4. Countability

1 Motivation

In topology as well as other areas of mathematics, we deal with a lot of infinite sets. However, as we will gradually discover, some infinite sets are bigger than others. Countably infinite sets, while infinite, are "small" in a very definite sense. In fact they are the "smallest infinite sets". Countable sets are convenient to work with because you can list their elements, making it possible to do inductive proofs, for example.

In the previous section we learned that the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$. In this section, we will learn that $\mathbb{Q}$ is countable. This is useful because despite the fact that $\mathbb{R}$ itself is a large set (it is *uncountable*), there is a countable subset of it that is "close to everything", at least according to the usual topology. Similarly the usual topology on $\mathbb{R}$ contains a lot of sets (uncountably many sets), but as you have already shown on a Big List question without exactly saying so, this topology has a countable basis, meaning that essentially all of the information about the topology can be specified by a "small" collection of open sets. This is not true of the Sorgenfrey Line, for example, as you will later prove.

With regard to this course in particular, the notion of countability comes up quite often in topology and so this set of notes is here to make sure students have a firm grasp of the concepts involved before we start throwing around the words "countable" and "uncountable" all the time.

2 Counting

The subject of countability and uncountability is about the "sizes" of sets, and how we compare those sizes. This is something you probably take for granted when dealing with finite sets.

For example, imagine we had a room with seven people in it, and a collection of seven hats. How would you check that we had the same number of hats as people? The most likely answer is that you would count the hats, then count the people. You would get an answer of 7 both times, you would note that $7 = 7$, and conclude that there are the same number of hats as there are people. You would be correct, of course.

Now imagine I asked you to do this but you had never heard of "7" - that you were *incapable* of describing the size of the collection of hats or the size of the group of people with a number. Could you still prove that there are the same number of hats as people?

One thing you could do is simply start putting hats on people; take your collection of hats and put one on each person's head. After you finish doing this, you would check whether (a) you have no unused hats remaining; and (b) everyone in the room is wearing precisely one hat. If both of these things are true, you would be justified in concluding that there are the same number of hats as there are people. You could even get more specific, and say that if after putting one hat on each person's head there are some left over hats, then there are more hats

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than people. On the other hand if after this some people still are not wearing hats, then there are more people than hats.

Surely this all seems trivial, but notice that what we have described here is a way of checking whether two collections of objects are the same size *without counting them*. Without even knowing what numbers are, for that matter. *This* is the notion of comparing sizes of sets that generalizes well to the infinite.

In particular, when dealing with infinite sets you *cannot* just count both sets and describe their sizes with integers you then can compare. The way of comparing sets described in the example above is the only strategy available to you in that situation.

To "mathematize" our above discussion a bit, notice that what you did with the people and hats amounts to constructing a function from the set of people to the set of hats, defined by

$$g(\text{Person X}) = \text{the hat you put on Person X's head.}$$

If you made sure to put at most one hat on each person's head, g is injective. If you put a hat on every person, g is surjective. If g is both injective and surjective (ie. if g is *bijective*), then we concluded that the set of hats and the set of people were the same size.

This is the idea that we use to compare the sizes of all sets.

Definition 2.1. *Given two sets A and B, we say A has the same cardinality as B if there exists a bijection $f : A \to B$. This is usually denoted by $|A| = |B|$.*

Those interested in fancy, old mathematical language might say that two sets of the same cardinality are *equipotent*.

Definition 2.2. *Given two sets A and B, we say that A has cardinality smaller than or equal to B if there exists an injection $f : A \to B$, or equivalently if there exists a surjection $g : B \to A$. This is usually denoted by $|A| \leq |B|$.*

If there is an injection $f : A \to B$ and there is no surjection from A to B, we say that A has smaller cardinality than B, and write $|A| < |B|$.

We will not go too much further into the abstract notion of cardinality, but the following theorem is somewhat satisfying.

Theorem 2.3 (Cantor-Schröder-Bernstein Theorem)**.** *Let A, B be sets. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

This theorem is much less trivial than it looks. It says that if there is an injection $f : A \to B$ and an injection $g : B \to A$, then there is a bijection $h : A \to B$. Try to prove it yourself, and you will find that it tricky even for finite sets. You are encouraged to look into the proof of this theorem, which is lovely, and the most basic example of a very useful proof technique called a "back and forth argument".

