MAT327 - Lecture 9

Wednesday, June 5th, 2019

Definition : Subspace Topology

Let $(X, \mathcal{T}_X)$ be a topological space, and let $Y \subseteq X$. We define the **subspace topology** on Y as:

$$\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}_X\}$$

The set $(Y, \mathcal{T}_Y)$ is called a **subspace** of $(X, \mathcal{T}_X)$.

That this is a topology is very easy to show. It does everything we want it to.

If $U \in \mathcal{T}_Y$, we say that U is *open in* Y.

Theorem :

If $(X, \mathcal{T}_X)$ is a topological space, and let $Y \subseteq X$ and let $\mathcal{B}_x$ be a basis for $\mathcal{T}_X$. Then:

$$\mathcal{B}_y = \{B \cap Y : B \in \mathcal{B}_x\}$$

is a basis on Y that generates $\mathcal{T}_Y$.

This is also very easy to show. Taking intersections of Y with the open sets in X does not disturb any of the properties we want these sets to satisfy.

Example : Subspaces

1. Subspaces of (in)discrete spaces are (in)discrete.

2. In $\mathbb{R}_7$, let $A \subseteq R$. If $7 \in A$, then $\mathcal{T}_A$ is the particular point topology on A, and if $7 \notin A$ then $\mathcal{T}_A$ is the discrete topology on A.

3. The subspace topology on $(0,1) \subseteq \mathbb{R}$ inherited from $\mathcal{T}_{\text{usual}}$ is generated by the following basis:

$$\mathcal{B}_{(0,1)} = \{(a,b) : 0 < a < b < 1\}$$

   Here we could also have a and b as rational numbers, as we've seen.

4. For all $a < b$ and $c < d$ in $\mathbb{R}$, we have that $(a,b)_{\text{usual}} \cong (c,d)_{\text{usual}}$.

5. $\mathbb{Z}$ as a subset of $\mathbb{R}_{\text{usual}}$ is discrete. To see this, note that for all $z \in \mathbb{Z}$, $z = (z - 1/2, z + 1/2) \cap \mathbb{Z}$.

   Subspaces that are discrete come up often enough that they have a name. We say that $\mathbb{Z}$ is a **discrete subspace** of $\mathbb{R}_{\text{usual}}$.

   It turns out the property of having a countable discrete subspace is a topological invariant.

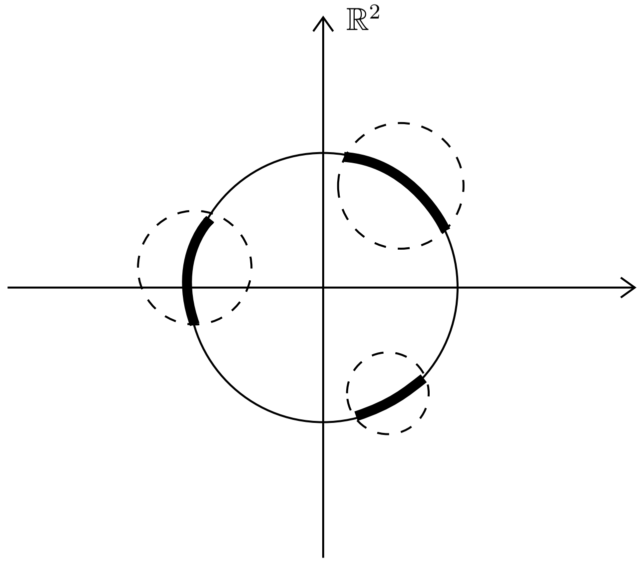
6. $\mathbb{Q}$ and $Y = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ are non-discrete subspaces of $\mathbb{R}_{\text{usual}}$. Y would be discrete if it didn't have 0.

7. Consider $[0,1] \subseteq \mathbb{R}_{\text{Sorgenfrey}}$. Here, $\{1\}$ is open. Indeed, $\{1\} = [0,1] \cap [1,2)$.

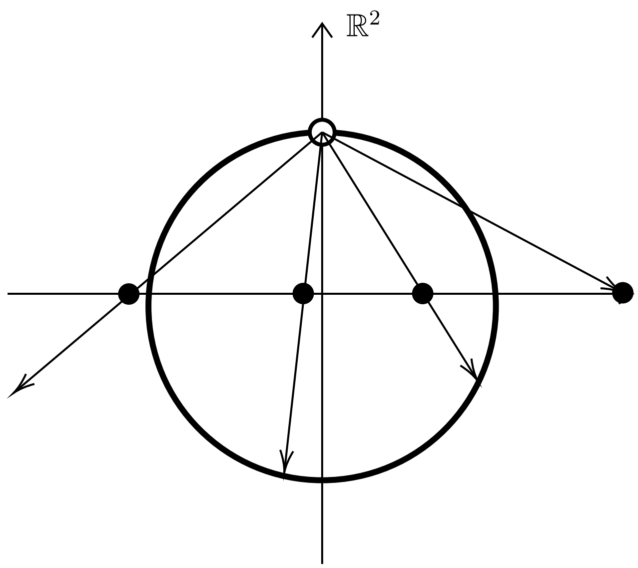
8. Let $Y = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subseteq \mathbb{R}^2_{\text{usual}}$. $\mathcal{T}_Y$ is generated by "open arcs".

