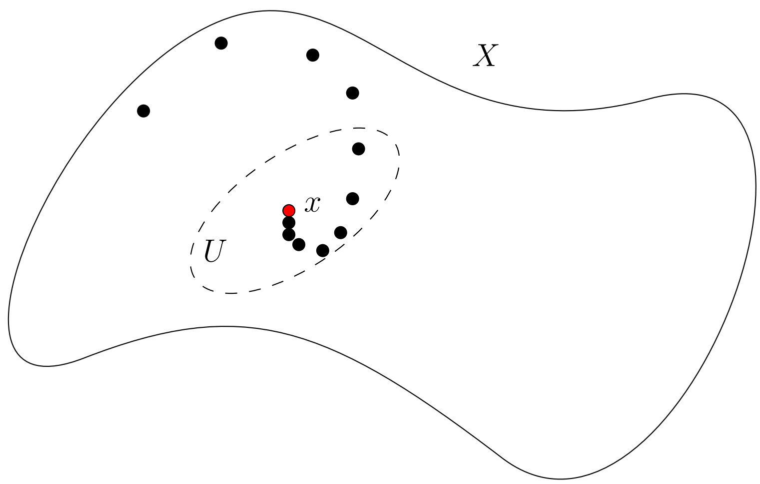
MAT327 - Lecture 6

Friday, May 24th, 2019

Definition :

Let $(X, \mathcal{T})$ be a topological space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to **converge** to a point $x \in X$ if for all open sets U containing x, there exists some $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in U$.



We write this as $x_n \to x$.

Now, some examples...

Example :

1. $\dfrac{1}{n} \to 0$ in $\mathbb{R}_{\text{usual}}$ and $\mathbb{R}_{\text{Sorgenfrey}}$
2. $-\dfrac{1}{n} \to 0$ in $\mathbb{R}_{\text{usual}}$ in $\mathbb{R}_{\text{usual}}$ but not in $\mathbb{R}_{\text{Sorgenfrey}}$.
3. In every topological space, the constant sequence $x, x, x, \ldots$ converges to x.
4. In X_{discrete}, a sequence converges to a point $x \in X$ if and only if it is eventually constantly equal to x. This is a consequence of the fact that x is open, and any open set containing x must be a superset of x.
5. In $(X, \mathcal{T}_p)$, eventually constant sequences still converge to the constant value. For any $x \neq p$, any sequence that is eventually only p or x, like $p, x, p, x \ldots$, converges to x (but not p, be careful here).
6. In $(X, \mathcal{T}_p)$, the constant sequence $p, p, p, \ldots$ converges to everything.
7. Taking it a step further, in $(X, \mathcal{T}_{\text{indiscrete}})$, every sequence converges to every point. Given a point, there is exactly one open set containing it, which is the entire space.

These last two examples are particularly bad. This is the first time we've seen that limits of sequences need not be unique. The natural question arises, what conditions can we put on a space to ensure that sequences in that space only converge to one point?

Example :

1. In $\mathbb{R}_{\text{ray}}$, $1/n$ and $-1/n$ both converge to 0. They also both converge to anything less than 0.
2. In $\mathbb{R}_{\text{co-finite}}$, the sequence $0, 1, 0, 1, \ldots$ does not converge as its image is finite.
3. in $\mathbb{R}_{\text{co-countable}}$, eventually constant sequences do converge to (only) their constant values. No other sequences converge. The reason for this is that for any non-constant sequence $\{x_n\}_{n\in\mathbb{N}}$, and for any potential limit point x, we can construct the open set:

   $$(\mathbb{R} \setminus \{x_n : n \in \mathbb{N}\}) \cup \{x\}$$

   Which contains x, and doesn't contain a tail of the sequence.

In that last example, notice how the convergent sequences in $\mathbb{R}_{\text{co-countable}}$ and $\mathbb{R}_{\text{discrete}}$ are exactly the same.

