19. Compactifications

1 Motivation

At this point in the course, we have explored two sorts of spaces that we determined were very well-behaved and pleasant: metrizable spaces and compact Hausdorff spaces. Both of these spaces allow us to prove theorems much more easily than usual. There are, of course, many spaces which are of neither type, and so the best we can hope for in general is that a given space can be embedded into a space of one of these two nice types. Recall that we say a topological space $(X, \mathcal{T})$ embeds into a topological space $(Y, \mathcal{U})$ if X is homeomorphic to a subspace of Y.

We learned that metrizability is hereditary, and therefore there is nothing to be gained from saying that a space embeds into a metrizable space: if X embeds into a metrizable space Y, then X is homeomorphic to a metrizable space and so is itself metrizable. This can be a convenient way of proving a space is metrizable (this is what we did when we proved Urysohn's metrization theorem, for example), but does not say anything new about how nice the space is.

On the other hand, subspaces of compact Hausdorff spaces need not be compact, and so in this section we will explore the idea of embedding topological spaces into compact Hausdorff spaces. The mathematical terminology for this process, and for the compact Hausdorff space in question, is a compactification.

While formalizing this idea, we will define a new idea: local compactness. The spaces we will explore will not in general be compact, but we will see that they need to have some local features of compactness in order to be sufficiently well-behaved as to admit a compactification. Consider for example the spaces $\mathbb{R}_{\text{usual}}$ and $\mathbb{Q}$ (with its subspace topology). Neither are compact, but $\mathbb{R}_{\text{usual}}$ "feels nicer" when it comes to compact sets. There are nice, simple compact sets everywhere in $\mathbb{R}_{\text{usual}}$. The basic open intervals of the form (a, b) even have compact closures! On the other hand, $\mathbb{Q}$ has very few compact sets. Open intervals like $(a, b) \cap \mathbb{Q}$ are not compact, their closures are not compact... they are not even contained in any compact subsets of $\mathbb{Q}$ (we will prove this later in this note).

There are a number of different ways to go about compactifying a space. More specifically, given a topological space $(X, \mathcal{T})$ there are a number of natural-feeling ways to define compact Hausdorff spaces into which X might embed. We will explore two of these sorts of compactifications in this course. For example consider $(0, 1)$, which is not compact with its usual topology. It is easy to embed this space into several different compact Hausdorff spaces: it has obvious embeddings into $[0, 1]$, $[0, 1]^{\mathbb{N}}$, and S^1 just to name three. We will discuss which of these and other compactifications are actually worth talking about, and which are superfluous.

2 Two guiding examples

In this section we are going to naively treat two examples: $(0,1)$ with its usual topology and $(\mathbb{N}, \mathcal{T}_{\text{discrete}})$.

There are many compact Hausdorff spaces into which we can embed $(0,1)$ easily. For example:

- $(0,1)$ embeds as a subspace of $[0,1]$ via the natural inclusion map. This map misses the two endpoints 0 and 1.
- $(0,1)$ embeds as a subspace of $[0,2]$ via the natural inclusion map. This map misses 0 and all the points in $[1,2]$.
- $(0,1)$ embeds as a subspace of S^1 via the function $f(x) = (\cos(2\pi x), \sin(2\pi x))$. This map misses only the point $(1,0)$ in S^1.
- $(0,1)$ embeds as a subspace of $H = [0,1]^{\mathbb{N}}$, which we know is compact by Tychonoff's theorem, via the map $g(x) = (x, 1, 1, 1, \ldots)$, for example. (There are many ways to do this.)
- $(0,1)$ embeds as a subspace of the Topologist's Sine Curve S, for example via $x \mapsto (0,x)$.

These are all fine, but it should seem reasonable to you that if our goal is to realize $(0,1)$ as a subspace of a compact Hausdorff space, some of these examples are overkill. There is no need to consider the embedding we specified into $[0,2]$ when the embedding into $[0,1]$ does essentially the same thing without all the extra points that are far away from the "copy" of $(0,1)$. The embeddings into H an S are even worse in this regard. To be a little more formal, notice the images of the embeddings we described into $[0,1]$ and S^1 are dense in their respective codomains, while the embeddings into $[0,2]$, H, and S leave big chunks of their codomains untouched. Said another way, to embed $(0,1)$ into $[0,2]$ as we did earlier, one first embeds it into $[0,1]$, then embeds $[0,1]$ into $[0,2]$ via the natural inclusion map.