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3 Countability

Definition 3.1. *A set A is said to be countably infinite if $|A| = |\mathbb{N}|$, and simply countable if $|A| \leq |\mathbb{N}|$.*

In words, a set is countable if it has the same cardinality as some subset of the natural numbers. In practise we will often just say "countable" when we really mean "countably infinite", when it is clear that the set involved is infinite. Note that $\emptyset$ is countable, since the empty function $f : \emptyset \to \mathbb{N}$ is vacuously an injection.

The prevailing intuition here should be that a set is countable provided that you can list its elements. After all, an injection $f : A \to \mathbb{N}$ is nothing more than a way of assigning a number to each element of A in a reasonable way (ie. in a way such that each element of A gets one number). In particular, a set A is countably infinite provided that you can construct a list

$$a_1, a_2, a_3, a_4, a_5, a_6, \ldots$$

of its elements and prove (a) that no element of A appears on the list more than once; and (b) that the list includes every element of A. (Given such a list, the function defined by $f(n) = a_n$ is a bijection $\mathbb{N} \to A$.) Many of our proofs that sets are countably infinite will just be given as lists like this.

It should already be apparent that this way of comparing sizes produces some results that feel unusual if one is only used to thinking about sizes of finite sets. For example, an obvious fact about finite sets is that if A is a finite set and B is a proper subset of A, then B is smaller than A. That is, if you start with a finite set A and take some things away, what iss left over is smaller than A. This is not the case with infinite sets.

Example 3.2. Let $E \subseteq \mathbb{N}$ be the set of even numbers. Then $|E| = |\mathbb{N}|$. Indeed:

$$2, 4, 6, 8, 10, 12, 14, 16, \ldots$$

4 Simple examples and facts

In this section we will look at some simple examples of countable sets, and from the explanations of those examples we will derive some simple facts about countable sets.

Example 4.1. The set $A = \{\, n \in \mathbb{N} : n > 7 \,\}$ is countable. We can certainly list its elements in a bijective way:

$$8, 9, 10, 11, 12, 13, \ldots$$

or think of the bijection $f : \mathbb{N} \to A$ given by $f(n) = n + 7$.

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It should be clear that nothing is special about the number 7 here. We started with a countably infinite set, removed finitely many elements, and were left with something that was still countably infinite. The fact that we removed the *first* seven elements allowed us to define f in a pretty convenient way, but even this was not necessary. This leads us to the following fact:

Proposition 4.2. *If A is a countable set and $B \subseteq A$ is finite, then $A \setminus B$ is countable.*

Proof. This result is obvious if A is finite, so we will treat the case in which A is countably infinite.

It is much easier to explain the idea of this proof than to write it down; to list the elements of $A \setminus B$, start with a list of the elements of A (which we have by the assumption that A is countable), delete the elements of B from the list, then squish everything down to fill the gaps created by the deletions.

For example, we explicitly treat the case in which B has two elements. Let the bijection $f : \mathbb{N} \to A$ witness that A is countable, and suppose $B = \{f(17), f(2523)\}$. Then the following function $g : \mathbb{N} \to A \setminus B$ is a bijection:

$$g(n) = \begin{cases} f(n) & n < 17 \\ f(n+1) & 17 \leq n < 2522 \\ f(n+2) & n \geq 2522 \end{cases}.$$

Check that this is a bijection between $\mathbb{N}$ and $A \setminus B$, and convince yourself of how you would write down a function like this for the general setting. □

We can also use the same idea, but backwards. That is, if we start with a countable set and add finitely many elements, the result is countable.

Example 4.3. The set $A = \{\, n \in \mathbb{Z} : n \geq -7 \,\} = \mathbb{N} \cup \{-7, -6, \ldots, -1, 0\}$ is countable. This is witnessed by the function $f : \mathbb{N} \to A$ defined by $f(n) = n - 8$.

Proposition 4.4. *Let A be a countable set, and B a finite set. Then $A \cup B$ is countable.*

Proof. Again, this result is obvious if A is finite, so assume it is countably infinite. Furthermore we may assume without loss of generality that A and B are disjoint, as any amount of overlap would effectively shrink B and "help" our cause. More formally, $A \cup B = A \cup (B \setminus A)$, and $B \setminus A$ must also be finite, so it suffices to consider disjoint sets A and B.

