MAT327 - Lecture 7

Wednesday, May 29th, 2019

Recall that a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to a point $x \in X$ if for all open sets U containing x, there exists an $N \in \mathbb{N}$ such that for all $n > N, x_n \in U$.

Recall also that:

$$T_2(\text{Hausdorff}) \Rightarrow T_1 \Rightarrow T_0$$

Exercise: Prove that $\mathbb{R}_{\text{co-finite}}$ is T_1 but not T_2. Also prove that $\mathbb{R}_{\text{ray}}$ is T_0 but not T_1.

Recall also that theorem that in a Hausdorff space, every sequence converges to at most one point. The converse was not true. In $\mathbb{R}_{\text{co-countable}}$, only the eventually constant sequences converge. Since this space is T_1, we know that the limits of all convergent sequences are unique. But the space is not Hausdorff.

The main question of this lecture, *how do we fix this?* There are two ways. One is to use something stronger than sequences, which Ivan talks about in his notes on Nets and Filters. The other way to fix this is to impose some additional condition on our space that magically makes sequences powerful enough to reverse this implication.

We'll explore the latter one in this lecture.

Correction from last class: We said that in $\mathbb{R}_{\text{co-finite}}$, if a sequence has an infinite range, it converges to all points. this is wrong by the counterexample:

$$a_n = 0, 1, 0, 2, 0, 3, 0, 4, \ldots$$

This converges to 0 only. (check this.)

Theorem :

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in A. If $a_n \to a$ for some $a \in X$, then $a \in \overline{A}$.

Proof. Every open set U around a contains points of a_n, but $a_n \in A$ for each $n \in \mathbb{N}$, so in particular $U \cap A \neq \emptyset$. ■

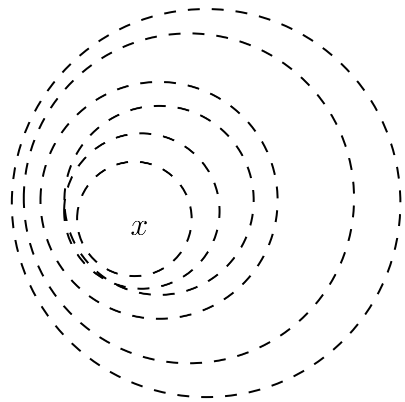
The converse is again false.

So sequences do not characterize every property of topological spaces. The direction we'll go now is to find out what property of topological spaces is stopping us, and isolate it.

Definition : Local Basis

Let $(X, \mathcal{T})$ be a topological space and let $x \in X$. A local basis at x is a collection of sets $\mathcal{B}_x \subseteq \mathcal{T}$ with the following properties:

1. $x \in B$ for all $B \in \mathcal{B}_x$
2. For all open sets U containing x, there exists some $B \in \mathcal{B}_x$ such that $x \in B \subseteq U$.



Example : Local Bases

Fix a point $x \in \mathbb{R}_{\text{usual}}$,

1. then the set $\{(a, b) : a < b \in \mathbb{R}\}$ is a local basis for x.
2. The set $\{(a, b) : a < b \in \mathbb{Q}\}$ is a local basis at x.
3. The set: $\left\{\left(x - \frac{1}{n}, x + \frac{1}{n}\right) : n \in \mathbb{N}\right\}$ is a local basis at x.
4. In $\mathbb{R}_{\text{Sorgenfrey}}$, the set:
$$\{[x, x + 1/n) : n \in \mathbb{N}\}$$
is a local basis at x.
5. In X_{discrete}, $\mathcal{B}_x = \{\{x\}\}$ is a local basis at x.
6. In X_p, the set $\{\{p\}\}$ is a local basis at p, and the set $\{\{x, p\}\}$ is local basis at a point x for any $x \neq p$.

Definition : First Countable

A topological space $(X, \mathcal{T})$ is said to be **first-countable** if every point has a countable local basis.

Example : First Countable

$\mathbb{R}^n_{\text{usual}}$, $\mathbb{R}_{\text{Sorgenfrey}}$, X_{discrete}, X_p, are all first countable.
$\mathbb{R}_{\text{co-finite}}$, $\mathbb{R}_{\text{co-countable}}$ are not first countable.

Note that every basis for a topological space admits a local basis around every point in the space.

That is, given a second-countable space $(X, \mathcal{T}_X)$ witnessed by a countable basis $\mathcal{B}$, given some $x \in X$ we can define:

$$\mathcal{B}_x = \{B \in \mathcal{B} : B \ni x\}$$

This shows that second countability implies first countability. An example of a space that is first countable but not second countable is $\mathbb{R}_{\text{Sorgenfrey}}$.