Additionally, we see that the embedding into S^1 seems to be the most "efficient", or "smallest" one, in the sense that the embedding only misses one point while the embedding into $[0,1]$ misses two points. We will shortly see that when $(0,1)$ is embedded into $[0,1]$ as we did above, the result is called a two-point compactification of $(0,1)$. The embedding into S^1 is a one-point compactification.

Now consider $\mathbb{N}$. It is even easier to embed $\mathbb{N}$ into a compact Hausdorff space. We can embed it into just about any infinite compact Hausdorff space.

- $\mathbb{N}$ embeds as a subspace of $\omega + 1$ via the natural inclusion. This map misses only the top point ω.
- $\mathbb{N}$ embeds as a subspace of $X = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\} \subseteq \mathbb{R}_{\text{usual}}$ via $n \mapsto \frac{1}{n}$. Again, this map misses just one point, in this case 0. X and $\omega + 1$ are homeomorphic, of course, so

this is what we expect. In this and the previous example, the image of $\mathbb{N}$ is dense in the codomain.

- $\mathbb{N}$ embeds as a subspace of $[0,1]$ via $n \mapsto \frac{1}{n}$. This map misses uncountably many points, and the image is certainly not dense in the codomain.
- $\mathbb{N}$ embeds as a subspace of ω_1+1, via $1 \mapsto m_1 := \min(\omega_1)$, $2 \mapsto m_2 := \min(\omega_1 \setminus \{m_1\})$, etc. This map misses uncountably many points and the image is not dense in the codomain.

Again, we can see here that the second two embeddings seem to involve a lot of superfluous points in the codomain, while the first two seem to use the least amount of new points possible (only adding one).

In the subsequent sections we will formalize what we mean by a compactification, and which compactifications are the largest and smallest ones.

3 One-Point compactifications

Definition 3.1. *A compactification of a topological space $(X,\mathcal{T})$ is an embedding of X as a dense subspace of a compact topological space. In other words, it is a compact topological space $(Y,\mathcal{U})$ and a map $f : X \to Y$ such that $f : X \to f(X)$ is a homeomorphism, and $\overline{f(X)} = Y$.*

We will only care about compactifications up to homeomorphism, so we will say two compactifications $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ are equivalent if there exists a homeomorphism $h : Y_1 \to Y_2$ that fixes the embedded elements of X, in the sense that $h(f_1(x)) = f_2(x)$ for all $x \in X$.

Note that this definition only cares about the "efficient" examples from above. The embeddings we defined that contained a large amount of superfluous information simply do not qualify as compactifications. They are not worth talking about.

This business of only caring about compactifications up to equivalence will ensure that we do not look at compactifications that look different but are really the same from a topological perspective. For example, it should be easy to see that the first two compactifications of $\mathbb{N}$ above are equivalent.

This definition does not mention Hausdorffness yet, but we will get there. First of all, we can immediately formalize the one-point compactifications we saw earlier.

Definition 3.2. *Let $(X,\mathcal{T})$ be a topological space. We define its one-point compactification in the following way:*

Let ∞ be a symbol which is not an element of X, and define the set $X^* := X \cup \{\infty\}$. We define a topology $\mathcal{T}^*$ on X^* by:

$$\mathcal{T}^* = \mathcal{T} \cup \{\, V \subseteq X^* : \infty \in V \text{ and } X \setminus V \text{ is closed and compact in } X \,\}.$$

For the sake of convenience, we will denote the topological space $(X^*,\mathcal{T}^*)$ by $\sigma(X)$.

Proposition 3.3. *If $(X, \mathcal{T})$ is not compact, then $\sigma(X)$ is a compactification of X.*

Proof. Suppose $(X, \mathcal{T})$ is not compact.

1. $\sigma(X)$ is compact. To see this, let $\mathcal{U}$ be an open cover of $\sigma(X)$. Then there is some $V \in \mathcal{U}$ that contains ∞. By definition of $\mathcal{T}^*$, $X^* \setminus V$ is a compact subset of X which is covered by $\mathcal{U} \setminus \{V\}$. Therefore there is some finite subcover of $X \setminus V$, which with the addition of V is a finite subcover of $\sigma(X)$.

   This argument should remind you of the proof that $\omega+1$ is compact (which is no accident).

