16. Compactness

1 Motivation

While metrizability is the analyst's favourite topological property, compactness is surely the topologist's favourite topological property. Metric spaces have many nice properties, like being first countable, very separative, and so on, but compact spaces facilitate easy *proofs*. They allow you to do all the proofs you wished you could do, but never could.

The definition of compactness, which we will see shortly, is quite innocuous looking. What compactness does for us is allow us to turn infinite collections of open sets into finite collections of open sets that do essentially the same thing. Compact spaces can be very large, as we will see in the next section, but in a strong sense every compact space acts like a finite space. This behaviour allows us to do a lot of hands-on, constructive proofs in compact spaces. For example, we can often take maxima and minima where in a non-compact space we would have to take suprema and infima. We will be able to intersect "all the open sets" in certain situations and end up with an open set, because finitely many open sets capture all the information in the whole collection.

We will specifically prove an important result from analysis called the Heine-Borel theorem that characterizes the compact subsets of $\mathbb{R}^n$. This result is so fundamental to early analysis courses that it is often given as the *definition* of compactness in that context.

2 Basic definitions and examples

Compactness is defined in terms of open covers, which we have talked about before in the context of bases but which we formally define here.

Definition 2.1. *Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{U} \subseteq \mathcal{T}$ be a collection of open subsets of X. We say $\mathcal{U}$ is an open cover of X if $X = \bigcup \mathcal{U}$.*

If $\mathcal{U}$ is an open cover of X and $\mathcal{V} \subseteq \mathcal{U}$ is a subcollection of $\mathcal{U}$ that is also an open cover of X, we say $\mathcal{V}$ is a subcover of $\mathcal{U}$.

Though the technical term is *open cover*, we will often refer to "covers" since open covers are the only sorts of covers we will discuss.

Example 2.2. Just a few examples here. We will save most of the discussion for after we have given the main definition.

1. In $\mathbb{R}_{\text{usual}}$, the following are both open covers.

$$\mathcal{U}_1 = \{ (-x, x) : x > 0 \} \quad \text{and} \quad \mathcal{U}_2 = \{ (n, n+2) : n \in \mathbb{Z} \}.$$

Note that $\mathcal{U}_1$ is an uncountable cover, and has many redundant sets from the point of view of covering $\mathbb{R}$. You can remove any finite number of sets, or even uncountably many sets, and still end up with a cover since for example $\mathcal{V}_1 = \{ (-n, n) : n \in \mathbb{N} \}$ is a subcover of $\mathcal{U}_1$. Note however that no *finite* subcollection of $\mathcal{U}_1$ can cover $\mathbb{R}$ (be sure to prove this to yourself).

$\mathcal{U}_2$ on the other hand is a countable cover and has no subcovers at all. As soon as you remove the interval $(6, 8)$, for example, the point 7 is no longer covered by any set in $\mathcal{U}_2$.

2. Along similar lines as $\mathcal{U}_1$ above, $\{ B_\epsilon(x) : x \in \mathbb{R}^n, \epsilon > 0 \}$ is a cover of $\mathbb{R}^n_{\text{usual}}$ that has countable subcovers but no finite subcovers.

3. In $\mathbb{R}_{\text{Sorgenfrey}}$, $\{ [-x, x) : x > 0 \}$ is a cover like $\mathcal{U}_1$ above that has many countable subcovers but no finite subcovers.

4. Let X be a nonempty set with its discrete topology. Then $\{ \{x\} : x \in X \}$ is a cover that has no subcovers.

Now, the main definition of this section.

Definition 2.3. *A topological space $(X, \mathcal{T})$ is said to be compact if every open cover of X has a finite subcover.*

We will often refer to subsets of topological spaces being compact, and in such a case we are technically referring to the subset as a topological space with its subspace topology. However in such situations we will talk about covering the subset with open sets from the larger space, so as not to have to intersect everything with the subspace at every stage of a proof.

The following is a related definition of a similar form.

Definition 2.4. *A topological space $(X, \mathcal{T})$ is said to be Lindelöf if every open cover of X has a countable subcover.*

Obviously every compact space is Lindelöf, but the converse is not true.