Let $f : \mathbb{N} \to A$ witness that A is countable, and enumerate $B = \{b_1, b_2, \ldots, b_k\}$ for some k. We can do this since B is finite. Now define:

$$g(n) = \begin{cases} b_n & n \leq k \\ f(n-k) & n > k \end{cases}.$$

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In the language of lists, we are simply prepending the elements of B to the beginning of our given list of elements of A, so that $A \cup B$ is listed as:

$$b_1, b_2, \ldots, b_k, f(1), f(2), f(3), f(4), \ldots,$$

then re-numbering this new list with g. $\square$

So adding or removing finitely many elements does not affect countability. However, in the previous section (Example 4.1) we saw that at least in some cases you can remove *infinitely* many elements of a countable set and still be left with a countable set. To reiterate:

Example 4.5. Let $E \subseteq \mathbb{N}$ be the set of even natural numbers. Then E is countable, as witnessed by $f(n) = 2n$.

Again from this example, we can extract a pair of more general facts.

Proposition 4.6. *Let A be a countably infinite set, and B an infinite subset of A. Then B is countable.*

Proof. The idea here is essentially the same as in Proposition 4.2; list the elements of A, delete the elements *not in* B from your list, then squish everything down to fill the gaps. We will take care to be a bit more rigorous with this proof though.

So let $f : \mathbb{N} \to A$ be a bijection witnessing that A is countable. We want to construct a bijection $g : \mathbb{N} \to B$.

What goes on in the proof that follows is essentially this: list the elements of A, then take the element of B with the smallest index in this list, and call that $g(1)$. Then take the element of B with the next smallest index, and call that $g(2)$. Repeat this process inductively, letting $g(k)$ equal the element of B with the k^{th}–smallest index. Then we prove that g is a bijection.

Let $k_1 = \min\{\, k \in \mathbb{N} : f(k) \in B \,\} = \min(f^{-1}(B))$. That is, k_1 is the smallest number that gets mapped into B by f. Define $g(1) := f(k_1)$. We proceed inductively from here.

Assume we have defined $g(1), g(2), \ldots, g(n)$. Let

$$k_{n+1} = \min\{\, k \in \mathbb{N} : f(k) \in B \setminus \{g(1), \ldots, g(n)\} \,\}.$$

That is, k_{n+1} is the smallest number that f maps to an element of B that we have not hit with g so far. (Note that since B is infinite, $B \setminus \{g(1), \ldots, g(n)\}$ is also infinite and in particular nonempty, so this minimum actually exists.)

Then, define $g(n+1) = f(k_{n+1})$. Continuing in this way we construct a function $g : \mathbb{N} \to B$ which is clearly injective. To see that it is surjective, fix $b \in B$, and assume for the sake of contradiction that b is not in the range of g. Define

$$X = \{\, n \in \mathbb{N} : f(n) \in B \text{ but } f(n) \text{ is not in the range of } g \,\}.$$

That is, X is the set of indices (according to f) of elements of B that are missed by g. Then X is nonempty, since $f^{-1}(b) \in X$ at least. Being a nonempty subset of the naturals, X must have

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a minimal element, which we call n_0. But then n_0 must be k_N for some $N \leq n$, by definition of the k_i.

To see this more carefully, note that at most n_0 of the k_i's can occur before n_0. Let M be the number of them that occur. In other words:

$$M = |\{1, 2, 3, \dots n_0\} \cap \{\, k_i \,:\, i \in \mathbb{N} \,\}|$$

This means $k_1, k_2, \dots, k_M$ are all $\leq n_0$ and there are no other k_i's below n_0. But then:

$$n_0 = \min \{\, n \in \mathbb{N} \,:\, f(n) \in B \setminus \{g(1), \dots, g(M)\} \,\}.$$

Therefore, by definition of the k_i's, we must have that $n_0 = k_{M+1}$. So $M + 1$ is the N we mentioned earlier.

This is a contradiction, since $g(n_0) = g(k_N) \in B$ by definition of g, but n_0 was a member of X. We conclude that X must be empty and in turn that g is surjective.