2. $\overline{X} = X^*$. In fact, it is easy to see that $\overline{X} = X^*$ if and only if $(X, \mathcal{T})$ is not compact. This is more or less immediate, since the only way $\{\infty\}$ can be open in $\sigma(X)$ is if X is compact.

3. The natural inclusion $i : X \to X^*$ given by $i(x) = x$ is an embedding. This is also more or less immediate, since the open subsets of X in $\sigma(X)$ are exactly the elements of $\mathcal{T}$.

□

This defines a sort of *minimal* compactification of a space, in the sense that it is formed by adding exactly one point. Every non-compact space can be compactified in this way. Recall though that our motivation was to embed spaces in compact *Hausdorff* spaces, and we have made no mention of that so far. Here is an example of this going wrong.

Example 3.4. Consider $\mathbb{Q}$ with its usual topology. Then $\sigma(\mathbb{Q})$ is not Hausdorff.

To see this, we will try to separate ∞ and 0 with open sets. (Any two elements of $\mathbb{Q}$ can be separated in $\sigma(\mathbb{Q})$ since $\mathbb{Q}$ is embedded.) Suppose for the sake of contradiction that U and V are open disjoint subsets of $\sigma(\mathbb{Q})$ containing 0 and ∞ respectively. This means $U \subseteq \mathbb{Q} \setminus V$, the latter of which is closed and compact by definition of the topology on $\mathbb{Q}^*$.

But this is impossible: given an open interval of rationals $(a, b) \cap \mathbb{Q} \subseteq U$ containing 0, we can find an irrational number in (a, b) and a sequence of rationals from (a, b) (and in particular from the compact subset $\mathbb{Q} \setminus V$) converging to it in the usual topology on $\mathbb{R}$. If we view this sequence as living in just $\mathbb{Q}^*$, it will have no convergent subsequences, contradicting the fact that compact subsets of $\mathbb{Q}$ are sequentially compact.

The property we define next is designed precisely to ensure that problems like in the example above do not occur. There are many equivalent ways of defining this, but we state the simplest one here.

Definition 3.5. *A topological space $(X, \mathcal{T})$ is said to be locally compact if for every $x \in X$ there is a compact subset K of X and an open subset U of X such that $x \in U \subseteq K$. (A topologist might express this by saying "every point in X has a compact neighbourhood", but I will avoid that language here.)*

In particular, if $(X, \mathcal{T})$ is Hausdorff, then it is locally compact if and only if every point is contained in an open set with a compact closure.

Some examples to get warmed up.

Example 3.6.

1. $\mathbb{R}_{\text{usual}}$ is locally compact (and Hausdorff). We already know this, since the closures of open intervals are closed intervals, which are compact.
2. Every discrete space is locally compact (and Hausdorff), since singletons themselves are both open and compact.
3. Every compact space $(X, \mathcal{T})$ is trivially locally compact, since you can just choose $U = K = X$ for any point.
4. $\mathbb{Q}$ is not locally compact (but is Hausdorff). We essentially proved this above.
5. $\sigma(\mathbb{Q})$ is locally compact (since it is compact) but not Hausdorff.
6. $\mathbb{R}_{\text{Sorgenfrey}}$ is also not locally compact (but is Hausdorff), though this is trickier to see. It follows immediately from the (surprising) fact that all compact subsets of the Sorgenfrey line are countable.

Some elementary properties of local compactness, whose proofs are mostly left as exercises:

Proposition 3.7.

1. *Local compactness is a topological invariant.*
2. *Continuous images of locally compact spaces are not necessarily locally compact.*
3. *Local compactness is not hereditary, but open subsets and closed subsets of locally compact Hausdorff spaces are locally compact.*
4. *Local compactness is finitely productive but not countably productive. In fact, a product of locally compact spaces is locally compact if and only if all but finitely many of the spaces are compact.*

Proof.

1. **Exercise.**
2. Recall that any topological space is a continuous image of a discrete space.
3. We know it is not hereditary since $\mathbb{Q} \subseteq \mathbb{R}$. The proof for open and closed subsets of locally compact Hausdorff spaces is an easy **exercise**.
4. The proof for finite products is an easy **exercise**. You may attempt the proof of the more general fact (it is not particularly difficult), but we will not need it for the remainder of this note.