Think about what condition each basis for $\mathbb{R}_{\text{Sorgenfrey}}$ needs to satisfy. This condition looks something like "For every real number $x \ldots$" Think about why this condition prevents any basis from being countable.

This is yet another example of $\mathbb{R}_{\text{Sorgenfrey}}$ and $\mathbb{R}_{\text{usual}}$ being worlds different despite looking so similar.

Theorem :

If $(X, \mathcal{T})$ is first countable, then every point has a countable nested local basis. That is, $\mathcal{B}_x = \{B_n\} : n \in \mathbb{N}$ is a local basis and $B_n \supseteq B_{n+1}$ for each $n \in \mathbb{N}$.

This theorem is used to prove the next two. The idea is that for each $n \in \mathbb{N}$, take the basis element B_n in our original basis, and let $B'_n = \bigcap_{i=1}^{n} B_n$.

Now, the payoffs:

Theorem :

Let $(X, \mathcal{T})$ be a first countable topological space and let $A \subseteq X$, and let $a \in A$. Then $a \in \overline{A}$ is and only if there is a sequence in A converging to a.

Proof. ($\Leftarrow$) Suppose that $a_n \to a$ for some sequence a_n, where $a_n \in A$ for each $n \in \mathbb{N}$. Let U be an open set containing a, for which U contains a tail of the sequence. But this tail of the sequence is in A, so $U \cap A \neq \emptyset$.

($\Rightarrow$) Let $a \in \overline{A}$. Fix a countable, nested, local basis for a, call it $\mathcal{B}_a$. As $a \in \overline{A}$ and each $B_{a_n} \in \mathcal{B}_a$ is open, we have that $B_{a_n} \cap A \neq \emptyset$ for all $n \in \mathbb{N}$. So for each $n \in \mathbb{N}$, take a_n to be in this intersection.

Let U be any open set containing a, for which there exists some $N \in \mathbb{N}$ such that $B_{a_n} \subseteq U$ for all $n > N$ (since the basis is nested).

But then $x_n \in U$ for all $n > N$, so U contains a tail of the sequence. Therefore the sequence converges to a. ∎

Theorem :

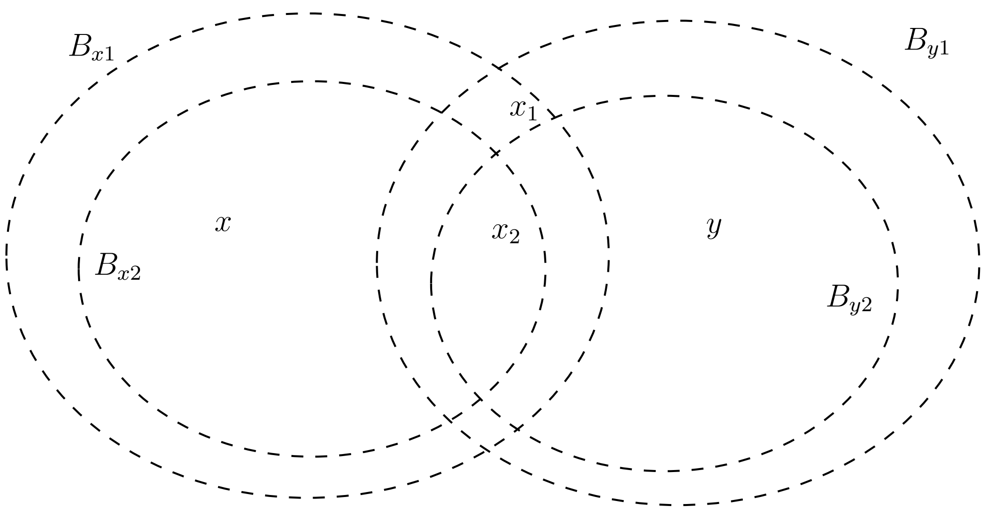
Let $(X, \mathcal{T})$ be a first countable topological space such that every sequence converges to at most one point $a \in X$. Then $(X, \mathcal{T})$ is Hausdorff.

Proof. We will proceed by contrapositive. Suppose that $(X, \mathcal{T})$ is first countable and not Hausdorff. Then there exists some $x \neq y \in X$ such that for all open U and V containing x and y respectively, we have that $U \cap V \neq \emptyset$.

By the fact that $(X, \mathcal{T})$ is first countable there exists a countable nested local basis at both x and y, call them

$\mathcal{B}_x = \{B_n : n \in \mathbb{N}\}$ and $\mathcal{B}_y = \{B_m : m \in \mathbb{N}\}$.

Construct the sequence by picking $x_1 \in B_{x1} \cap B_{y1}$, pick $x_2 \in B_{x2} \cap B_{y2}$, and continuing in this manner. None of these intersections are empty by assumption, so our sequence is well-defined. This sequence converges to both x and y, which is impossible.