□

One way to interpret this result is as a sort of dichotomy theorem. A subset of a countably infinite set must be either finite or infinite, and this result says that if it is not finite, it must be *countably* infinite. In other words, there are no sizes available for a set to be "between" finite and countably infinite.

Just as in the two "matching" propositions we proved earlier, Proposition 4.6 leads to another fact if we go backwards. That is, if we start with a countable set and add countably many elements, the result is countable.

Proposition 4.7. *Let A, B be countable sets. Then $A \cup B$ is countable.*

Proof. If A and B are finite, this is obvious. If just one of them is finite, this is Proposition 4.4. So assume both are countably infinite. And again, without loss of generality, we may assume A and B are disjoint. We can do this since $A \cup B = A \cup (B \setminus A)$, and by the previous proposition $B \setminus A$ must be either finite or countably infinite.

Let $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$ be bijections witnessing that A and B are countable. Define $h : \mathbb{N} \to A \cup B$ by:

$$h(n) = \begin{cases} f(k) & \text{if } n = 2k \\ g(k) & \text{if } n = 2k - 1 \end{cases}$$

In the language of lists, take the list of elements of A and the list of elements of B and interweave them.

Convince yourself that h is a bijection. □

Corollary 4.8. *$\mathbb{Z}$ is countable.*

Proof. Notice that $\mathbb{Z} = (\mathbb{N} \cup \{0\}) \cup (-\mathbb{N})$, where $-\mathbb{N}$ is an abuse of notation denoting $\{\, -n \,:\, n \in \mathbb{N} \,\}$, the negative integers. The result then follows from Propositions 4.7 and 4.4. □

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Exercise 4.9. Construct an explicit bijection $f : \mathbb{N} \to \mathbb{Z}$.

Exercise 4.10. Prove that any finite union of countable sets is countable. (Hint: Use the previous Proposition, and induction.)

At this point we state a proposition that will free us up just a bit. Every proof that something is countably infinite that we have done so far has involved bijections from $\mathbb{N}$. Bijections are pretty nice functions though, and in particular they have inverses which are also bijections, and the composition of two bijections is again a bijection.

Proposition 4.11. *Let A be an infinite set, and C a fixed countably infinite set. Then the following are equivalent.*

1. *A is countable.*
2. *There exists a bijection $f : \mathbb{N} \to A$.*
3. *There exists a bijection $g : A \to \mathbb{N}$.*
4. *There exists a bijection $h : A \to C$, or $h : C \to A$.*
5. *There exists an injection $i : A \to \mathbb{N}$, or $i : A \to C$.*
6. *There exists a surjection $s : \mathbb{N} \to A$, or $s : C \to A$.*

Proof. **Exercise.** (Just about everything here follows from elementary properties of functions or the Propositions we have proved above. The only one that should require any work to show is equivalent to the others is (6)). □

Exercise 4.12. Later in this course we will learn about the Axiom of Choice. After we have learned about it (or right now if you are already familiar with it) try to spot whether any of the implications between these facts rely on some amount of Choice.

5 More interesting examples and facts

Earlier we proved the odd fact that infinite sets can be the same size as proper subsets of themselves. We even learned that $\mathbb{N}$ and $\mathbb{Z}$ are the same size as one another, even though by most intuitive measures $\mathbb{Z}$ is at least "twice the size" of $\mathbb{N}$. There are, after all, two perfect-looking copies of $\mathbb{N}$ inside $\mathbb{Z}$. This is weird.

We will start off this section by proving that $\mathbb{N}$ is the same size as something with *infinitely many* perfect copies of itself inside it.

Theorem 5.1. *$\mathbb{N} \times \mathbb{N}$ is countable.*

Proof. There are two ways to prove this. One is by drawing a picture and pointing at it, and the other is by explicitly defining a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. The former way is the best way. Here is the picture:

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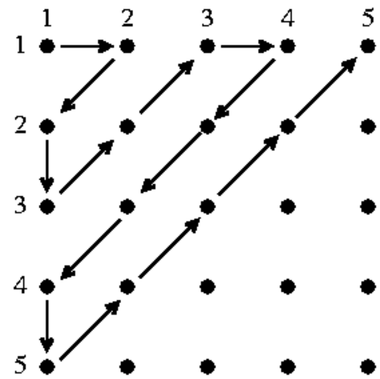

Figure 1: Convince yourself that this is a proof that $\mathbb{N} \times \mathbb{N}$ is countable. (Source: College Math Teaching, 10 April, 2015.)