Exercise 2.5. Show that every compact space is Lindelöf, and find an example of a topological space that is Lindelöf but not compact.

Some examples:

Example 2.6.

1. $\mathbb{R}_{\text{usual}}$ is not compact. We have already shown this, since the covers $\mathcal{U}_1$ and $\mathcal{U}_2$ defined in Example 2.2.1 have no finite subcovers. $\mathbb{R}_{\text{usual}}$ turns out to be Lindelöf, though the proof is not obvious.

2. An open interval in $\mathbb{R}_{\text{usual}}$, such as $(0,1)$, is not compact. You should expect this since even though we have not mentioned it, you should expect that compactness is a topological invariant.

3. Similarly, $\mathbb{R}^n_{\text{usual}}$ is not compact, as we have also already seen. It is Lindelöf, though again this is not obvious.

4. If X is a set, then $(X, \mathcal{T}_{\text{discrete}})$ is compact if and only if X is finite, and Lindelöf if and only if X is countable. More generally, any finite topological space is compact and any countable topological space is Lindelöf.

5. For any set X, $(X, \mathcal{T}_{\text{indiscrete}})$ is compact.

6. $[0,1]$ with its usual topology is compact. This is not obvious at all, but we will prove it shortly.

7. $\omega + 1$ is compact. To see this, first recall that we have already seen that any nontrivial basic open set containing the top point ω must be of the form $(n, \infty) = (n, \omega]$ for some $n \in \mathbb{N}$. Now let $\mathcal{U}$ be any open cover of $\omega + 1$, and let $U_\omega \in \mathcal{U}$ be any set that contains ω. By the previous discussion, U_ω must contain a basic open set of the form $(n, \omega]$ for some $n \in \mathbb{N}$. Then for each $k \leq n$, pick some $U_k \in \mathcal{U}$ that contains k. Then $\{U_1, U_2, \ldots, U_n, U_\omega\}$ is a finite subcover of $\mathcal{U}$.

8. As a generalization of the previous example, let $(X, \mathcal{T})$ be a topological space, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X that converges to a point $x \in X$. Then $\{\, x_n : n \in \mathbb{N} \,\} \cup \{x\}$ is compact (with its subspace topology inherited from $\mathcal{T}$). Spaces like this are somehow the "minimal" infinite compact spaces.

9. ω_1 with its order topology is not Lindelöf, and therefore not compact. To see this, consider the cover $\{\, (-\infty, \alpha] : \alpha \in \omega_1 \,\}$. Convince yourself that this is an open cover, and that it has no countable subcovers.

10. $\omega_1 + 1$ with its order topology is compact. Proving this will be an exercise on the Big List.

3 Basic results

As you would expect, both of the properties defined in the previous section are topological invariants. In fact, we can do even better:

Proposition 3.1. *Let $(X, \mathcal{T})$ be a compact (respectively Lindelöf) topological space, let $(Y, \mathcal{U})$ be a topological space, and suppose $f : X \to Y$ is a continuous surjection. Then $(Y, \mathcal{U})$ is compact (respectively Lindelöf).*

Proof. We give the proof for compactness. Suppose $\mathcal{U}$ is an open cover of Y. Then since f is continuous, $\mathcal{V} = \{ f^{-1}(U) : U \in \mathcal{U} \}$ is an open cover of X. Since $(X, \mathcal{T})$ is compact there is a finite subcover

$$\{f^{-1}(U_1), f^{-1}(U_2), \ldots, f^{-1}(U_n)\}.$$

of $\mathcal{V}$. Then $\{U_1, U_2, \ldots, U_n\}$ is a finite subcover of $\mathcal{U}$, since f is surjective. □

A topologist would describe the result of the previous proposition as "continuous images of compact sets are compact", and so on.

Proposition 3.2. *Compactness is not hereditary.*

Proof. We already know this from previous examples. For example $(0, 1)$ is a non-compact subset of the compact space $[0, 1]$. Also $\mathbb{N}$ is a non-compact subset of the compact space $\omega + 1$. □

The previous exercise should lead you to think about defining "hereditary compactness". That property does come up occasionally, but it is *extremely* strong. So strong as to be almost useless. You will explore this a little bit during a Big List problem.