First countability allows us to reverse both of these theorems we mentioned earlier.

Continuous Functions

We have topological spaces, and we have topological properties. We are now interested in investigating how these properties change between spaces.

Definition : Continuous Functions

Let $(X, \mathcal{T}_x)$, $(Y, \mathcal{T}_Y)$ be topological spaces and let $f : X \to Y$ be a function. We say that f is **continuous** if $f^{-1}(U) \in \mathcal{T}_X$ for all $U \in \mathcal{T}_Y$.

More concisely, we say that f is continuous if the preimage of open sets is open.

Example : Continuous Functions

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous in the $\epsilon - \delta$ sense. Then $f : \mathbb{R}_{\text{usual}} \to \mathbb{R}_{\text{usual}}$ is continuous in the topological sense.
2. Fix some $a \in \mathbb{R}^n$. Let $f : \mathbb{R}^n_{\text{usual}} \to \mathbb{R}_{\text{usual}}$ be given by $f(x) = ||x - a||$. Then f is continuous.
3. The first projection function $\pi_1 : \mathbb{R}^2_{\text{usual}} \to \mathbb{R}_{\text{usual}}$ given by $\pi_1(x, y) = x$ is continuous.
4. Any function whose domain is a discrete space is continuous. This is a simple consequence of the fact that the preimage of any set whatsoever is open, so the preimage of open sets is definitely open.
5. Similarly, any function whose co-domain is indiscrete is continuous.
6. Any constant function is continuous.
7. The compositions of continuous functions are continuous.
8. For any set X, the identity function $I : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous if and only if $\mathcal{T}_1$ refines $\mathcal{T}_2$.

Proof. (1) Fix some $U \in \mathbb{R}_{\text{usual}}$, we want to show that $f^{-1}(U)$ is open in $\mathbb{R}_{\text{usual}}$.

Pick some $x \in f^{-1}(U)$. Then $f(x) \in U$. Then there exists some $\epsilon > 0$ such that $(f(x) - \epsilon, f(x) + \epsilon) \subseteq U$. But by definition there exists a $\delta > 0$ such that if $y \in (x - \delta, x + \delta)$, then $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$. But this means that $(x - \delta, x + \delta) \subseteq f^{-1}(U)$. □

Theorem :

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $\mathcal{B}$ be a basis on Y that genereates $\mathcal{T}_Y$. Let $\mathcal{S}$ be a sub-basis on Y that also generates $\mathcal{T}_Y$.
Let $f : X \to Y$, then the following are equivalent:

1. f is continuous.
2. For every $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{T}_X$.
3. For every $S \in \mathcal{S}$, $f^{-1}(S) \in \mathcal{T}_X$.

The proofs here follow from the fact that preimages play nicely with unions and intersections. Here we can prove a function mapping to $\mathbb{R}_{\text{usual}}$ is continuous just by proving it for open intervals, or even for rays (which form a sub-basis).

Definition : Continuity at a Point

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f : X \to Y$ be a function and let $x \in X$. We say that f is continuous at x if for all open sets $V \in \mathcal{T}_Y$ containing $f(x)$, there exists an open set $U \in \mathcal{T}_X$ containing x such that $f(U) \subseteq V$.

Theorem : Equivalent Definitions of Continuity

The following are equivalent:

1. f is continuous.
2. The preimages of closed sets are closed.
3. f is continuous at x for all $x \in X$.
4. For all $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

Proving that the first 3 are equivalent is very easy. I'll do the proof that $1 \Rightarrow 4$.

Proof. ($\Rightarrow$) Suppose that f is continuous. Pick some $y \in f(\overline{A})$. Then $y = f(x)$ for some $x \in \overline{A}$.

Let $V \subseteq Y$ be an open set containing y. Then $f^{-1}(V)$ is an open set containing x, and hence $f^{-1}(V) \cap A \neq \emptyset$. Therefore,

$$f(f^{-1}(V) \cap A) \subseteq f(f^{-1}(V)) \cap f(A) \subseteq V \cap f(A)$$

The leftmost set in this inequality is non-empty by assumption, so $V \cap f(A) \neq \emptyset$.

Therefore, $y \in \overline{f(A)}$. ■

After this lecture, I asked Ivan about sequential continuity, a concept covered in MAT237 and MAT257. The theorem goes:

Theorem :

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if for all $x \in \mathbb{R}^n$, whenever $x_n \to x$, then $f(x_n) \to f(x)$.

My question was if this generalizes to all functions $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ between topological spaces. The $\Rightarrow$ direction still holds, but again, sequences are not strong enough for the converse to hold.

This is fixed, again, by making $(X, \mathcal{T}_X)$ first countable. This is explored in big list 6.9.