□

With that out of the way, here is why we defined it:

Proposition 3.8. *A topological space $(X, \mathcal{T})$ is locally compact and Hausdorff if and only if $\sigma(X)$ is Hausdorff.*

Proof. ($\Rightarrow$). Let $x, y \in X^*$ be two distinct points. If $x, y \in X$, then we can separate them with open sets since X is Hausdorff. So without loss of generality assume $y = \infty$. Since X is locally compact we can find a compact set $K \subseteq X$ and an open set $U \subseteq X$ such that $x \in U \subseteq K$. Then U and $X^* \setminus K$ are disjoint open subsets of X^* separating x and y, respectively.

($\Leftarrow$). Suppose $\sigma(X)$ is Hausdorff. Then X is Hausdorff since it is homeomorphic to a subspace of a Hausdorff space. X is locally compact since it is homeomorphic to an open subset of the locally compact space $\sigma(X)$. □

A bonus fact: recall that a topological space $(X, \mathcal{T})$ is called completely regular if given a point $x \in X$ and a closed set C not containing x, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(c) = 1$ for all $c \in C$. $(X, \mathcal{T})$ is called $T_{3.5}$ or Tychonoff if it is completely regular and Hausdorff. On the Big List you examined this property and saw that the property of being Tychonoff is stronger than T_3. Urysohn's Lemma easily shows us that every T_4 space is Tychonoff. It is also easy to see that the property of being Tychonoff is hereditary.

Proposition 3.9. *Every locally compact Hausdorff space is Tychonoff, and the reverse implication is not true in general.*

Proof. Suppose $(X, \mathcal{T})$ is locally compact Hausdorff. Then its one-point compactification $\sigma(X)$ is a compact Hausdorff space, which is T_4 and so is Tychonoff. Then X is homeomorphic to $\sigma(X) \setminus \{\infty\}$, which is Tychonoff since the property of being Tychonoff is hereditary.

Conversely, $\mathbb{Q}$ is T_4 and therefore Tychonoff, but not locally compact. □

4 Examples of one-point compactifications

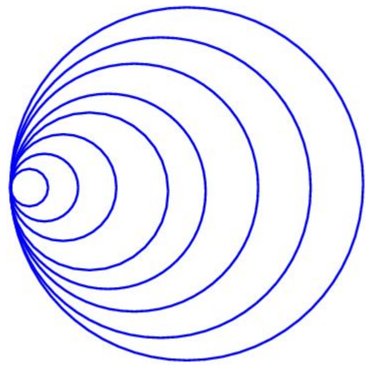
This is the fun part, where we describe some one point compactifications. Recall that we are only interested in these spaces up to homeomorphism, so we often describe them in terms of other, more familiar spaces.

1. $\sigma(\mathbb{N}_{\text{discrete}}) \simeq \omega + 1$. Given what we have defined in the previous section, this is easy to see.

2. $\sigma((0, 1)_{\text{usual}}) \simeq \sigma(\mathbb{R}_{\text{usual}}) \simeq S^1$. This is also easy to see.

3. Let $X = (0, 1) \cup (7, 8)$, as a subspace of $\mathbb{R}_{\text{usual}}$. Then $\sigma(X)$ is homeomorphic to a figure-eight, thought of as a subspace of $\mathbb{R}^2_{\text{usual}}$.

4. More generally, if X is a disjoint union of n open intervals in $\mathbb{R}$, then $\sigma(X)$ is homeomorphic to n circles in $\mathbb{R}^2_{\text{usual}}$ that are disjoint except for a single common point. In other words, a shape that looks something like this:



5. $\sigma(\omega_1) = \omega_1 + 1$.

6. If U is any open ball in $\mathbb{R}^2_{\text{usual}}$, then $\sigma(U) \simeq \sigma(\mathbb{R}^2_{\text{usual}}) \simeq S^2$, where S^2 is the two-dimensional unit sphere in $\mathbb{R}^3$:
$$S^2 = \left\{ (x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 + z^2 = 1 \right\}.$$
The map that witnesses this is the usual stereographic projection map, which for convenience we describe going in the other direction: $s : S^2 \setminus \{(0, 0, 1\} \to \mathbb{R}^2$ given by
$$s((x, y, z)) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right).$$
It is easy to check that this is a continuous bijection with a continuous inverse.

7. If X is the union of two disjoint open balls in $\mathbb{R}^2_{\text{usual}}$, then $\sigma(X)$ is homeomorphic to a subspace of $\mathbb{R}^3_{\text{usual}}$ consisting of two spheres touching at only a single point. These are sometimes called "kissing spheres".

