MAT327 - Lecture 8

Friday, May 31st, 2019

Last time on Dragonball Z, we defined continuous functions. Continuous functions took preimages of open sets in the co-domain to open sets in the domain.

The following class of functions does the opposite, it takes an open set in our domain and maps it to an open set in the co-domain.

Definition : Open and Closed Functions

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. We say that f is **open** if for all $U \in \mathcal{T}_X$, we have that $f(U) \in \mathcal{T}_Y$. We say that f is **closed** if for all closed $C \subseteq X$, we have that $f(C)$ is closed in Y.

Ivan offers a good explanation for why continuous functions are defined the way they are in his notes, despite open functions seeming like the more natural definition.

Recall that "the preimage of open sets is open" was equivalent to saying "the preimage of closed sets is closed". Either of these can be used to show a function is continuous. However, a function being closed and open is not equivalent. The reason for this, basically, is that preimages are better behaved than images.

The complement of a preimage is equal to the preimage of the complement, but the complement of an image is only a subset of the image of the complement.

That closed and open are not equivalent is demonstrated by the following examples:

Example : Open and Closed Functions

1. $f : \mathbb{R}_{\text{usual}} \to \mathbb{R}_{\text{usual}}$ given by $f(x) = 7$ is closed but not open.
2. $\pi_1 : \mathbb{R}^2_{\text{usual}} \to \mathbb{R}_{\text{usual}}$ given by $\pi_1(x, y) = x$ is open but not closed.
3. Any function with a discrete codomain is open and closed.
4. The absolute value function $|\cdot| : \mathbb{R}_{\text{Sorg}} \to \mathbb{R}_{\text{Sorg}}$ is neither open nor closed. To see this, note that $|\cdot|([-11, -7)) = (7, 11]$, and that $|\cdot|((-\infty, 0)) = (0, \infty)$.

   The first example shows that the image of an open set need not be open, and the second example shows that the image of a closed set need not be closed.

Theorem :

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. An injection $f : X \to Y$ is open if and only if it is closed.

Definition : Homeomorphisms

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. We say that $f : X \to Y$ is a **homeomorphism** if f is bijective and continuous, and f^{-1} is continuous. When such a function exists, we say that X and Y are **homeomorphic** and we write $(X, \mathcal{T}_X) \cong (Y, \mathcal{T}_Y)$.

Homeomorphisms are the biggest payoff of this section. These are the functions that preserve topological structure. Not only are the points in X and Y in 1-1 correspondence, so too are the open sets in the topologies on X and Y.

We have the following equivalent definitions of homeomorphism:

Theorem :

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and suppose $f : X \to Y$ is a bijection. The following are equivalent.

1. f is a homeomorphism
2. f is both continuous and open.
3. f is continuous and closed.
4. $U \subseteq X$ is open if and only if $f(U)$ is open. That is, $\mathcal{T}_Y = \{f(U) : U \in \mathcal{T}_X\}$

Example : Homeomorphisms

1. If $m \neq 0$, then $f(x) = mx + b$ is a homeomorphism between $\mathbb{R}_{\text{usual}}$ and itself.
2. Let $p \neq q \in X$. Then $f : X_p \to X_q$ is a homeomorphism if and only if it is a bijection and $f(p) = q$.
3. A function $f : X_{\text{discrete}} \to Y_{\text{discrete}}$ is a homeomorphism if and only if it is a bijection.
4. $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)_{\text{usual}} \to \mathbb{R}_{\text{usual}}$ is a homeomorphism.
5. $\mathbb{R}_{\text{usual}}$ is never homeomorphic to $\mathbb{Q}$ under any topology. There does not exist a bijection from $\mathbb{R}$ to $\mathbb{Q}$, let alone a homeomorphism.

We immediately see that homeomorphisms are much 'looser' than vector space isomorphisms and even diffeomorphisms. We don't have a lot to say about homeomorphisms yet. But we know that they preserve certain properties of topological spaces. This is the meat of this course.

Definition : Topological Invariants

A property ϕ of topological spaces is a **topological inveriant** if whenever $(X, \mathcal{T}_X)$ is homeomorphic to $(Y, \mathcal{T}_Y)$, one has the property ϕ if and only if the other does.

Just like dimension was a vector-space invariant, (i.e, isomorphic vector spaces always had the same dimension), we can now find a lot out about a topological space just by drawing a homeomorphism between it and a space we know more about.

Theorem :

The following are topological invariants:

1. T_0, T_1, T_2
2. separable
3. First and Second countable
4. ccc
5. Being uncountably infinite, being uncountable, having size n.
6. There exists an open singleton.
7. There exists a singleton that is the intersection of 7 open sets.

The significance of these is that if you can find that one topological space is, say, separable, and the other is not, you have shown that they are not homeomorphic.

Theorem : Second Last Problem in Big List Section 6

$\mathbb{R}_{\text{usual}}$ is not homeomorphic to $\mathbb{R}^2_{\text{usual}}$.

This theorem is not that easy to show with our current knowledge. They both satisfy most of the invariants we've discussed. In general, $\mathbb{R}^m_{\text{usual}} \not\cong \mathbb{R}^n_{\text{usual}}$ but this is *much* harder to show.

In general, homeomorphic topological spaces are much more loosely similar than isomorphic vector spaces. To show this, consider:

$$\mathcal{T}_{\mathbb{Q}} = \{U \cap \mathbb{Q} : U \text{ is open in } \mathbb{R}_{\text{usual}}\}$$

$$\mathcal{T}_{\mathbb{Q}^2} = \{U \cap \mathbb{Q}^2 : U \text{ is open in } \mathbb{R}^2_{\text{usual}}\}$$

$\mathbb{Q} \cong \mathbb{Q}^2$ with these topologies. Highly counterintuitive.