6. Continuity and homeomorphisms

1 Motivation

Up to now we have defined just a few topological properties, like the first three T-axioms and the "countability properties" (separable, ccc, first and second countable). More than anything else, in my opinion, the focus of point set topology is the study of properties like these and others; how they relate to one another, and how well they are preserved through different operations like taking continuous images, subspaces, products, etc. We will find that a relatively small number of new definitions in this and the next two sections will open up quite a lot of new territory for us to explore. We are now beginning to study topological properties themselves, rather than just particular topological spaces.

In this note, we will focus on how these properties transfer to other sets and spaces via functions. In particular we will define a special type of function—a *continuous* function—between topological spaces in such a way that some amount of the topological structure of the domain space is preserved in the co-domain space. Then we will ask questions about whether certain properties of topological spaces are preserved by these nice functions.

Of particular note, we will be able to say that if a continuous function is quite nice—if it is a bijection that perfectly preserves the topological structure of its domain, roughly speaking—that the two spaces on either end of the function are essentially "the same", from a topological standpoint.

We will also take a look at some different ways of constructing continuous functions.

2 Continuous functions

Definition 2.1. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $f : X \to Y$ be a function. We say that f is continuous if $f^{-1}(U) \in \mathcal{T}$ for every $U \in \mathcal{U}$.*

In words, we say that f is continuous if "the preimage of every open set is open".

Strictly speaking we should refer to a function $f : X \to Y$ as being continuous or not with respect to specific topologies on X and Y, but we will just talk about functions between sets when there can be no confusion about the topologies in question. When confusion could arise, we will be clear and say something like $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$, though we will understand that such a function has domain X and co-domain Y.

Before we do anything with continuous functions, here are some examples. We will start with examples in and around the real numbers, where the reading is probably most comfortable.

Example 2.2.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Then f is a continuous function from $\mathbb{R}_{\text{usual}}$ to $\mathbb{R}_{\text{usual}}$.

To see this, fix an open set $U \subseteq \mathbb{R}$. We want to show that $f^{-1}(U)$ is open. Our tool here will be the fact that we know f satisfies the first year calculus definition of continuity.

Fix a point $x \in f^{-1}(U)$. Then $f(x) = x^3 \in U$, and so by definition of the usual topology there is an $\epsilon > 0$ such that $(x^3 - \epsilon, x^3 + \epsilon) \subseteq U$. By the (first year calculus) definition of continuity, there exists a $\delta > 0$ such that if $y \in (x - \delta, x + \delta)$, then $f(y) \in (x^3 - \epsilon, x^3 + \epsilon) = (f(x) - \epsilon, f(x) + \epsilon)$. But then $x \in (x - \delta, x + \delta) \subseteq f^{-1}(U)$, and so $f^{-1}(U)$ is open as required.

2. Fix a point $a \in \mathbb{R}^n$, and define $f : \mathbb{R}^n_{\text{usual}} \to \mathbb{R}_{\text{usual}}$ by $f(x) = d(x, a)$ (where d is the usual Euclidian distance). Then f is continuous.
3. The projection function $\pi_1 : \mathbb{R}^2_{\text{usual}} \to \mathbb{R}_{\text{usual}}$ given by $\pi_1(x, y) = x$ is continuous. These sorts of functions will be very important for us soon in the course.
4. More generally, let $f : \mathbb{R}^n \to \mathbb{R}^k$ be continuous in the calculus sense. Then f is a continuous function from $\mathbb{R}^n_{\text{usual}}$ to $\mathbb{R}^k_{\text{usual}}$. Show this.
5. Any function from a discrete space to any other topological space is continuous.
6. Any function from any topological space to an indiscrete space is continuous.
7. Any constant function is continuous (regardless of the topologies on the two spaces). The preimage under such a function of any set containing the constant value is the whole domain, and the preimage of any set not containing the constant value is empty.
8. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.
9. If X is a set, the function $\text{id} : X \to X$ defined by $\text{id}(x) = x$ is called the identity function. This is useful tool for comparing two topologies on a set.

   For example, consider $\text{id} : \mathbb{R}_{\text{Sorgenfrey}} \to \mathbb{R}_{\text{usual}}$. Here, id is continuous.

   On the other hand, the function $\text{id} : \mathbb{R}_{\text{usual}} \to \mathbb{R}_{\text{Sorgenfrey}}$ is not continuous.

   (Make sure to verify both of these assertions.)