Proposition 3.3. *Lindelöf-ness is not hereditary.*

Proof. $\omega_1 + 1$ is a compact space with ω_1 as a non-Lindelöf subspace. □

Again, this should lead you to considering "hereditarily Lindelöf" spaces. This is a very interesting property indeed. The relationship between this property and hereditary separability is of particular interest, but it falls outside the scope of this introductory course.

How productive these properties are is an interesting question, which we will explore in a later section. For now, here are a few very simple results that showcase the usefulness of compactness.

Proposition 3.4. *Let $(X, \mathcal{T})$ be a compact topological space and $C \subseteq X$ a closed subset. Then C is compact (with its subspace topology).*

Proof. Let $\mathcal{U}$ be an open cover of C. Then by definition of the subspace topology, each $U \in \mathcal{U}$ is of the form $U = C \cap V_U$ for some open set $V_U \in \mathcal{T}$. But then $\mathcal{V} := \{ V_U : U \in \mathcal{U} \} \cup \{ X \setminus C \}$ is an open cover of X. Since X is compact $\mathcal{V}$ has a finite subcover of the form $\{ V_{U_1}, V_{U_2}, \ldots, V_{U_n}, X \setminus C \}$. But then $\{ U_1, U_2, \ldots, U_n \}$ is a finite subcover of $\mathcal{U}$, as required. □

Proposition 3.5. *Let $(X, \mathcal{T})$ be a compact Hausdorff space. Then $(X, \mathcal{T})$ is regular.*

Proof. Let $x \in X$ and let $C \subseteq X$ be a closed set not containing x. For each $c \in C$, let U_c and V_c be disjoint open sets containing c and x, respectively. We can find these because $(X, \mathcal{T})$ is

Hausdorff. Then $\mathcal{U} = \{ U_c : c \in C \}$ is an open cover of C. By the previous proposition, C is compact, and therefore there is a finite subcover $\{ U_{c_1}, U_{c_2}, \ldots, U_{c_n} \} \subseteq \mathcal{U}$. But then

$$U := U_{c_1} \cup \cdots \cup U_{c_n} \quad \text{and} \quad V := V_{c_1} \cap \cdots \cap V_{c_n}$$

are disjoint open sets containing C and x, as required. □

The previous proof seems simple, but the notable feature should be what compactness did for us. This is the same proof we wished we could do to show a Hausdorff space is regular, but in the general case we could have infinitely many V_c's to intersect, which might result in a set V that is not open. Since C is compact, we were able to reduce what could have been a very large intersection to a finite one.

In fact, we can do even better than this.

Proposition 3.6. *Let $(X, \mathcal{T})$ be a compact Hausdorff space. Then $(X, \mathcal{T})$ is normal.*

Proof. **Exercise.** Use the previous result, and do what you wish would work. Then notice that it actually does work because the space is compact. □

If we carefully examine the argument in the proof of Proposition 3.5, we find that we actually proved the following fact as well.

Proposition 3.7. *Let $(X, \mathcal{T})$ be a Hausdorff topological space, and let $K \subseteq X$ be compact. Then K is closed.*

Proof. **Exercise.** We essentially did this in the previous proof already. □

The following result should strike you as impressive, given how nice homeomorphisms are. Its proof is a simple combination of the results in this section.

Proposition 3.8. *Let $(X, \mathcal{T})$ be a compact topological space and let $(Y, \mathcal{U})$ be a Hausdorff topological space. Then any continuous bijection $f : X \to Y$ is a homeomorphism.*

Proof. The statement of the proposition amounts to saying that any continuous bijection $f : X \to Y$ is open, or equivalently closed.

So let $C \subseteq X$ be a closed set. We want to show that $f(C)$ is closed in Y. By Proposition 3.4 C is compact, and therefore $f(C)$ is compact by Proposition 3.1. By Proposition 3.7, we have that $f(C)$ is closed, finishing the proof. □

These are just a sampling of the results compactness allows us to prove.

