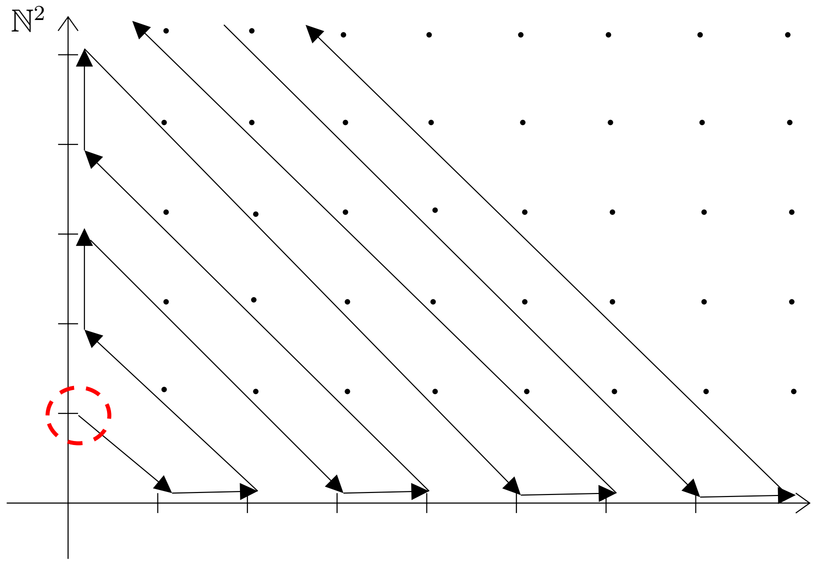
MAT327 - Lecture 5

Wednesday, May 22nd, 2019

Theorem :

$\mathbb{N} \times \mathbb{N}$ is countable.

Proof. The idea of the proof is sketched by the following figure.



More explicitly, the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(m, n) = 2^n 3^m$ is an injection by uniqueness of prime factorizations.

Corollary 1: If A, B are countable, so is $A \times B$.

Corollary 2: Any finite Cartesian produce of countable sets is countable (By induction).

Corollary 3: $\mathbb{Q}$ is countable.

Define the mapping:

$$f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}; f\left(\frac{p}{q}\right) = (p, q)$$

Where it is understood that p/q is in lowest terms before we plug it into the function. This mapping is an injection.

Corollary 4: Countable unions of countable sets are countable.

Assume without loss of generality that the sets $\{A_n : n \in \mathbb{N}\}$ are disjoint, and assign each of them to one column of $\mathbb{N}$ of the form (c, n), letting n vary through $\mathbb{N}$.

This last proof requires AOC.

Definition : Uncountable

A set A is **uncountable** if it is not countable.

Wow!

Theorem : Cantor's Theorem

$(0,1) \subseteq \mathbb{R}$ is uncountable.

Proof. Suppose not. Certainly $(0,1)$ is not finite, so fix a bijection $f : \mathbb{N} \to (0,1)$. This function has the following form:

1	$0.a_{11}a_{12}a_{13}a_{14}\ldots$
2	$0.a_{21}a_{22}a_{23}a_{24}\ldots$
3	$0.a_{31}a_{32}a_{33}a_{34}\ldots$
4	$0.a_{41}a_{42}a_{43}a_{44}\ldots$
.	.
.	.

Now for each $n \in \mathbb{N}$, let a_n be a digit other than $f(n)_n$, where the subscript denotes the nth digit of this real number.

By construction, $a_k = 0.a_1a_2a_3a_4\ldots$ differs from $f(k)$ in exactly one digit. In particular, it is not equal to any of the real numbers on this list. We've constructed a real number that wasn't on the list.

There is a subtle problem. It doesn't matter unless we're trying to be extremely rigorous, but we might have two representations of the same number. Recall that $0.1 = 0.999999\ldots$

This problem is remedied by defining a_n in the following way:

$$a_k = \begin{cases} 1 \text{ if } x_{kk} \neq 1 \\ 2 \text{ if } x_{kk} = 1 \end{cases}$$

This avoids the problem of repeating 9's occurring. ■

Remark: This is a famous proof called Cantor's Diagonalization Argument. We will be doing more proofs like this.

Now here's Ivan's favourite proof of all time.

Theorem :

Let A be a set. There is no surjection $f : A \to \mathcal{P}(A)$.

Proof. Suppose that such a function $f : A \to \mathcal{P}(A)$ exists. We will show that f is not surjective.

Define the set:

$$D = \{x \in A : x \notin f(x)\}$$

Don't get confused here by "$x \notin f(x)$", remember we are mapping to $\mathcal{P}(A)$, so $f(x)$ is a subset of A. x is in D if and only if it happens to be a member of the set that it maps to.

Note that $D \in \mathcal{P}(A)$, so suppose for the sake of contradiction assume that f is surjective. Here, we could also assume the weaker condition that there exists some $a \in A$ such that $f(a) = D$.

Now there's two cases to consider. In the first case where $a \in D$, then $a \notin f(a)$, but $f(a) = D$, so $a \notin D$, which is a contradiction.

Otherwise, $a \notin D$, but $D = f(a)$, which implies that $a \in D$. The ol' switcheroo. ■

*"Suppose D is the set of all people in MAT327. If I ask one of them not to come to class, they're not in the set anymore. *laughs*"* ∼ **Ivan**

The cleverness here was only in defining D in such a way that it breaks our assumption.

So now we have that

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| \dots$$

In particular, there are infinitely many infinities, and no largest infinity.

Exercise: (Big List) Prove that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

Now, does there exist a set A such that $|\mathbb{N}| < |A| < |\mathbb{R}|$? There does, but the question of if there exists some B such that $|\mathbb{R}| < |B| < |P(\mathbb{R})|$ is undecidable.

We denote the cardinality of $\mathbb{N}$ as $\aleph_0$. We denote the cardinality of $\mathbb{R}$ as $\mathfrak{C}$.

Now, some definitions.

Definition : Separable

A topological space $(X, \mathcal{T})$ is called **separable** if it has a countable dense subset.

Definition : Second Countable

A topological space $(X, \mathcal{T})$ is called **second countable** if X has a countable basis $\mathcal{B}$ which generates $\mathcal{T}$.

Intuitively, a space is second countable if it can be specified with a countable amount of information. This definition does not say that every basis is countable, only that one exists.

Example : Second-Countable

Recall we found that the set $\mathcal{B}_{\mathbb{Q}}$, defined as:

$$\mathcal{B}_{\mathbb{Q}} = \{(a,b) : a < b \in \mathbb{Q}\},$$

forms a basis for the usual topology on $\mathbb{R}$. It's not too much of a stretch to show that this set is countable. We can define an obvious bijection between it and $\mathbb{Q} \times \mathbb{Q}$, which is countable since it is a cartesian product of two countable sets, by corollary 1.
Hence, $\mathbb{R}_{\text{usual}}$ is second-countable.

Example : Separable sets

$\mathbb{R}$ with its usual, Sorgenfrey, particular point, and ray topologies are all separable.

The following definition is one we use to talk about how much 'room' is in a space. We'll see it come up a lot in the next few lectures.

Definition :

A topological space $(X, \mathcal{T})$ has the **countable chain condition** (CCC) if there are no uncountable collections of pairwise disjoint open sets.

Exercise: Show that $\mathbb{R}_{\text{usual}}$ and $\mathbb{R}_{\text{Sorgenfrey}}$ are ccc.

Exercise: Show that X_{discrete} is ccc if and only if X is countable.