You will get some more questions of this sort on the Big List.

5 The Stone-Čech compactification

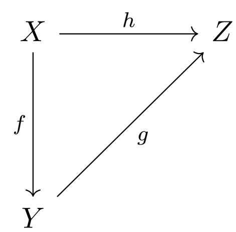
When defining one-point compactifications we remarked that they were necessarily the smallest possible compactification of a topological space, since they are obtained by adding exactly one point. In this section we discuss the opposite: the *largest* compactification of a topological space. We will approach this idea through some simpler results.

The following result is what actually justifies talking about compact spaces into which a space X embeds *densely*. It essentially says that if X embeds as a subspace of a compact Hausdorff space Z, then there is actually a compactification of X embedded in Z.

Proposition 5.1. *Let $(X, \mathcal{T})$ be a topological space, and suppose $h : X \to Z$ is an embedding into some compact Hausdorff space Z. Then there is a compactification $f : X \to Y$ of X such that there is an embedding $g : Y \to Z$ with the property that $h(x) = g(f(x))$ for all $x \in X$.*

Moreover, the compactification Y is unique up to equivalence.

Remark 5.2. This theorem This result looks complicated, but it is actually quite simple. The statement of the theorem is best illustrated with a diagram, which is a picture like the following:



The map h across the top is the given embedding of X into a compact Hausdorff space. The theorem asserts the existence of a compactification $f : X \to Y$ and an embedding g of Y into Z. The final condition in the theorem says that the diagram commutes, meaning that if you get from X to Z along either of the two paths of arrows, the resulting function compositions are the same.

Proof. Let $h : X \to Z$ be an embedding into a compact Hausdorff space. Then $h : X \to h(X)$ is a homeomorphism. Let $Y = \overline{h(X)} \subseteq Z$. Then Y is a closed subset of a compact Hausdorff space, and is therefore a compactification of X via the map h. Clearly Y embeds into Z via the natural inclusion map $i : Y \to Z$, which has the required properties.

It is more interesting (and trickier) to prove the uniqueness of the compactification Y up to equivalence. Suppose $f : X \to Y'$ is another compactification of X that embeds into Z via a map $g : Y' \to Z$ such that $h(x) = g(f(x))$. Then $g(f(X)) = h(X)$.

We first show that $g(Y') = \overline{h(X)}$. On the one hand, g is continuous and $Y' = \overline{f(X)}$ since Y' is a compactification. Therefore $g(Y') = g(\overline{f(X)}) \subseteq \overline{h(X)}$. Conversely, $g(Y')$ is a continuous image of a compact set, and so is compact and therefore closed since Z is Hausdorff. Clearly $h(X) = g(f(X)) \subseteq g(Y')$, and therefore $\overline{h(X)} \subseteq g(Y')$ since $g(Y')$ is closed.

Having established this, the map $g^{-1} \circ i : Y \to Y'$ is easily seen to be a homeomorphism with the required property. That is, we showed $Y = g(Y')$ and since g is an embedding, there's not much to do. □

Again, all this result says is that any embedding of a space $(X, \mathcal{T})$ into a compact Hausdorff space Z must "pass through" some compactification along the way. You are really embedding X into one of its compactifications, and then embedding the compactification into Z.

There is no hope of finding an interesting "largest" compactification Y that can always act as an intermediary compactification as in the above result; a space can be embedded into its one-point compactification, and so any such Y could not be larger than that.

We can do pretty well for ourselves though. For a Tychnoff space X, there is a *unique* (up to equivalence) compactification that in some sense "knows about" all continuous maps from X into all compact Hausdorff spaces. This is called the Stone-Čech compactification. A great deal of theory surrounds this construction, and it can be done in varying levels of generality. In these notes, we give a general treatment and a specific treatment of the case of the natural numbers.

Definition 5.3. *Let $(X, \mathcal{T})$ be a Tychonoff space (that is, a completely regular Hausdorff space). The Stone-Čech compactification of X is the unique (up to equivalence) Hausdorff compactification of X, usually denoted by βX, satisfying the following universal property:*

If $f : X \to Z$ is a continuous map into a compact Hausdorff space Z, there is a unique continuous function $\beta f : \beta X \to Z$ such that $f = \beta f \circ i$. A mathematician might say that any such map f "lifts" or "factors through" βX.