The phenomenon in the last example above is worth stating on its own, simple as it is:

Proposition 2.3. *Let X be a set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Then the identity function $\text{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous if and only if $\mathcal{T}_1$ refines $\mathcal{T}_2$.*

Proof. **Exercise.** □

3 Equivalent conditions

Before going any further towards exploring the properties of continuous functions, we state several conditions that are equivalent to the definition of continuity, to make our future work easier.

The first condition says that it suffices to consider basic and even subbasic open sets in the definition of continuity. This should not come as a surprise, since considering only basic open sets has worked just fine for the definitions of closure, density, sequence convergence, etc.

Proposition 3.1. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, let $f : X \to Y$ be a function, and let $\mathcal{B}$ and $\mathcal{S}$ be a basis and subbasis on Y generating $\mathcal{U}$, respectively. (Note that we are not assuming $\mathcal{S}$ generates $\mathcal{B}$.) Then the following are equivalent:*

1. *The preimages of open sets are open: f is continuous.*
2. *The preimages of basic open sets are open: $f^{-1}(U) \in \mathcal{T}$ for every $U \in \mathcal{B}$.*
3. *The preimages of subbasic open sets are open: $f^{-1}(U) \in \mathcal{T}$ for every $U \in \mathcal{S}$.*

Proof. It is immediate that (1) implies (2) and (3). To see that (2) implies (1), fix an open set $U \in \mathcal{U}$. Since $\mathcal{U}$ is generated by the basis $\mathcal{B}$, we know that $U = \bigcup \mathcal{C}$ for some $\mathcal{C} \subseteq \mathcal{B}$. By assumption, $f^{-1}(B) \in \mathcal{T}$ for all $B \in \mathcal{B}$, and therefore:

$$f^{-1}(U) = f^{-1}\left(\bigcup_{B \in \mathcal{C}} B\right) = \bigcup_{B \in \mathcal{C}} f^{-1}(B),$$

which is a union of open subsets of X, and is therefore open.

Showing that (3) implies (1) or (2) is left as an easy exercise. □

That last one—at least the equivalence between (1) and (2)—is probably something you would have taken for granted. Now we state some more interesting equivalences that may not be so intuitive, beginning with a definition that should be familiar from calculus.

Definition 3.2. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, let $f : X \to Y$ be a function, and let $x \in X$. f is said to be continuous at x if for every open $V \in \mathcal{U}$ containing $f(x)$, there is an open set $U \in \mathcal{T}$ containing x such that $f(U) \subseteq V$.*

Exercise 3.3. Convince yourself that in the context of $\mathbb{R}^n_{\text{usual}}$, this is almost exactly the same as the definition of continuity at a point given in terms of ϵ-balls.

The following proposition will be our "master list" of ways of characterizing continuity.

Proposition 3.4. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and let $f : X \to Y$ be a function. The following are equivalent:*

1. *f is continuous.*

2. *For every closed set $C \subseteq Y$, $f^{-1}(C)$ is closed in X.*
3. *For every $x \in X$, f is continuous at x.*
4. *For every subset $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.*

Proof. We will prove the equivalence in this order: (3) $\Rightarrow$ (1) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). Fix a nonempty open set $V \in \mathcal{U}$. We want to show that $f^{-1}(V)$ is open in $(X, \mathcal{T})$. Fix a point $x \in f^{-1}(V)$. Then $f(x) \in V$, and since f is continuous at x there is an open set $U_x \subseteq X$ containing x such that $f(U_x) \subseteq V$. But then $U_x \subseteq f^{-1}(V)$. Since x was chosen arbitrarily, this shows that $f^{-1}(V)$ is open.

(1) $\Rightarrow$ (4). Suppose f is continuous, and fix a subset $A \subseteq X$. Let $x \in \overline{A}$. We want to show that $f(x) \in \overline{f(A)}$. So pick an open set $V \in \mathcal{U}$ containing $f(x)$. Then by assumption $f^{-1}(V)$ is an open set containing x, and therefore $f^{-1}(V) \cap A \neq \emptyset$ by the definition of closure. So let y be an element of this intersection. Then $f(y) \in V \cap f(A)$, and in particular $V \cap f(A) \neq \emptyset$, as required.

(4) $\Rightarrow$ (2). Fix a closed subset $C \subseteq Y$, and let $A = f^{-1}(C)$. We want to show that A is closed, which we will do by showing $A = \overline{A}$ (more specifically that $\overline{A} \subseteq A$, since the opposite containment is always true). So fix $x \in \overline{A}$. Then

$$f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{C} = C.$$

(Here, the first $\subseteq$ is by assumption of (4), the second $\subseteq$ is since $f(A) = f(f^{-1}(C)) \subseteq C$, and the $=$ is because C is closed.)