For a more explicit proof, define a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$f(n, m) = 2^n 3^m.$$

This function is injective by the uniqueness of prime decompositions, and so $\mathbb{N}\times\mathbb{N}$ is countable by Proposition 4.11, part 5. □

Corollary 5.2. *Let A and B be countable sets. Then $A \times B$ is countable.*

Proof. **Exercise.** □

Exercise 5.3. Show that the Cartesian product of finitely many countable sets is countable in two different ways. Take note of why each of your two proofs does not extend to even countable products of countable sets.

From Corollary 5.2 follows one of the most important results of this section for our purposes.

Corollary 5.4. *$\mathbb{Q}$ is countable.*

Proof. $\mathbb{Q}$ is obviously infinite, so we will show it is countable by constructing an injection into the set $\mathbb{Z} \times \mathbb{N}$, which is countable by the previous corollary.

Recal that $\mathbb{Q}$ can be expressed as

$$\left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}, \text{ and the fraction is in lowest terms} \right\}.$$

Define a function $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ by $f(\frac{p}{q}) = (p, q)$. This map is an injection (check this!) and therefore $\mathbb{Q}$ is countable by Proposition 4.11, part 5. □

We will talk a great deal more about this fact throughout the course, in many different contexts. For now, take a moment to come to terms with it. Thus far you have likely been picturing countable sets as sparse collections of points, like $\mathbb{N}$, $\mathbb{Z}$, etc. $\mathbb{Q}$ on the other hand is

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very densely packed. Wherever you look in the real numbers, there are infinitely many rationals there. Still, the set of all rationals is countable.

Another corollary of Theorem 5.1 is the following. At this point you should not be too surprised by it, though it would have been surprising at the beginning of our discussion on countability.

Corollary 5.5. *A countable union of countable sets is countable. That is, if A_n, $n \in \mathbb{N}$ are countable sets, then $\bigcup_{n\in\mathbb{N}} A_n$ is countable.*

Proof. As before, we may assume that the A_n are all infinite and mutually disjoint.

Let $f_n : \mathbb{N} \to A_n$ be a bijection witnessing that A_n is countable, and define $g : \mathbb{N} \times \mathbb{N} \to \bigcup_{n\in\mathbb{N}} A_n$ by $g(n,i) = f_n(i)$.

Exercise 5.6. Show that the function g defined above is a bijection, and conclude that $\bigcup_{n\in\mathbb{N}} A_n$ is countable.

□

Exercise 5.7. Show that the set $\mathcal{B}_{\mathbb{Q}} = \{ (a,b) \subseteq \mathbb{R} : a, b \in \mathbb{Q} \}$ is countable. This means that $\mathbb{R}_{\text{usual}}$ has a countable basis (you showed in Big List 2.2 that $\mathcal{B}_{\mathbb{Q}}$ is a basis for $\mathbb{R}_{\text{usual}}$).

This property of having a countable basis is useful enough (and *invariant* enough, in senses we will discuss later) to earn a name. We will do that at the end of this note.

To summarize what we have learned thus far:

- A subset of a countable set is either finite or countably infinite.
- Finite and countably infinite unions of countable sets are countable.
- Finite Cartesian products of countable sets are countable.

6 Are all sets countable!?

No.

Definition 6.1. *A set A that is not countable is called uncountable.*

By this point, the following should not be surprising.

Proposition 6.2. *If A is uncountable, B is a set, and $f : A \to B$ is a bijection, then B is uncountable.*

Proof. **Exercise.** □

Fine, so what sets are actually uncountable? Lots of them, it turns out. We will give one specific example and its consequences, and one method for producing as many uncountable sets as you want.

The following is a very famous and lovely proof, first given by Cantor in the 1880s. It was quite a surprise at the time. It is often called "Cantor's Diagonalization Proof".