Theorem 5.4. *Let $(X, \mathcal{T})$ be a Tychonoff space. Then its Stone-Čech compactification exists.*

Proof. **Uniqueness**: We first prove that if a compactification satisfying the given universal property exists, then it is unique up to equivalence. Suppose $i : X \to Z$ and $i' : X \to Z'$ are two compactifications of X satisfying the given universal property. Then in particular i and i' are continuous functions from X into compact Hausdorff spaces, and so by the universal property there must exist continuous functions $g : Z \to Z'$ and $g' : Z' \to Z$ such that $g \circ i = i'$ and $g' \circ i' = i$. From this it immediately follows that $g' \circ g$ is the identity map on Z, and that $g \circ g'$ is the identity map on Z'. Therefore g is a continuous function with a continuous inverse, making it the homeomorphism we require.

Existence: Now we construct a compactification satisfying the required property. This is another example of embedding a space into a large product. Let $(X, \mathcal{T})$ be a Tychonoff space. Let C be the collection of all continuous functions $X \to [0, 1]$ (which is certainly nonempty since X is completely regular).

The product space $[0, 1]^C$ is Hausdorff, and compact by Tychonoff's theorem. Let $i : X \to [0, 1]^C$ be the map that sense $x \in X$ to the evaluation at x. That is, for $x \in X$, define

$$i(x) = e_x : C \to [0, 1] \quad \text{given by} \quad e_x(f) = f(x).$$

Finally, define $\beta X = \overline{i(X)}$, with its subspace topology inherited from $[0, 1]^C$. We first check that βX is a compactification, then check that it has the required universal property.

The map $i : X \to [0, 1]^C$ is an embedding by BL 15.2, which follows an argument essentially identical to the proof of Urysohn's metrization theorem. The collection of functions C can separate points from closed sets in X since X is completely regular, and the map i is the same as the map we used in that proof (except that its range is possibly an uncountable product this time). Clearly $i(X)$ is dense in βX, and βX is a compact Hausdorff space since it is a closed subspace of the compact Hausdorff space $[0, 1]^C$. Therefore $i : X \to \beta X$ is a compactification.

We now check that this compactification has the required universal property. This proof is somewhat tedious in general, but all the work was done on the Big List already.

Let $f : X \to K$ be a continuous function into a compact Hausdorff space K. We want to show that there is a unique continuous function $\beta f : \beta X \to K$ such that $f = \beta f \circ i$.

We first treat the special case in which $K = [0,1]$. In this case, $f \in C$, and therefore we can simply let $\beta f : \beta X \to [0,1]$ be the projection π_f onto the f^{th} coordinate (recalling that $\beta X \subseteq [0,1]^C$). Then we can check that for any $x \in X$:

$$(\pi_f \circ i)(x) = \pi_f(i(x)) = \pi_f(e_x) = e_x(f) = f(x).$$

It is also easy to check that π_f is the unique map which does this.

Returning to the general situation, suppose K is some compact Hausdorff space. Then K is $T_{3.5}$, and so by BL 17.3 it is homeomorphic to a subset of $[0,1]^J$ for some indexing set J, so for the remainder of this proof we will treat K as though it *is* a subset of this product. Seeing K this way, we have that $f : X \to K$ is in particular a continuous function to a product space, and therefore all of its coordinate functions are continuous functions $X \to [0,1]$. For each $\alpha \in J$ let f_α be the corresponding coordinate function. Then by the previous discussion, the projection $\pi_{f_\alpha} : \beta X \to [0,1]$ satisfies $\pi_{f_\alpha} \circ i = f_\alpha$. Let $\beta f : \beta X \to K$ be the product of these functions:

$$\beta f(g) = h_g : J \to [0,1] \quad \text{given by} \quad h_g(\alpha) = \pi_{f_\alpha}(g).$$

(I'm sure this looks very confusing, so we will go through it slowly. Given $g \in \beta X \subseteq [0,1]^C$, we are trying to define a function $h_g \in K \subseteq [0,1]^J$. We define it by saying how h_g acts on each $\alpha \in J$. For each α, we have these special projection functions $\pi_{f_\alpha} : \beta X \to [0,1]$ onto the coordinates of βX corresponding to the coordinate functions of f. Therefore we say that $h_g(\alpha) = \pi_{f_\alpha}(g)$ because that is the only thing we can do.)