4 Compactness and $\mathbb{R}^n$

The main goal of this section is to prove the Heine-Borel theorem, which says that a subset of $\mathbb{R}^n_{\text{usual}}$ is compact if and only if it is closed and bounded. This is a condition that any student who has taken multivariable calculus has likely seen as the *definition* of compactness in that context. We now know the true definition of compactness, but it will still be useful to see that this condition is equivalent. We will approach this result through a few smaller results.

In particular, we start by proving something we promised in Example 2.6.6, which is that $[0,1]$ is compact. This result is of course subsumed by the Heine-Borel theorem, but its proof is beautiful and worthy of independent study and appreciation.

Theorem 4.1. $[0,1]$ *with its usual topology is compact.*

Proof. This proof is sometimes called a "creeping along proof', for reasons that will soon become clear.

Let $\mathcal{U}$ be an open cover of $[0,1]$, which we think of as a collection of open subsets of $\mathbb{R}_{\text{usual}}$. Our task is to find a finite subcover. Define the following set:

$$B := \{\, s \in [0,1] \,:\, [0,s] \text{ can be covered by finitely many elements of } \mathcal{U} \,\}\,.$$

Having defined this, our task is now to simply show that $1 \in B$. Obviously $0 \in B$ (since $[0,0] = \{0\}$ and some element of $\mathcal{U}$ must contain 0) and B is bounded above by 1, so B has a least upper bound. Call this least upper bound b.

Claim. $b \in B$.

Proof. Obviously $b \in [0,1]$, and so some $U \in \mathcal{U}$ must contain b. Then by definition of the usual topology there is some $\epsilon > 0$ such that $(b-\epsilon, b+\epsilon) \subseteq U$. There must exist an $s \in (b-\epsilon, b] \cap B$, or else any such s would be an upper bound of B smaller than b. So fix such an s. Then by definition of B there is some finite subcollection $\{U_1, U_2, \ldots, U_N\} \subseteq \mathcal{U}$ that covers $[0,s]$. But then $\{U_1, U_2, \ldots, U_N, U\}$ covers $[0,b]$, and therefore $b \in B$ as required. □

Here is the "creeping along" part, from which we will deduce that $b = 1$.

Claim. *Suppose* $s \in B$ *and* $s < 1$*. Then there is a* $t > s$ *such that* $t \in B$*.*

Proof. To see this, fix $s \in B$ such that $s < 1$. Then by definition of B there is some finite subcollection $\{U_1, U_2, \ldots, U_N\} \subseteq \mathcal{U}$ that covers $[0,s]$. Without loss of generality, assume $s \in U_N$. By definition of the usual topology there is some $\epsilon > 0$ such that $s \in B_\epsilon(s) \subseteq U_N$, and by using a smaller ϵ if necessary, we may assume that this ϵ is small enough that $B_\epsilon(x) \subseteq (0,1)$. Let $t = s + \frac{\epsilon}{2}$. Then we have that $[s,t] \subseteq U_N$, and therefore

$$[0,t] = [0,s] \cup [s,t] \subseteq U_1 \cup \cdots \cup U_N,$$

which implies that $t \in B$. □

Claim. $b = 1$.

Proof. We know $b \leq 1$, so suppose for the sake of contradiction that it is strictly less than 1. $b \in B$ by the first claim, and therefore by the second claim there is some $t > b$ in B. But this contradicts the fact that $b = \sup B$. □

□

This is a very simple-looking argument. Read it closely, and try to see all the moving parts. In particular, try to do the same proof with other bounded sets that are not compact, like $(0, 1)$, $[0, 1] \cap \mathbb{Q}$, etc. and see where the techniques used in the proof break down.

If we examine this proof carefully, we can abstract away from the specifics of the usual metric topology on $\mathbb{R}$ and prove a more general result about complete linear orders ("complete" in this context means something different than it does in the context of metric spaces). We will save this for the Big List.