Theorem 6.3 (Cantor)**.** $(0,1) \subseteq \mathbb{R}$ *is uncountable.*

Proof. This is a proof by contradiction. Certainly $(0,1)$ is not finite, so we begin by assuming that $(0,1)$ is countably infinite, and fixing a bijection $f : \mathbb{N} \to (0,1)$ witnessing this. Every real number between 0 and 1 has an infinite decimal expansion of the form:

$$0.a_1a_2a_3a_4a_5a_6a_7\dots$$

(where we imagine a tail of zeros on a real number with a terminating decimal expansion). In this way, we will index every digit appearing in the decimal expansion of every real number on our list (the list given by f). That is, for $x \in (0,1)$, we have by assumption that there is a unique $n \in \mathbb{N}$ such that $f(n) = x$. So we're going to write:

$$x = f(n) = 0.x_1^n x_2^n x_3^n x_4^n x_5^n x_6^n x_7^n \dots$$

In your head you should now be imagining that we have constructed an infinite table that looks something like:

$$\begin{array}{lcccccccc}
f(1) = & 0. & \color{red}{x_1^1} & x_2^1 & x_3^1 & x_4^1 & x_5^1 & \dots \\
f(2) = & 0. & x_1^2 & \color{red}{x_2^2} & x_3^2 & x_4^2 & x_5^2 & \dots \\
f(3) = & 0. & x_1^3 & x_2^3 & \color{red}{x_3^3} & x_4^3 & x_5^3 & \dots \\
f(4) = & 0. & x_1^4 & x_2^4 & x_3^4 & \color{red}{x_4^4} & x_5^4 & \dots \\
f(5) = & 0. & x_1^5 & x_2^5 & x_3^5 & x_4^5 & \color{red}{x_5^5} & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

where I have coloured the diagonal elements in the interesting part of our table red to call attention to them.

We will now construct a real number $y = 0.y_1y_2y_3y_4y_5\dots$ that cannot be on our list (ie. cannot be in the range of f), contradicting the assumption that f is a bijection.

Indeed, let $y_1 = x_1^1 - 1$, unless $x_1^1 = 0$, in which case let $y_1 = 9$. The specifics of this are not so important, but what is important is that we fix a concrete way of defining y_1 to not equal x_1^1. Now do the same for each y_k. That is, let $y_k = x_k^k - 1$, unless $x_k^k = 0$, in which case let $y_k = 9$. Also, at the end of this process, we have to make sure we did not construct $y = 0 = 0.0000\dots$ or $y = 1 = 0.9999\dots$. So at the end, we just make any change to one digit that ensures this.

I claim that $y = 0, y_1y_2y_3y_4y_5\dots$ is not in the range of f.

To see this, suppose it was in the image of f. Then there would have to exist an $n \in \mathbb{N}$ such that $f(n) = y$. But then by construction of y, the n^{th} decimal place of y differs from the n^{th} decimal place of $f(n)$.

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This is a contradiction, proving that y is not in the image of f, which in turn shows that f was not a bijection. Since the f we started with was arbitrary, we conclude that there can be no bijection between $\mathbb{N}$ and $(0,1)$. □

Amazing! I hope some of the beauty of this result is apparent even to the most jaded math specialists reading this. Both of $\mathbb{N}$ and $(0,1)$ are infinite, but $(0,1)$ is *bigger*!

Of course, this also means that $\mathbb{R}$ is uncountable.

Corollary 6.4. $\mathbb{R}$ *is uncountable.*

Proof. All we need to do is exhibit a bijection $h : (0,1) \to \mathbb{R}$. $f(x) = \pi(x - \frac{1}{2})$ is a bijection between $(0,1)$ and $(-\frac{\pi}{2}, \frac{\pi}{2})$ (check this!), and $g(x) = \tan(x)$, when restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$, is a bijection from this set to $\mathbb{R}$. Their composition $h := g \circ f : (0,1) \to \mathbb{R}$ is therefore a bijection, as required. □

We conclude our discussion of uncountable sets with what is undoubtedly my personal favourite proof in mathematics. I can trace my decision to pursue mathematics professionally directly to when I first read and understood this proof and its implications.

Theorem 6.5. *Let A be a set. There is no surjection $f : A \to \mathcal{P}(A)$. Since $g : A \to \mathcal{P}(A)$ given by $g(x) = \{x\}$ is clearly an injection, this means $|A| < |\mathcal{P}(A)|$. In particular, $\mathcal{P}(\mathbb{N})$ is uncountable.*

Proof. Let A be a set, and let $f : A \to \mathcal{P}(A)$ be a function. We will show that f cannot be surjective. Define a set:

$$D = \{\, x \in A \,:\, x \notin f(x) \,\}\,.$$

Clearly D is a subset of A, so $D \in \mathcal{P}(A)$.