It is routine if tedious to check that this βf is unique (all of the work is done in the case where $K = [0,1]$, since uniquely specifying the coordinates of a function to a product uniquely specifies the function itself). It is easy to check that it does what we want though, as long as we carefully follow through the definitions. Remember that we want to show that $\beta f \circ i = f$. So let $x \in X$. Then:

$$(\beta f \circ i)(x) = \beta f(i(x)) = \beta f(e_x) = h_{e_x},$$

where $h_{e_x} : J \to [0,1]$ is defined by

$$h_{e_x}(\alpha) = \pi_{f_\alpha}(e_x) = e_x(f_\alpha) = f_\alpha(x).$$

In other words, $\beta f \circ i : X \to K$ is the function $J \to [0,1]$ whose α^{th} coordinate agrees with the α^{th} coordinate of f. Therefore $\beta f \circ i = f$, as required. □

Phew! This is a very powerful result, and is one of the "ultimate" uses of our technique of embedding spaces into large products. You should look at this proof and try to find analogues to the proof of Urysohn's metrization theorem. Both arguments are really applications of this particular way of embedding things into large products via evaluation maps.

6 The Stone-Čech compactification of a discrete space

When dealing with the Stone-Čech compactification of a discrete topological space, a much nicer characterization in terms of ultrafilters is available. We will treat the case of $\mathbb{N}$ with its discrete topology, though just about everything we say here will apply to any discrete space. It is possible to characterize the Stone-Čech compactification of any $T_{3.5}$ space in terms of ultrafilters, but it is much more annoying for non-discrete spaces.

Recall that an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is called principal if either of the following two equivalent properties is true:

- $\mathcal{U}$ contains a finite set.
- There is an $n \in \mathbb{N}$ such that $\mathcal{U} = \mathcal{U}_n := \{ A \subseteq \mathbb{N} : n \in A \}$.

An ultrafilter that is not principal is called non-principal. Equivalently, $\mathcal{U}$ is non-principal if and only if for all $n \in \mathbb{N}$, $\{n, n+1, n+2, \dots\} \in \mathcal{U}$. (Make sure to convince yourself of this equivalence.)

Proposition 6.1. *Let $\beta\mathbb{N}$ be the set of all ultrafilters on $\mathbb{N}$. For each $A \subseteq \mathbb{N}$, let*

$$B_A = \{ \mathcal{U} \in \beta\mathbb{N} : A \in \mathcal{U} \},$$

be the collection of all ultrafilters that contain A. Then $\mathcal{B} = \{ B_A : A \subseteq \mathbb{N} \}$ is a basis for a topology $\mathcal{T}$ on $\beta\mathbb{N}$. Let $i : \mathbb{N} \to \beta\mathbb{N}$ be defined by $i(n) = \mathcal{U}_n$. Then this space along with the map i is the Stone-Čech compactification of $\mathbb{N}_{discrete}$.

Proof. **Exercise.** You will be guided through this on the Big List. Most of the things we have to show here are routine and easy. □

The topological space $\beta\mathbb{N}$, and even more so the space $\beta\mathbb{N} \setminus \mathbb{N}$ of all non-principal ultrafilters on $\mathbb{N}$ with its subspace topology from $\mathcal{T}$, is a relatively poorly-understood space, with bizarre behaviour from a set theoretic perspective.

Some properties of note:

- $\beta\mathbb{N}$ is obviously compact, Hausdorff, and separable.
- Its cardinality is $2^{\mathfrak{c}}$, or in other words equal to the cardinality of $\mathcal{P}(\mathbb{R})$.
- It is totally disconnected and therefore zero-dimensional.
- The only sequences that converge in $\beta\mathbb{N}$ are eventually constant. In particular $\beta\mathbb{N}$ is not sequentially compact, and therefore is not metrizable.
- No point of $\beta\mathbb{N} \setminus \mathbb{N}$ has a countable local basis. In particular $\beta\mathbb{N}$ and $\beta\mathbb{N} \setminus \mathbb{N}$ are not first countable.
- Any dense subset of $\beta\mathbb{N} \setminus \mathbb{N}$ must contain at least $\mathfrak{c} := |\mathbb{R}|$ many points. (In particular $\beta\mathbb{N} \setminus \mathbb{N}$ is very much not separable.)

- $\beta\mathbb{N} \setminus \mathbb{N}$ contains a (in fact many) homeomorphic copy of $\beta\mathbb{N}$.
- No point in $\beta\mathbb{N} \setminus \mathbb{N}$ is isolated (which is particularly interesting given how disconnected the space feels).