That is, $f(x) \in C$, or in other words $x \in f^{-1}(C) = A$, as required.

(2) $\Rightarrow$ (3). Suppose the preimages of closed sets are closed. Fix $x \in X$, and an open set $V \in \mathcal{U}$ containing $f(x)$. Then $Y \setminus V$ is closed, and therefore $f^{-1}(Y \setminus V)$ a closed subset of X by assumption, and it does not contain x. But then the complement of this set, $X \setminus f^{-1}(Y \setminus V)$, is open and does contain x. So fix an open set U such that

$$x \in U \subseteq X \setminus f^{-1}(Y \setminus V).$$

Then $f(U) \subseteq f\left(X \setminus f^{-1}(Y \setminus V)\right) = f(X) \setminus (Y \setminus V) \subseteq V$, as required. □

Having access to all of these different characterizations of continuity allows us to much more easily prove that certain functions are continuous, and to prove facts about continuous functions.

Exercise 3.5. Here are just a few exercises about continuous functions, now that we have characterized them thoroughly. There will be these and more on the Big List.

1. Check that addition and multiplication, thought of as functions $\mathbb{R}^2 \to \mathbb{R}$ with their usual topologies, are both continuous. This can be done from the original definition of continuity alone, but it is much easier with one of the five equivalent ones above.
2. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous when the domain and codomain have the usual topology, but not continuous when they both have the ray topology or when they both have the Sorgenfrey topology.
3. Characterize the continuous functions from $\mathbb{R}_{\text{co-countable}}$ to $\mathbb{R}_{\text{usual}}$.
4. Characterize the continuous functions from $(\mathbb{R}, \mathcal{T}_7)$ to $\mathbb{R}_{\text{usual}}$.
5. Describe what properties a function $f : X \to Y$ must have in order to be continuous if Y has the discrete topology. This vague-seeming question will come up again quite late in the course.

4 Open and closed functions

Just a brief interlude to mention another sort of function. We mention it for a few reasons. First, it is a useful enough property of functions to give a name, as we will soon see. Second, it is a natural property to consider. So natural, in fact, that a student new to the field might expect that *this* is the sort of function that "preserves topological structure". Thirdly, it is a concept students often conflate with continuity, so we would like to call specific attention to the differences between this and continuity.

Definition 4.1. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $f : X \to Y$ be a function.*

- *f is said to be an* <u>*open function*</u> *if $f(U) \in \mathcal{U}$ for all $U \in \mathcal{T}$.*
- *f is said to be a* <u>*closed function*</u> *if $f(C)$ is a closed subset of Y whenever C is a closed subset of X.*

So a function is open if the images of open sets are open, whereas a function is continuous if the preimages of open sets are open. These sound similar, but open and continuous are very different.

Here is some intuition for why "continuity" has the definition it does. The idea is that the function $f : X \to Y$ should tell you about the topology on Y in terms of the topology on X. So we require that given some information about the topology on Y—an open subset of Y—the function returns an open subset of X that we know about.

Very colloquially speaking, a continuous function gathers information about Y and brings it back for analysis, whereas an open function just yells things at Y.

Also, **caution**! It seems like "open function" and "closed function" might be equivalent somehow (via taking complements, or something?), but they are not. The first two examples below illustrate this.

Example 4.2.

1. Consider the constant function $f : \mathbb{R}_{\text{usual}} \to \mathbb{R}_{\text{usual}}$ given by $f(x) = 7$. We already know that f is continuous. The image of any nonempty set is $\{7\}$, and therefore f is closed but not open.

2. The projection function defined in Example 2.2.3 continuous and open, but not closed. Try to show this.

3. Any function from any space to a discrete space is both open and closed.

4. Let $f : \mathbb{R}_{\text{Sorgenfrey}} \to \mathbb{R}_{\text{Sorgenfrey}}$ be the absolute value function $f(x) = |x|$. Then f is not open, since for example $f([-7, -1)) = (1, 7]$, which is not open.

Just to wrap up the story about open and closed not being equivalent even though they might seem like they should be:

Proposition 4.3. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $f : X \to Y$ be a bijection. Then f is open if and only if it is closed.*

Proof. **Exercise.** □

5 Homeomorphisms

Arguably, this section is the ultimate payoff of studying continuous functions. Here we define a particularly nice sort of continuous function that provides a way of detecting whether two topological spaces are "the same" from the point of view of topological structure. If you have studied some other branches of math, these functions are the analogues of bijective linear transformations for vector spaces, isomorphisms between groups/rings/fields/modules/graphs/linear orders/etc., diffeomorphisms between manifolds, etc.