Now suppose for the sake of contradiction that f is surjective. Then in particular its range must contain D, so let $a \in A$ be such that $f(a) = D$.

Question: Is $a \in D$?

Either it is or it is not, surely. We examine both cases.

Case 1: Yes. If $a \in D$, then $a \in f(a)$ (since $f(a) = D$). However looking back at the definition of D, we see that this means a cannot be in D. So this is impossible.

Case 2: No. If $a \notin D$, then $a \notin f(a)$. Looking back at the definition of D again, this precisely means $a \in D$. Again, this is impossible.

In either case we get a contradiction. Since these are the only two cases, the assumption that f is surjective must be false, finishing the argument. □

The implications of this proof are far-reaching. Just for a taste, it implies that if you keep iterating the power set operation, you get sets with strictly larger and larger cardinalities. That is, for example:

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \cdots$$

So there are *infinitely many sizes of infinity*! I love this stuff.

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7 Enough cool proofs, give me some definitions!

Fine, fine. Here are two definitions that we have alluded to as being useful in previous discussions, and that I define here since we have now formalized the notion of countability.

To reiterate some of that discussion, countability is a notion of "smallness" for us. Even though $\mathbb{R}_{\text{usual}}$ is a big, complicated space, the fact that a countable set ($\mathbb{Q}$, for example) is dense in it serves to simplify many things. The space itself is big, but there is a countable (read: small) set that is "close" to everything.

On a similar note, we know that $\mathbb{R}_{\text{usual}}$ has a tonne of open sets that can get very complicated. We already decided that we prefer to specify a simpler basis of open sets rather than specify *every* open set in the topology. But even the usual basis of $\mathbb{R}_{\text{usual}}$—the collection of ϵ-balls around all the points—is big, in the sense that it is uncountable. However, you proved in a Big List problem and also in Exercise 5.7 above that the set of intervals with rational endpoints is a countable basis for the usual topology. This is very useful, because now we know that all the information in this big, complicated topology can be recovered from a small amount of information.

As the course progresses, these ideas will crop up several more times. For the moment, we give them names.

Definition 7.1. *A topological space $(X, \mathcal{T})$ is called separable if there is a countable, dense subset of X.*

Definition 7.2. *A topological space $(X, \mathcal{T})$ is called second countable if there is a countable basis on X that generates $\mathcal{T}$.*

Two notes about second countability:

First, take care to note that the definition does not say that *every* basis of the topology is countable, just that there is at least one countable basis. In particular the usual basis of the usual topology on $\mathbb{R}$ is very much uncountable, but the existence of $\mathcal{B}_{\mathbb{Q}}$ shows that $\mathbb{R}_{\text{usual}}$ is second countable.

Second, it may seem weird to define "second countable" without having defined "first countable". These properties both have terrible names. The concept of "first countable" is something I may have hinted at in lecture, but which we are not quite ready to discuss. You will soon find that it is a subtle and useful property, but only after we have discussed sequences.

Finally, here is one more topological property that involves countability. We will not prove anything with this property at the moment, but you will do a little bit in a Big List problem for this section, and we will revisit this property several times throughout the course.

Definition 7.3. *A topological space $(X, \mathcal{T})$ is said to have the countable chain condition if there are no uncountable collections of mutually disjoint, nonempty, open subsets of X. Another way*

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to characterize this property is to say that every collection of mutually disjoint, nonempty, open subsets of X is countable.

Remark 7.4. If $(X, \mathcal{T})$ has the countable chain condition, we will usually say "$(X, \mathcal{T})$ has the ccc" or "$(X, \mathcal{T})$ is ccc".

You proved that $\mathbb{R}_{\text{usual}}$ is ccc with a problem at the end of the first section of the Big List, though of course you did not have this name at the time.

For now we will simply add this property to our growing catalogue of topological properties, but we make special note that the relationships between this property, separability, and second countability will be interesting to explore. Whenever we prove something about one of these three properties, you should think about the other ones in the same context.

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