Now, as promised, the Heine-Borel theorem. At the moment, we only state and prove it for $\mathbb{R}_{\text{usual}}$. It turns out that we have done essentially all of the work already.

Theorem 4.2 (Heine-Borel theorem for $\mathbb{R}$)**.** *A subset of $\mathbb{R}_{usual}$ is compact if and only if it is closed and bounded.*

Proof. ($\Rightarrow$). Suppose $K \subseteq \mathbb{R}$ is compact. Since $\mathbb{R}_{\text{usual}}$ is Hausdorff, Proposition 3.7 implies that K is closed. It only remains to show that it is bounded.

The collection $\{ (-n, n) : n \in \mathbb{N} \}$ is an open cover of $\mathbb{R}$, and so $\mathcal{U} = \{ K \cap (-n, n) : n \in \mathbb{N} \}$ is an open cover of K (in its subspace topology). Since K is compact, there is some finite collection $F \subseteq \mathbb{N}$ of natural numbers such that $K \subseteq \bigcup_{n \in F}(-n, n)$. Of course, these open balls around the origin are all nested, so if we let $N = \max(F)$, we have that $K \subseteq (-N, N)$. In other words, K is bounded.

($\Leftarrow$). Suppose $K \subseteq \mathbb{R}$ is closed and bounded. Since it is bounded, there is an $N \in \mathbb{N}$ such that $K \subseteq [-N, N]$. By Theorem 4.1 and the fact that compactness is a topological invariant, $[-N, N]$ is compact. Then K is a closed subset of a compact space, and so is compact by Proposition 3.4 □

This allows us to conclude some useful things. For example:

Corollary 4.3. *Let $(X, \mathcal{T})$ be a compact topological space, and let $f : X \to \mathbb{R}_{usual}$ be a continuous function. Then f is bounded, and in fact it achieves a minimum and a maximum.*

Proof. **Exercise.** □

Corollary 4.4 (Extreme Value Theorem from first year calculus)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f achieves a minimum and a maximum.*

Proof. **Exercise.** □

The full general form of the Heine-Borel theorem, which says that a subset of $\mathbb{R}^n_{\text{usual}}$ is compact if and only if it is closed and bounded, requires a result analogous to Theorem 4.1 about $[0,1]^n$. We discuss that result here, because its proof is also interesting. We could have given this sort of proof for $[0,1]$, but the creeping along proof was too good to pass up.

Theorem 4.5. *$[0,1]^n \subseteq \mathbb{R}^n_{usual}$ is compact.*

Proof. Let $K_0 = [0,1]^n$, and suppose for the sake of contradiction that K_0 is not compact. Then there is an open cover $\mathcal{U}$ of K_0 with no finite subcover. K_0 should be thought of as an n-dimensional cube with side length 1, and so by bisecting each side of this cube we can divide K_0 into 2^n n-dimensional cubes of side length $\frac{1}{2}$.

If each of these 2^n cubes can be covered by finitely many sets from $\mathcal{U}$, then all of K_0 could be covered by finitely many sets from $\mathcal{U}$, contradicting our assumption that $\mathcal{U}$ has no finite subcovers. So let K_1 be one of these 2^n cubes with the property that no finite collection of sets from $\mathcal{U}$ covers K_1.

Continuing in the same way, by bisecting each side of the n-dimensional cube K_1 we produce 2^n even smaller n-dimensional cubes of side length $\frac{1}{4}$, one of which must not be able to be covered by finitely many sets from $\mathcal{U}$. Call this smaller cube K_2.

Inductively continuing this process, we define a sequence of n-dimensional cubes

$$K_0 \supset K_1 \supset K_2 \supset \cdots$$

where K_n is a cube with side length $\frac{1}{2^n}$ with the property that no finite collection of sets from $\mathcal{U}$ can cover any of the K_n's. For each n, pick any point $x_n \in K_n$. Then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}^n$ and therefore converges to some point $x \in \bigcap_{n\in\mathbb{N}} K_n$.