Definition 5.1. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $f : X \to Y$ be a bijection. f is said to be a homeomorphism if f is continuous and its inverse f^{-1} is continuous.*

In this case we say that $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are homeomorphic, and write $(X, \mathcal{T}) \simeq (Y, \mathcal{U})$, or more often simply $X \simeq Y$ if the topologies are understood from context.

In the spirit of how we approached continuity, here are some equivalent characterizations of homeomorphisms. Since this is such a strong property, these proofs are quite straightforward.

Proposition 5.2. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $f : X \to Y$ be a bijection. Then the following are equivalent.*

1. *f is a homeomorphism.*
2. *f is continuous and open.*

3. *f is continuous and closed.*
4. *$U \subseteq X$ is open if and only if $f(U) \subseteq Y$ is open.*

Proof. **Exercise.** □

Example 5.3.

1. Any non-constant affine linear function is a homeomorphism from $\mathbb{R}_{\text{usual}}$ to itself. For which other topologies on $\mathbb{R}$ that we know is this true?

2. Let X be a set with two or more elements, and let $p \neq q \in X$. A function $f : (X, \mathcal{T}_p) \to (X, \mathcal{T}_q)$ is a homeomorphism if and only if it is a bijection such that $f(p) = q$.

3. A function $f : X \to Y$ where X and Y are discrete spaces is a homeomorphism if and only if it is a bijection.

4. $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a homeomorphism. (Technically we have never defined a topology on that open interval, but you should be able to imagine what it looks like.)

5. There exist no homeomorphisms $f : \mathbb{Q} \to \mathbb{R}_{\text{usual}}$, since these sets cannot be put into bijection with one another.

6 Topological invariants

As we said earlier, homeomorphisms are functions that preserve all topological properties (that is, all properties that can be described in terms of open sets). Often when studying topological spaces we will want to show that two spaces are or are not homeomorphic.

Showing that two spaces are homeomorphic requires that we construct a homeomorphism between them. There is no way around that. Showing that two spaces are *not* homeomorphic amounts to showing that *no bijection* from one to the other can be a homeomorphism, which is tricky to do in general. That is where topological invariants step in.

Definition 6.1. *A property ϕ of topological spaces is called a topological invariant if whenever $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are homeomorphic topological spaces, one has property ϕ if and only if the other has property ϕ.*

We can see that topological invariants furnish us with a quick series of checks on whether two spaces can be homeomorphic: if we can identify a topological invariant that one space has and another does not, they cannot be homeomorphic.

Every property we have studied so far is a topological invariant.

Proposition 6.2. *The following properties are topological invariants.*

1. *T_0 and T_1.*

2. *Hausdorff.*
3. *Separable.*
4. *Having a particular cardinality. (in particular, being countable or uncountable is a topological invariant)*
5. *First countable.*
6. *Second countable.*
7. *The countable chain condition (ccc).*

Proof. These proofs are all very straightforward, and are left to you as exercises. We will just do one here.

(Separable) Suppose X is separable and $f : X \to Y$ is a homeomorphism. Let $D \subseteq X$ be a countable dense set. We will show that $f(D) = \{ f(d) \in Y \, : \, d \in D \}$ is a countable dense subset of Y.

Clearly $f(D)$ is countable, since D is countable and f is a bijection. It remains to show that $f(D)$ intersects every nonempty open subset of Y. So fix such an open set $U \subseteq Y$. Then $f^{-1}(U)$ is an open subset of X, and therefore there is a $d \in D \cap f^{-1}(U)$. But then we immediately have that $f(d) \in f(D \cap f^{-1}(U)) = f(D) \cap U$, as required.

(Note that the last $=$ there is since f is a bijection. If it were not, we would instead have a $\subseteq$ there, which would still suffice for the proof. So really, this proof shows that the image of a dense set under a continuous, surjective function is dense. We will see some more specific problems like this on the Big List, especially after we define subspaces in the next section.) □

Exercise 6.3. Once you have filled out the table in BL 5.8, take note of which of those spaces we have proved are not homeomorphic to one another.

Exercise 6.4. Show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic with their usual topologies. This is surprisingly challenging at this point in the course. Later we will be able to do it easily, but at this point you will have to be creative.

Showing that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ (when $n \neq m$) in general is even more difficult, as the trick we will learn for the earlier case above will not work here. These spaces are much more alike topologically than they might seem, at least until we have some more powerful topological invariants to work with.