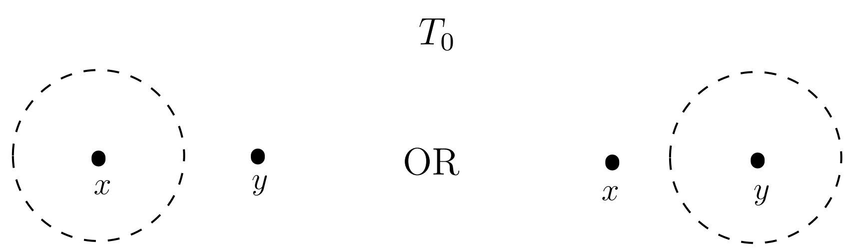
As Ivan mentions in his notes on Nets and Filters, there is no way to determine that these two spaces are any different if we're only analyzing sequence convergence in these spaces. See these notes for how we can fix this.

We now define some properties that will help us diagnose how good a space behaves with respect to sequence convergence.

Definition : T_0

A topological space $(X, \mathcal{T})$ is said to be T_0 if for every pair of distinct points $x_1, x_2 \in X$, there is an open set that contains one but not the other.

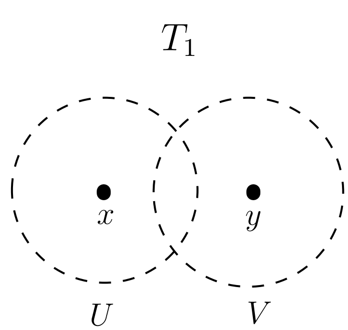
Note that this is not a symmetric property. It's also not a very powerful property. All of the spaces we've seen so far are T_0 except the indiscrete topology.



More importantly,

Definition : T_1

$(X, \mathcal{T})$ is said to be T_1 if for every $x \neq y \in X$, there exists U and V open such that $x \in U$, $y \notin U$, $y \in V$, and $x \notin V$.



This is just T_0 but made symmetric. That is, U and V must both exist with the desired properties. In T_0, only one of them had to exist. This definition becomes useful for (constant) sequence convergence.

Example : T_0

$\mathbb{R}_{\text{ray}}$ and $\mathbb{R}_p$ are both T_0 but not T_1.

Theorem :

In a topological space $(X, \mathcal{T})$, the following are equivalent:

1. $(X, \mathcal{T})$ is T_1
2. For every $x \in X$, the set $\{x\}$ is closed. This last one is often given as the first definition of T_1.
3. Every finite set $F \subseteq X$ is closed.
4. For every $S \subseteq X$, $S = \bigcap\{U \in \mathcal{T} : S \subseteq U\}$
5. Constant sequences converge (only) to their constant values.

The proof of all of these isn't too bad. For each of them, try to think about why the proof fails if we only have a T_0 space.

For some of these proofs, like $(1 \Rightarrow 5)$, it may help you understand the T_1 property a bit better if you think about what step of the proof fails if the space is only T_0. For me, this helps me understand it better than a simple counterexample.

Example : T_1

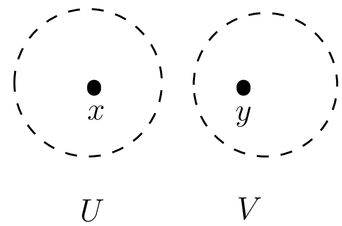
$\mathbb{R}_{\text{co-countable}}$ is T_1. But sequences are still weird. For example, $1, 2, 3, 4, \dots$ still converges to everything.

We've fixed the constant sequences, but we still have a problem. As most sequences are not constant, We still need something stronger to make sequences behave nicely.

Definition : Hausdorff

We say that a topological space $(X, \mathcal{T})$ is **Hausdorff** (or T_2) if for all $x \neq y \in X$ there exist disjoint open sets U, V such that $x \in U$ and $y \in V$.

T_2



A Hausdorff space ensures that the sets U and V are disjoint, whereas in T_1 spaces there might have been overlap.

The reason we're being told all this now, is because now we can state the following theorem:

> **Theorem :**
>
> In a Hausdorff space, every sequence converges to at most one point.

So being Hausdorff is the weakest property we can put on a space so that limits of sequences are unique. It's another name we can give to a property that was true for $\mathbb{R}^n$ and that we took for granted.

In doing this, we strip away another layer of $\mathbb{R}^n$. We have the ability to say "Ah, that's why limits are unique."

The natural question here is if the converse is true. It is not, as demonstrated by this annoying counterexample.

In $\mathbb{R}_{\text{co-countable}}$, every two (non-empty) open sets intersect a lot, the space is T_1 but not Hausdorff, yet limits are unique.

Next lecture, what additional condition can we put on the space to ensure that the converse does hold?