Since $\mathcal{U}$ covers K_0, there is some $U \in \mathcal{U}$ that contains x, and by definition of the usual topology on $\mathbb{R}^n$ there is some $\epsilon > 0$ such that $x \in B_\epsilon(x) \subseteq U$. Then we can pick N so large that $K_N \subseteq U$, contradicting the fact that K_N cannot be covered by finitely many sets from $\mathcal{U}$. □

Corollary 4.6. *For all $a > 0$, $[-a,a]^n \subseteq \mathbb{R}^n_{usual}$ is compact.*

With these tools established, we are ready to prove the full Heine-Borel theorem.

Theorem 4.7 (Heine-Borel theorem)**.** *A subset of $\mathbb{R}^n_{usual}$ is compact if and only if it is closed and bounded.*

Proof. **Exercise.** □

5 Productivity

In the next section of notes we are going to prove a tremendously powerful result about the productivity of compactness. In contrast to the somewhat non-constructive proof to come, we will give a very hands-on proof here that compactness is finitely productive.

Theorem 5.1. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be compact topological spaces. Then $X \times Y$ with its product topology is compact. In other words, compactness is finitely productive.*

We will prove this in two parts. The first is a general fact about finite products involving a compact space. Then we will use that fact to prove the theorem. The reader is strongly encouraged to draw some pictures along the way, as the proof makes a lot more sense with a picture accompanying it.

The following lemma says that if we have an open subset O of a product of two spaces containing a "slice" of the space, then there is an entire open "tube" containing that slice inside O.

Lemma 5.2. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and suppose Y is compact. Let $a \in X$, and let O be an open subset of $X \times Y$ that contains $\{a\} \times Y$, which is the "slice" at a. Then there is an open set $U \subseteq X$ containing a such that $U \times Y \subseteq O$.*

Proof. O is a union of basic open sets, so write $O = \bigcup \{ U_\alpha \times V_\alpha : \alpha \in I \}$, where I is some indexing set. The subspace $\{a\} \times Y$ is homeomorphic to Y, and therefore is compact. That means some finite subcollection $U_{\alpha_1} \times V_{\alpha_1}, \ldots, U_{\alpha_n} \times V_{\alpha_n}$ covers $\{a\} \times Y$. We may assume without loss of generality that $a \in U_{\alpha_k}$ for all $k = 1, \ldots, n$, since any other sets would be disjoint from the slice.

Let $U = U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$. Then $a \in U$ and $Y = V_{\alpha_1} \cup \cdots \cup V_{\alpha_1}$, so it is easy to see that

$$U \times Y \subseteq \bigcup_{k=1}^{n} (U_{\alpha_k} \times V_{\alpha_k}) \subseteq O$$

as required. This open set $U \times Y$ is usually called an "open tube". □

Before we go on, note that what we did with O is another notable use of compactness turning an infinite intersection of open sets into a finite intersection. O was an open set that contained some compact set, and O is necessarily a union of basic open sets, but compactness allowed us to focus only on finitely many basic open sets, and in turn to take a finite intersection.

Proof of Theorem 5.1. Let $\mathcal{U}$ be an open cover of $X \times Y$. For each $a \in X$, the slice $\{a\} \times Y$ is compact and therefore some finite subcollection $\{ U_1, U_2, \ldots, U_n \} \subseteq \mathcal{U}$ covers it. The union of these sets:

$$O_a := U_1 \cup \cdots \cup U_n$$

is an open set containing $\{a\} \times Y$, and therefore by the previous lemma there is an open set $U_a \subseteq X$ containing a such that $U_a \times Y \subseteq O_a$. In particular, note that the open tube $U_\alpha \times Y$ can be covered by finitely many sets from $\mathcal{U}$.

Now consider the collection $\{ U_a : a \in X \}$. This is an open cover of X. Since X is compact there is a finite set $a_1, a_2, \ldots, a_k$ such that $X = \bigcup_{i=1}^{k} U_{a_i}$. But then $X \times Y$ is covered by the corresponding tubes:

$$X \times Y = \bigcup_{i=1}^{k} (U_{a_i} \times Y).$$

Since each of these tubes can be covered by finitely many sets from $\mathcal{U}$, we are done. □