9. $S^1 \setminus \{(0,1)\}$ as a subspace of $\mathbb{R}^2_{\text{usual}}$ is homeomorphic to $\mathbb{R}_{\text{usual}}$.

The following figure demonstrates what we mean in example 8 by "open arcs". Each arc is of the form $U \cap S^1$ where U is an open ball in $\mathbb{R}^2$. Proving that the open arcs generate a topology on S^1 is hard, and would requires us to use polar coordinates.



And the following figure demonstrates the idea of the homeomorphism between $S^1 \setminus \{(0,1)\}$ as a subspace $\mathbb{R}^2_{\text{usual}}$, and $\mathbb{R}_{\text{usual}}$. For every point on the circle, draw the straight line from $\{(0,1)\}$ to that point, and we can map each point on the circle to the point in $\mathbb{R}$ that this line touches. This idea is known as **stereographic projection**.



What follows in Ivan's notes is a long sequence of propositions. Most of these aren't too hard. But there's a few that are important.

Theorem :

Let $(X, \mathcal{T}_X)$ be a topological space and let $Y \subseteq X$. Then the map $i : Y \to X$ given by $i(x) = x$ is continuous.

We recall that theorem that the identity function $i : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous if and only if $\mathcal{T}_1 \supseteq \mathcal{T}_2$. This fact does not contradict this claim. It doesn't make sense to compare the topologies when one is over Y and one is over X.

Theorem :

If $f : X \to Z$ is continuous, then $f|_y : Y \to Z$ is continuous. More concisely, the restriction of a continuous map to a subspace is continuous.

Theorem :

Suppose $(X, \mathcal{T}_X)$ is a topological space and let $(Y, \mathcal{T}_Y)$ be a subspace. If $f : Z \to Y$ is continuous, then $f : Z \to X$ is continuous.
More concisely, extending the codomain does not affect continuity.

Now a slightly more interesting/useful theorem.

If we can define a continuous function on A, and a continuous function on B, under what circumstances can we glue them together to form a continuous function on $A \cup B$?

Theorem : Pasting Lemma

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $A, B \subseteq X$ be either both closed or both open such that $X = A \cup B$ (they do not have to be disjoint).
If $f(x) = g(x)$ for all $x \in A \cap B$, then $h : X \to Y$ defined by:

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous.

Hereditary Properties

We now continue our studies of the properties of topological spaces. In particular, how a subspace 'inherits' a property from its ambient space. These are called **hereditary properties**.

Definition : Hereditary Properties

A topological property ψ is **hereditary** if for any subspace $(Y, \mathcal{T}_Y)$ of a topological space $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ has ψ whenever $(X, \mathcal{T}_X)$ does.

Example :

1. T_2 is hereditary. That is, any subspace of a Hausdorff space is Hausdorff (as proven in the notes.) Similarly, T_0 and T_1 are also both hereditary by a very similar proof.

2. 1st countability and 2nd countability are hereditary.

3. Separability and ccc are not hereditary. To show this, we need a separable/ccc topological space with a subspace that is not separable/ccc.

   Consider $\mathbb{R}_7$, which is both separable ($\{7\}$ is dense) and ccc (no two open sets are disjoint, forget about an uncountable collection of them.)

   Let $Y = \mathbb{R} \setminus \{7\} \subseteq \mathbb{R}_7$. This space is conveniently not ccc nor separable.

Definition :

A topological space $(X, \mathcal{T}_X)$ is hereditarily ψ if it and all of its subspaces has ψ.

This is a weaker condition that is sometimes useful, particularly when ψ itself is not a hereditary property.

As an example, the usual topology on $\mathbb{R}$ is hereditarily separable.

Theorem :

Every second countable space is hereditarily separable.

Proof. Let $(X, \mathcal{T}_X)$ be a second countable space. X is immediately separable as second countable implies separable. Now let $Y \subseteq X$ be a subspace.

Second countability is hereditary, so $(Y, \mathcal{T}_Y)$ is second countable and hence separable. ■

Theorem :

If ψ is a topological invariant, then "hereditarily ψ" is also a topological invariant.

This theorem is much easier to prove than it looks. The proof is just a matter of unravelling definitions.