing to do would be to count the set, but this falls apart when we get to infinite sets. The solution: bijections.

Definition :

Given two sets A, B, we say that A **has the same cardinality as** B if there exists a bijection $f : A \to B$. We denote this $|A| = |B|$.

If there is an injection $g : A \to B$, we say that $|A| \leq |B|$.

If there is a surjection $g : A \to B$, we say that $|A| \geq |B|$.

The notation of $|A|$ is just the number of elements of A if A is finite.

Theorem : Cantor-Shroder-Bernstein Theorem

If A and B are sets and $|A| \leq |B|$, and $|A| \geq |B|$, then $|A| = |B|$.

The proof of this theorem is omitted, but it involves constructing a bijection from the injection and surjection.

At first glance, this theorem may look obvious. But note that this theorem generalizes to infinite sets. In the finite case, we say that $|A| = n$ if there is a bijection $f : \{1, 2, 3 \ldots n\} \to A$.

Now we examine the smallest type of infinite sets, called **countable sets.**

Definition : Countability

A set A is said to be **countably infinite if** $|A| = |\mathbb{N}|$

Example :

1. The set $E = \{2n : n \in \mathbb{N}\}$ of all even numbers is countable. Consider the bijection $f : E \to \mathbb{N}; f(n) = n/2$.
2. The set $O = \{2n - 1 : n \in \mathbb{N}\}$ of all odd numbers is countable. Consider the bijection $f : O \to \mathbb{N}; f(n) = (n + 1)/2$.

Definition : Countably

A set A is said to be **countable** if it is finite or countably infinite. Equivalently, if there exists an injection $f : A \to \mathbb{N}$.

Theorem :

Let A be countable, and let B be finite. Then $A \cup B$ and $A \setminus B$ are countable.

The proof of this fact is given in Ivan's notes. It involves doing the obvious thing.

Theorem :

If a set A is countably infinite, and $B \subseteq A$ is infinite, then B is countable. More consisely, any subset of a countable set is countable.

Proof. (Sketch) Take the following listing of A:

$\mathbb{N}$	A
1	a_1
2	a_2
3	a_3
.	.
.	.

Remove all the a_i's that are in B, and 'push' everything up. This function remains a surjection, at the very least. ■

Now, a similar, but stronger result.

Theorem :

If A and B are countable, then $A \cup B$ is countable.

Proof. By the theorem before the last one, we've already "proven" this in the case where one set is infinite and one is finite. In the case where they're both finite, the proof is immediate. So it suffices to only prove this for when both A and B are countably infinite.

We will look at $A \cup B$ as $A \cup (B \setminus A)$. In the case where $B \setminus A$ is finite, we are done, as the theorem before the last takes care of this case. Otherwise, the last theorem implies that $B \setminus A$ is countably infinite.

That is, there exists bijections $f_1 : \mathbb{N} \to A$ $f_2 : \mathbb{N} \to B \setminus A$. Define the function:

$$f(n) = \begin{cases} f_1(n) & \text{if} \quad n \text{ is even} \\ f_2(n) & \text{if} \quad n \text{ is odd} \end{cases}$$

Which is also a bijection. ■

Proposition 4.11 in Ivan's notes is an important exercise to look at. The proofs aren't very hard to do, but they do give some idea of how countability works.